Instability in the Gel'fand inverse problem at high energies

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Abstract

We give an instability estimate for the Gel'fand inverse boundary value problem at high energies. Our instability estimate shows an optimality of several important preceeding stability results on inverse problems of such a type.

1 Introduction

In this paper we continue studies on the Gel'fand inverse boundary value problem for the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2, \\ &\text{with } \partial D \in C^2, \end{aligned} \tag{1.2}$$

$$v \in \mathbb{L}^\infty(D). \tag{1.3}$$

As boundary data we consider the map $\hat{\Phi} = \hat{\Phi}(E)$ such that

$$\hat{\Phi}(E)(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}|_{\partial D} \tag{1.4}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \tag{1.5}$$

The map $\hat{\Phi} = \hat{\Phi}(E)$ is known as the Dirichlet-to-Neumann map.

We consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. Given $\hat{\Phi}$ for some fixed E, find v.

This problem is known as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see [7], [19]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [5], [19]). Problem 1.1 can be also considered as an example of ill-posed problem: see [14], [3] for an introduction to this theory.

There is a wide literature on the Gel'fand inverse problem at fixed energy. In a similar way with many other inverse problems, Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness results and global reconstruction methods for Problem 1.1 were obtained for the first time in [19] in dimension $d \geq 3$ and in [4] in dimension $d = 2$.

Global logarithmic stability estimates for Problem 1.1 were obtained for the first time in [1] in dimension $d \geq 3$ and in [25] in dimension $d = 2$. A principal improvement of the result of [1] was obtained recently in [24] (for the zero energy case): stability of [24] optimally increases with increasing regularity of v.

Note that for the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [30] for $d \geq 3$ and in [17] for $d = 2$. Global logarithmic stability estimates for this problem were obtained for the first time in [1] for $d \geq 3$ and [15] for $d = 2$. Principal increasing of global stability of [1], [15] for the regular coefficient case was found in [24] for $d \geq 3$ and [28] for $d = 2$. In addition, for the case of piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [13]. Lipschitz stability estimate for the case of piecewise constant conductivity was obtained in [2] (see [27] for additional studies in this direction).

The optimality of the logarithmic stability results of [1], [15] with their principal effectivizations of [24], [28] (up to the value of the exponent) follows from [16]. An extention of the instability estimates of [16] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was obtained in [8].

On the other hand, it was found in [20], [21] (see also [23], [26]) that for inverse problems for the Schrödinger equation at fixed energy E in dimension $d \geq 2$ (like Problem 1.1) there is a Hölder stability modulo an error term rapidly decaying as $E \to +\infty$ (at least for the regular coefficient case). In addition, for Problem 1.1 for $d = 3$, global energy dependent stability estimates changing from logarithmic type to Hölder type for high energies were obtained in [12], [11]. However, there is no efficient stability increasing with respect to increasing coefficient regularity in the results of [12]. An additional study, motivated by [12], [24], was given in [18].

The following stability estimate for Problem 1.1 was recently proved in [11]:

Theorem 1.1 (of [11])**.** *Let D satisfy* (1.2)*, where $d \geq 3$. Let $v_j \in W^{m,1}(D)$, $m > d$, supp $v_j \subset D$ and $||v_j||_{W^{m,1}(D)} \leq N$ for some $N > 0$, $j = 1, 2$, (where $W^{m,p}$ denotes the Sobolev space of m-times smooth functions in $\mathbb{L}^p$). Let v_1, v_2 satisfy* (1.5) *for some fixed $E \geq 0$. Let $\hat{\Phi}_1(E)$ and $\hat{\Phi}_2(E)$ denote the DtN maps for v_1 and v_2, respectively. Let $s_1 = (m - d)/d$. Then, for any $\tau \in (0, 1)$ and any $\alpha, \beta \in [0, s_1]$, $\alpha + \beta = s_1$,*

$$||v_2 - v_1||_{L^\infty(D)} \leq A(1 + \sqrt{E})\delta^\tau + B(1 + \sqrt{E})^{-\alpha} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-\beta}, \qquad (1.6)$$

where $\delta = ||\hat{\Phi}_2(E) - \hat{\Phi}_1(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$ and constants $A, B > 0$ depend only on N, D, m, τ.

In particular cases, Hölder-logarithmic stability estimate (1.6) becomes coherent (although less strong) with respect to results of [21], [23], [24]. In this connection we refer to [11] for more detailed infromation. Concerning two-dimensional analogs of results of Theorem 1.1, see [20], [26], [28], [29].

In a similar way with results of [9], [10], estimate (1.6) can be extended to the case when we do not assume that condition (1.5) is fulfiled and consider an appropriate impedance boundary map (or Robin-to-Robin map) instead of the Dirichlet-to-Neumann map.

In the present work we prove optimality of estimate (1.6) (up to the values of the exponents α, β) in dimension $d \geq 2$. Our related instability results for Problem 1.1 are presented in Section 2, see Theorem 2.1 and Proposition 2.1. Their proofs are given in Section 4 and are based on properties of solutions of the Schrödinger equation in the unit ball given in Section 3.

2 Main results

In what follows we fix $D = B^d(0,1)$, where

$$B^d(x^0,\rho) = \{x \in \mathbb{R}^d : ||x - x^0||_{\mathbb{E}^d} < \rho\}, \quad x_0 \in \mathbb{R}^d,\ \rho > 0. \tag{2.1}$$

Let

$$\begin{aligned} ||F|| &\text{ denote the norm of an operator} \\ &F : \mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D). \end{aligned} \tag{2.2}$$

We recall that if v_1, v_2 are potentials satisfying (1.3), (1.5) for some fixed E, then

$$\hat\Phi_2(E) - \hat\Phi_1(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D), \tag{2.3}$$

where $\hat\Phi_1$, $\hat\Phi_2$ are the DtN maps for v_1, v_2, respectively, see [19], [22].

Our main result is the following theorem:

Theorem 2.1. *Let $D = B^d(0,1)$, where $d \geq 2$. Then for any fixed constants $A, B, \kappa, \tau, \varepsilon > 0$, $m > d$ and $s_2 > m$ there are some energy level $E > 0$ and some potential $v \in C^m(D)$ such that condition (1.5) holds for potentials v and $v_0 \equiv 0$, simultaneously, supp $v \subset D$, $\|v\|_{\mathbb{L}^\infty(D)} \leq \varepsilon$, $\|v\|_{C^m(D)} \leq C_1$, where $C_1 = C_1(d,m) > 0$, but*

$$||v - v_0||_{L^\infty(D)} > A(1+\sqrt{E})^\kappa \delta^\tau + B(1+\sqrt{E})^{2(s-s_2)}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s} \tag{2.4}$$

for any $s \in [0, s_2]$, where $\hat\Phi$, $\hat\Phi_0$ are the DtN map for v and v_0, respectively, and $\delta = ||\hat\Phi(E) - \hat\Phi_0(E)||$ is defined according to (2.2).

Theorem 2.1 shows, in particular, the optimality (at least for potentials in the neighborhood of zero) of estimate (1.6) (up to the values of the exponents α, β). As a corollary of Theorem 2.1, one can obtain an optimality of the stability results of [20], [21], [23], [26].

In the present work Theorem 2.1 is proved by explicit instability example with complex potentials. Examples of this type were considered for the first time in [16] for showing the exponential instability in Problem 1.1 in the zero energy case. An extention to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was obtained in [8].

Let us consider the cylindrical variables:

$$\begin{aligned} (r_1,\theta,x') &\in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{d-2}, \\ r_1\cos\theta = x_1,& \quad r_1 \sin\theta = x_2, \\ x' &= (x_3,\ldots,x_d). \end{aligned} \tag{2.5}$$

Take $\phi \in C^\infty(\mathbb{R}^2)$ with support in $B^2(0,1/3)\cap\{x_1 > 1/4\}$ and with $\|\phi\|_{\mathbb{L}^\infty} = 1$. For integers $m, n > 0$, define the complex potential

$$v_{nm} = n^{-m} e^{in\theta}\phi(r_1, |x'|). \tag{2.6}$$

We recall that

$$\|v_{nm}\|_{\mathbb{L}^\infty} = n^{-m}, \quad \|v_{nm}\|_{C^m} \le C_1, \tag{2.7}$$

where $C_1 = C_1(d,m) > 0$. Note that C_1 is the same as in Theorem 2.1. Estimates (2.7) were given in [16] (see Theorem 2 of [16]).

To prove Theorem 2.1 we use, in partucular, the following proposition:

Proposition 2.1. *Let $D = B^d(0,1)$, where $d \ge 2$. Let condition (1.5) hold with $v \equiv v_{nm}$ (of (2.6)) and $v \equiv v_0 \equiv 0$ for some $E > 0$ and some integers $m > 0$, $n > 20(1+\sqrt{E})^2$. Then, for any $\sigma > 0$,*

$$\|\hat\Phi_{nm}(E) - \hat\Phi_0(E)\|_{H^{-\sigma}(\mathbb{S}^{d-1})\to H^\sigma(\mathbb{S}^{d-1})} \le C_2(1+Q+EQ)2^{-n/4}, \tag{2.8}$$

where $\hat\Phi_{nm}$, $\hat\Phi_0$ are the DtN map for v_{nm} and v_0, respectively, $C_2 = C_2(d,\sigma) > 0$,

$$Q = \|(-\Delta + v_0 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} + \|(-\Delta + v_{nm} - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}, \tag{2.9}$$

where $(-\Delta + v_0 - E)^{-1}$, $(-\Delta + v_{nm} - E)^{-1}$ are considered with the Dirichlet boundary condition in D and $H^{\pm\sigma} = W^{\pm\sigma,2}$ denote the standart Sobolev spaces.

Analogs of estimate (2.8) (but without dependence of the energy) were given in Theorem 2 of [16] for the zero energy case and in Theorem 2.4 of [8] for the case of the non-zero energy and the case of the energy intervals.

We obtain Theorem 2.1, combining known results on the spectrum of the Laplace operator in the unit ball (see formula (4.9) below), Proposition 2.1, estimates (2.7) and the fact that

$$\|F\|_{L^\infty(\mathbb{S}^{d-1})\to L^\infty(\mathbb{S}^{d-1})} \le c(d,\sigma)\|F\|_{H^{-\sigma}(\mathbb{S}^{d-1})\to H^\sigma(\mathbb{S}^{d-1})} \tag{2.10}$$

for sufficiently large σ. The detailed proof of Theorem 2.1 and the proof of Proposition 2.1 are given in Section 4. These proofs use, in particular, results, presented in Section 3.

Remark 2.1. In a similar way with [16], [8], using a ball packing and covering by ball arguments (see also [6]), the instability result of Theorem 2.1 can be extended to the case when only real-valued potentials are considered and in the neighborhood of any potential (not only $v_0 \equiv 0$).

3 Some properties of solutions of the Schrödinger equation in the unit ball

In this section we continue assume that $D = B^d(0,1)$, where $d \ge 2$. We fix an orthonormal basis in $\mathbb{L}^2(\mathbb{S}^{d-1}) = \mathbb{L}^2(\partial D)$

$$\begin{aligned}&\{f_{jp} : j \ge 0,\ 1 \le p \le p_j\},\\ &f_{jp} \text{ is a spherical harmonic of degree } j,\end{aligned} \tag{3.1}$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$p_j = \binom{j+d-1}{d-1} - \binom{j+d-3}{d-1}, \tag{3.2}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for } n \geq 0 \tag{3.3}$$

and

$$\binom{n}{k} = 0 \quad \text{for } n < 0. \tag{3.4}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere and so $|x|^j f_{jp}(x/|x|)$ is harmonic. We use also the polar coordinates $(r,\omega) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$, with $x = r\omega \in \mathbb{R}^d$.

Lemma 3.1. *Let $D = B^d(0,1)$, where $d \geq 2$. Let potential v satisfy* (1.3) *and* (1.5) *for some fixed E. Let $||v||_{\mathbb{L}^\infty(D)} \leq N$, for some $N > 0$. Then for any solution $\psi \in C(D \cup \partial D)$ of equation* (1.1) *the following inequality holds:*

$$\|\psi\|_{\mathbb{L}^2(D)} \leq \Big(1 + (N+|E|)\|(-\Delta + v - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}\Big)\|f\|_{\mathbb{L}^2(\partial D)}, \tag{3.5}$$

where $f = \psi|_{\partial D}$, $(-\Delta + v - E)^{-1}$ is considered with the Dirichlet boundary condition in D.

Proof of Lemma 3.1. We expand the function f in the basis $\{f_{jp}\}$:

$$f = \sum_{j,p} c_{jp} f_{jp}. \tag{3.6}$$

We have that

$$\|f\|^2_{\mathbb{L}^2(\partial D)} = \sum_{j,p} |c_{jp}|^2. \tag{3.7}$$

Let

$$\psi_0(x) = \sum_{j,p} c_{jp} r^j f_{jp}(\omega). \tag{3.8}$$

Note that

$$\begin{aligned} \|\psi_0\|^2_{\mathbb{L}^2(D)} &= \sum_{j,p} |c_{jp}|^2 \|r^j f_{jp}(\omega)\|^2_{\mathbb{L}^2(D)} = \\ &= \sum_{j,p} |c_{jp}|^2 \int_0^1 r^{2j+d-1} dr \leq \sum_{j,p} |c_{jp}|^2 \end{aligned} \tag{3.9}$$

Using (1.1) and the fact that ψ_0 is harmonic, we get that

$$(-\Delta + v - E)(\psi - \psi_0) = (E - v)\psi_0. \tag{3.10}$$

Since $\psi|_{\partial D} = \psi_0|_{\partial D} = f$, using (3.10), we find that

$$\|\psi - \psi_0\|_{\mathbb{L}^2(\partial D)} \leq (N+|E|)\|(-\Delta + v - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}\|\psi_0\|_{\mathbb{L}^2(D)}. \tag{3.11}$$

Combining (3.7), (3.9), (3.11), we obtain (3.5). ∎

Let $< \cdot, \cdot >$ denote the scalar product in the Hilbert space $\mathbb{L}^2(\partial D)$:

$$< f, g >= \int\limits_{\partial D} f(x)\bar{g}(x)dx, \tag{3.12}$$

where $f, g \in \mathbb{L}^2(\partial D)$.

Lemma 3.2. *Let $D = B^d(0,1)$, where $d \geq 2$. Let potentials v_1, v_2 satisfy (1.3) and (1.5) for some fixed E. Let v_1, v_2 be supported in $B^d(0,1/3)$ and $||v_i||_{\mathbb{L}^\infty(D)} \leq N$, $i = 1,2$, for some $N > 0$. Then for any $j_1, j_2 \in \mathbb{N} \cup \{0\}$, $1 \leq p_1 \leq p_{j_1}$, $1 \leq p_2 \leq p_{j_2}$ and $j_{max} = \max\{j_1, j_2\} \geq 10(1 + \sqrt{|E|})^2$ the following inequality holds:*

$$\left|\left\langle f_{j_1p_1}, \left(\hat{\Phi}_1(E) - \hat{\Phi}_2(E)\right) f_{j_2p_2}\right\rangle\right| \leq C(d)\Big(1 + (N + |E|)Q\Big)2^{-j_{max}}, \tag{3.13}$$

where

$$Q = \|(-\Delta + v_1 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} + \|(-\Delta + v_2 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}, \tag{3.14}$$

$\hat{\Phi}_1$, $\hat{\Phi}_2$ are the DtN map for v_1 and v_2, respectively, and $(-\Delta + v_1 - E)^{-1}$, $(-\Delta + v_2 - E)^{-1}$ are considered with the Dirichlet boundary condition in D.

Analogs of estimate (3.13) (but without dependence of the energy) were given in Lemma 1 of [16] for the zero energy case and in Lemma 3.4 of [8] for the case of the non-zero energy and the case of the energy intervals.

We prove Lemma 3.2 for $E \neq 0$ in Section 5, using expression of solutions of equation $-\Delta\psi = E\psi$ in $B^d(0,1) \setminus B^d(0,1/3)$ in terms of the Bessel functions J_α and Y_α with integer or half-integer order α.

4 Proofs of Proposition 2.1 and Theorem 2.1

We continue to assume that $D = B^d(0,1)$, where $d \geq 2$ and to use the orthonormal basis $\{f_{jp} : j \in \mathbb{N} \cup \{0\},\ 1 \leq p \leq p_j\}$ in $\mathbb{L}^2(\mathbb{S}^{d-1}) = \mathbb{L}^2(\partial D)$. The Sobolev spaces $H^\sigma(\mathbb{S}^{d-1})$ can be defined by

$$\begin{gathered}\left\{\sum_{j,p} c_{jp}f_{jp} : \Big\|\sum_{j,p} c_{jp}f_{jp}\Big\|_{H^\sigma} < +\infty\right\},\\ \Big\|\sum_{j,p} c_{jp}f_{jp}\Big\|^2_{H^\sigma} = \sum_{j,p}(1+j)^{2\sigma}|c_{jp}|^2,\end{gathered} \tag{4.1}$$

see, for example, [16].

Consider an operator $A : H^{-\sigma}(\mathbb{S}^{d-1}) \to H^\sigma(\mathbb{S}^{d-1})$. We denote its matrix elements in the basis $\{f_{jp}\}$ by

$$a_{j_1p_1j_2p_2} =< f_{j_1p_1}, Af_{j_2p_2} > . \tag{4.2}$$

We identify in the sequel an operator A with its matrix $\{a_{j_1p_1j_2p_2}\}$. In this section we always assume that $j_1, j_2 \in \mathbb{N} \cup \{0\}$, $1 \leq p_1 \leq p_{j_1}$, $1 \leq p_2 \leq p_{j_2}$.

We recall that (see formula (12) of [16])

$$\|A\|_{H^{-\sigma}(\mathbb{S}^{d-1})\to H^{\sigma}(\mathbb{S}^{d-1})} \le 4 \sup_{j_1,p_1,j_2,p_2} (1+\max\{j_1,j_2\})^{2\sigma+d}|a_{j_1p_1j_2p_2}|. \quad (4.3)$$

Proof of Proposition 2.1. In a similar way with the proof of Theorem 2 of [16] we obtain that

$$< f_{j_1p_1}, \left(\hat\Phi_{mn}(E)-\hat\Phi_0(E)\right) f_{j_2p_2} >= 0 \quad (4.4)$$

for $j_{max} = \max\{j_1,j_2\} \le \left[\frac{n-1}{2}\right]$ (the only difference is that instead of the operator $-\Delta$ we consider the operator $-\Delta - E$), where $[\cdot]$ denotes the integer part of a number. Note that

$$\left[\frac{n-1}{2}\right] + 1 \ge n/2 > 10(1+\sqrt{E})^2, \quad \|v_{nm}\|_{\mathbb{L}^\infty(D)} \le 1. \quad (4.5)$$

Combining (4.3), (4.4), (4.5) and Lemma 3.2, we get that

$$\begin{aligned} &\|\hat\Phi_{mn}(E)-\hat\Phi_0(E)\|_{H^{-\sigma}(\mathbb{S}^{d-1})\to H^{\sigma}(\mathbb{S}^{d-1})} \le \\ &\quad\le 4C(d)\Big(1+(1+E)Q\Big) \sup_{j_{max}\ge n/2} (1+j_{max})^{2\sigma+d}2^{-j_{max}} \le \\ &\quad\le C_2(d,\sigma)(1+Q+EQ)2^{-n/4}, \end{aligned} \quad (4.6)$$

where

$$Q = \|(-\Delta+v_0-E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} + \|(-\Delta+v_{nm}-E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}. \quad (4.7)$$

■

Let $N(\rho)$ denote the counting function of the Laplace operator in D

$$N(\rho) = |\{\lambda < \rho^2 : \lambda \text{ is a Dirichlet eigenvalue of } -\Delta \text{ in } D\}|, \quad (4.8)$$

where $|\cdot|$ is the cardinality of the corresponding set. We recall that according to the Weyl formula (of [31]):

$$N(\rho) \le c_1(d)\rho^d. \quad (4.9)$$

Lemma 4.1. *Let $D = B^d(0,1)$, where $d \ge 1$. Then for any $\rho > 1$ there is some $E = E(\rho) \in (\rho^2, 2\rho^2)$ such that the interval*

$$\left(E(\rho) - c_2\rho^{2-d}, E(\rho) + c_2\rho^{2-d}\right) \quad (4.10)$$

does not contain Dirichlet eigenvalues of $-\Delta$ in D, where $c_2 = c_2(d) > 0$.

Proof of Lemma 4.1. We put $c_2 = 2^{d-1}/(c_1(d)+1)$. Then we can select k disjoint intervals of the length $2c_2\rho^{2-d}$ in the interval $(\rho^2, 2\rho^2)$, where

$$k = \left[\frac{\rho^2}{2c_2\rho^{2-d}}\right] = [(c_1(d)+1)\rho^d] > N(\rho). \quad (4.11)$$

Thus, we have that at least one of these intervals does not contain Dirichlet eigenvalues of $-\Delta$ in $D = B^d(0,1)$. ■

Proof of Theorem 2.1. Let $E = E(\rho)$ be the number of Lemma 4.1 for some $\rho > 1$. Using (4.10), we find that the distance from E to the Dirichlet spectrum of the operator $-\Delta$ in D is not less than $c_2\rho^{2-d}$. Using also that $E \in (\rho^2, 2\rho^2)$, we get that

$$\|(-\Delta - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} \le \frac{1}{c_2\rho^{2-d}} \le E^{(d-2)/2}/c_2, \tag{4.12}$$

where $(-\Delta - E)^{-1}$ is considered with the Dirichlet boundary condition in D. Let

$$n = [20(1+\sqrt{E})^2] + 1. \tag{4.13}$$

Using (2.7) and (4.10), we find that the distance from E to the Dirichlet spectrum of the operator $-\Delta + v_{nm}$ in D is not less than $c_2\rho^{2-d} - n^{-m}$, where v_{nm} is defined according to (2.6). Since $m > d$ and $E \in (\rho^2, 2\rho^2)$, using (4.13), we get that

$$\begin{aligned} \|(-\Delta + v_{nm} - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} \le c_3 E^{(d-2)/2}, \\ E = E(\rho), \quad \rho \ge \rho_1(d,m) > 1, \\ c_3 = c_3(d,m) > 0, \end{aligned} \tag{4.14}$$

where $(-\Delta + v_{nm} - E)^{-1}$ is considered with the Dirichlet boundary condition in D.

Combining Proposition 2.1 and estimates (2.10), (4.12), (4.14), we find that

$$\begin{aligned} \delta = \|\hat\Phi_{nm}(E) - \hat\Phi_0(E)\|_{\mathbb{L}^\infty(\mathbb{S}^{d-1})\to\mathbb{L}^\infty(\mathbb{S}^{d-1})} \le c_4 E^{d/2} 2^{-n/4}, \\ E = E(\rho), \quad \rho \ge \rho_1(d,m) > 1, \\ n = [20(1+\sqrt{E})^2] + 1 \\ c_4 = c_4(d,m) > 0. \end{aligned} \tag{4.15}$$

Since $s_2 > m$, taking ρ big enough and using (4.15), we obtain the following inequalities:

$$n^{-m} < \varepsilon, \tag{4.16}$$

$$A(1+\sqrt{E})^{\kappa}\delta^{\tau} < \frac{1}{2}n^{-m}, \tag{4.17}$$

$$\begin{aligned} B(1+\sqrt{E})^{2(s-s_2)}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s} < \frac{1}{2}n^{-m}, \\ 0 \le s \le s_2, \end{aligned} \tag{4.18}$$

where

$$E = E(\rho), \quad n = [20(1+\sqrt{E})^2] + 1. \tag{4.19}$$

Combining (2.6), (2.7), (4.16) - (4.19), we get that

$$\begin{aligned} A(1+\sqrt{E})^{\kappa}\delta^{\tau} + B(1+\sqrt{E})^{2(s-s_2)}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s} < \\ < \frac{1}{2}n^{-m} + \frac{1}{2}n^{-m} = \|v_{nm} - v_0\|_{\mathbb{L}^\infty(D)} \\ \|v_{nm}\|_{\mathbb{L}^\infty(D)} = n^{-m} < \varepsilon, \\ \|v_{nm}\|_{C^m(D)} < C_1, \\ \operatorname{supp} v_{nm} \subset D. \end{aligned} \tag{4.20}$$

■

5 Proof of Lemma 3.2

To prove Lemma 3.2 we need some preliminaries. Consider the problem of finding solutions of the form $\psi(r,\omega) = R(r)f_{jp}(\omega)$ of equation (1.1) with $v \equiv 0$ and $D = B^d(0,1)$, where $d \geq 2$. We recall that:

$$\Delta = \frac{\partial^2}{(\partial r)^2} + (d-1)r^{-1}\frac{\partial}{\partial r} + r^{-2}\Delta_{S^{d-1}}, \tag{5.1}$$

where $\Delta_{S^{d-1}}$ is Laplace-Beltrami operator on S^{d-1},

$$\Delta_{S^{d-1}} f_{jp} = -j(j+d-2)f_{jp}. \tag{5.2}$$

Then we obtain the following equation for $R(r)$:

$$-R'' - \frac{d-1}{r}R' + \frac{j(j+d-2)}{r^2}R = ER. \tag{5.3}$$

Taking $R(r) = r^{-\frac{d-2}{2}}\tilde{R}(r)$, we get

$$r^2\tilde{R}'' + r\tilde{R}' + \left(Er^2 - \left(j + \frac{d-2}{2}\right)^2\right)\tilde{R} = 0. \tag{5.4}$$

This equation is known as the Bessel equation. For $E = k^2 \neq 0$ it has two linearly independent solutions $J_{j+\frac{d-2}{2}}(kr)$ and $Y_{j+\frac{d-2}{2}}(kr)$, where

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\alpha}}{\Gamma(m+1)\Gamma(m+\alpha+1)}, \tag{5.5}$$

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos\pi\alpha - J_{-\alpha}(z)}{\sin\pi\alpha} \text{ for } \alpha \notin \mathbb{Z}, \tag{5.6}$$

and

$$Y_\alpha(z) = \lim_{\alpha' \to \alpha} Y_{\alpha'}(z) \text{ for } \alpha \in \mathbb{Z}. \tag{5.7}$$

We recall also that the system of functions

$$\begin{gathered}\left\{\psi_{jp}(r,\omega) = R_j(k,r)f_{jp}(\omega) : j \in \mathbb{N}\cup\{0\}, 1 \leq p \leq p_j\right\}, \\ \text{is complete orthogonal system (in the sense of } \mathbb{L}^2\text{) in the space} \\ \text{of solutions of equation (1.1) in } D' = B(0,1)\setminus B(0,1/3) \\ \text{with } v \equiv 0,\ E = k^2 \text{ and boundary condition } \psi|_{r=1} = 0,\end{gathered} \tag{5.8}$$

where

$$R_j(k,r) = r^{-\frac{d-2}{2}}\Big(Y_{j+\frac{d-2}{2}}(kr)J_{j+\frac{d-2}{2}}(k) - J_{j+\frac{d-2}{2}}(kr)Y_{j+\frac{d-2}{2}}(k)\Big). \tag{5.9}$$

For the proof of (5.8) see, for example, [8].

Lemma 5.1. *For any $\rho > 0$, integers $d \geq 2$, $n \geq 10(\rho+1)^2$ and $z \in \mathbb{C}$, $|z| \leq \rho$, the following inequalities hold:*

$$\frac{1}{2}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)} \leq |J_\alpha(z)| \leq \frac{3}{2}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)}, \tag{5.10}$$

$$|J'_\alpha(z)| \le 3\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}, \tag{5.11}$$

$$\frac{1}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \le |Y_\alpha(z)| \le \frac{3}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \tag{5.12}$$

$$|Y'_\alpha(z)| \le \frac{3}{\pi}(|z|/2)^{-\alpha-1}\Gamma(\alpha+1) \tag{5.13}$$

where $'$ denotes derivation with respect to z, $\alpha = n + \frac{d-2}{2}$ and $\Gamma(x)$ is the Gamma function.

In fact, the proof of Lemma 5.1 is given in [8] (see Lemma 3.3 of [8]). It was shown in [8] that inequalities (5.10) - (5.13) hold for any $n > n_0$, where n_0 is such that

$$\begin{cases} n_0 > 3, \\ \exp\left(\dfrac{\rho^2/4}{n_0+1}\right) - 1 \le 1/2, \\ 3\pi\dfrac{\max\left(1, (\rho/2)^{2n_0+1}\right)}{\Gamma(n_0)} + \dfrac{\rho^2}{2n_0-\rho^2} + \dfrac{(\rho/2)^{2n_0}e^{\rho^2/4}}{\Gamma(n_0)} \le 1/2, \end{cases} \tag{5.14}$$

(see formula (6.18) of [8]). The only thing to check is that $n_0 = [10(\rho+1)^2] - 1$ satisfy (5.14), where $[\cdot]$ denotes the integer part of a number, The first two inequalities are obvious. The third follows from the estimate

$$\Gamma(n_0) = (n_0-1)! \ge \left(\frac{n_0-1}{e}\right)^{n_0-1}. \tag{5.15}$$

The final part of the proof of Lemma 3.2 consists of the following: first, we consider the case when $E = k^2 \neq 0$ and

$$j_1 = \max\{j_1, j_2\} \ge 10(1+|k|)^2. \tag{5.16}$$

Let ψ_1, ψ_2 denote the solutions of equation (1.1) with boundary condition $\psi|_{\partial D} = f_{j_2p_2}$ and potentials v_1 and v_2, respectively. Using Lemma 3.1 for v_1 and v_2, we get that

$$\|\psi_1 - \psi_2\|_{\mathbb{L}^2(D)} \le 2\Big(1 + (N+|E|)Q\Big), \tag{5.17}$$

where

$$Q = \|(-\Delta + v_1 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} + \|(-\Delta + v_2 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}, \tag{5.18}$$

Note that $\psi_1 - \psi_2$ is the solution of equation (1.1) in $D' = B(0,1)\setminus B(0,1/3)$ with potential $v \equiv 0$ and boundary condition $\psi|_{r=1} = 0$. According to (5.8), we have that

$$\psi_1 - \psi_2 = \sum_{j,p} c_{jp}\psi_{jp} \quad \text{in } D' \tag{5.19}$$

for some c_{jp}, where

$$\psi_{jp}(r,\omega) = R_j(k,r)f_{jp}(\omega). \tag{5.20}$$

Since $R_j(k,1)=0$, we find that

$$\left.\frac{\partial R_j(k,r)}{\partial r}\right|_{r=1} = \left.\frac{\partial\left(r^{\frac{d-2}{2}}R_j(k,r)\right)}{\partial r}\right|_{r=1}. \quad (5.21)$$

For $j\ge 10(1+|k|)^2$, using Lemma 5.1, we have that

$$\begin{gathered}\left|\frac{\left.\frac{\partial R_i(k,r)}{\partial r}\right|_{r=1}}{Y_\alpha(k)J_\alpha(k)}\right| = |k|\left|\frac{Y_\alpha'(k)}{Y_\alpha(k)}-\frac{J_\alpha'(k)}{J_\alpha(k)}\right| \le \\ \le 6|k|\left(\frac{(|k|/2)^{-\alpha-1}\Gamma(\alpha+1)}{(|k|/2)^{-\alpha}\Gamma(\alpha)}+\frac{(|k|/2)^{\alpha-1}\Gamma(\alpha+1)}{(|k|/2)^{\alpha}\Gamma(\alpha)}\right)=6\alpha,\end{gathered} \quad (5.22)$$

$$\begin{gathered}\left(\frac{||r^{-\frac{d-2}{2}}Y_\alpha(kr)||_{\mathbb{L}^2(\{1/3<|x|<2/5\})}}{|Y_\alpha(k)|}\right)^2 \ge \\ \ge \int_{1/3}^{2/5}\left(\frac{1}{3}\frac{(|k|r/2)^{-\alpha}\Gamma(\alpha)}{(|k|/2)^{-\alpha}\Gamma(\alpha)}\right)^2 r\,dr \ge \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(\frac{1}{3}(5/2)^\alpha\right)^2,\end{gathered} \quad (5.23)$$

$$\begin{gathered}\left(\frac{||r^{-\frac{d-2}{2}}J_\alpha(kr)||_{\mathbb{L}^2(\{1/3<|x|<2/5\})}}{|J_\alpha(k)|}\right)^2 \le \\ \le \int_{1/3}^{2/5}\left(3\frac{(|k|r/2)^{\alpha}\Gamma(\alpha)}{(|k|/2)^{\alpha}\Gamma(\alpha)}\right)^2 r\,dr \le \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(3(2/5)^\alpha\right)^2,\end{gathered} \quad (5.24)$$

where $\alpha = j+\frac{d-2}{2}$. Note that if $j\ge 10(1+|k|)^2$ then $j+\frac{d-2}{2}>3$. Combining (5.23) and (5.24), we get that

$$\begin{gathered}\frac{||\psi_{jp}||_{L^2(\{1/3<|x|<2/5\})}}{|Y_\alpha(k)J_\alpha(k)|}\ge \\ \ge \left(\left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\right)^{1/2}\left(\frac{1}{3}(5/2)^\alpha-3(2/5)^\alpha\right) > \frac{6}{1000}(5/2)^\alpha.\end{gathered} \quad (5.25)$$

Combining (5.22) and (5.25), we find that

$$\left|\left.\frac{\partial R_j(k,r)}{\partial r}\right|_{r=1}\right| \le 1000\alpha(5/2)^{-\alpha}||\psi_{jp}(E)||_{\mathbb{L}^2(\{1/3<|x|<1\})}. \quad (5.26)$$

Proceeding from (5.19) and using the Cauchy-Schwarz inequality, we get that

$$|c_{jp}| = \left|\frac{\left\langle\psi_{jp},\psi_1-\psi_2\right\rangle_{\mathbb{L}^2(\{1/3<|x|<1\})}}{||\psi_{jp}(E)||^2_{\mathbb{L}^2(\{1/3<|x|<1\})}}\right| \le \frac{||\psi(E)-\psi_0(E)||_{\mathbb{L}^2(B(0,1))}}{||\psi_{jp}(E)||_{\mathbb{L}^2(\{1/3<|x|<1\})}}. \quad (5.27)$$

Using (5.19), we find that

$$\begin{gathered}\left\langle f_{j_1p_1},\left(\hat\Phi_1(E)-\hat\Phi_2(E)\right)f_{j_2p_2}\right\rangle = \left\langle f_{j_1p_1},\left.\frac{\partial(\psi_1-\psi_2)}{\partial\nu}\right|_{\partial D}\right\rangle = \\ = \left\langle f_{j_1p_1},\left.\frac{\partial R_{j_1}(k,r)}{\partial r}\right|_{r=1} f_{j_1p_1}\right\rangle = c_{j_1p_1}\left.\frac{\partial R_{j_1}(k,r)}{\partial r}\right|_{r=1}\end{gathered} \quad (5.28)$$

Combining (5.16), (5.26), (5.27) and (5.28), we obtain that

$$\left\langle f_{j_1 p_1}, \left(\hat{\Phi}_1(E) - \hat{\Phi}_2(E)\right) f_{j_2 p_2} \right\rangle \leq C(d) 2^{-j_1} ||\psi_1 - \psi_2||_{\mathbb{L}^2(B(0,1))}. \tag{5.29}$$

Combining (5.17) and (5.29), we get (3.13) for $j_1 \geq j_2$ and $E \neq 0$.

For $j_1 < j_2$ we use the fact that $\Phi_v^*(E) = \Phi_{\bar{v}}(\bar{E})$ in order to swap j_1 and j_2, where Φ_v^* denotes the adjoint operator to Φ_v. Thus we complete the proof of Lemma 3.2 for the non-zero energy case.

Estimate (3.13) for the zero energy case follows from Lemma 1 of [16].

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