Convex Hierarchical Analysis for the Performances of Uncertain Large-Scale Systems

Technical Report

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Abstract

The performance analysis of uncertain large-scale systems is considered in this paper. It is performed via a hierarchical modelling and analysis of the systems thanks to the recursive application of a propagation of dissipativity properties result. At each step of the analysis, the local part of the system is viewed as the interconnection of sub-systems. The propagation is used to obtain 'propagated' dissipativity properties of this local part from 'sub' dissipativity properties of the sub-systems. At the next step, the former 'propagated' porperties are used as 'sub' properties. This is in contrast with an one-step approach such as (upper bound) μ-analysis which computation time can be prohibitive for large-scale systems even if the associated optimization problem is convex: the trade-off between conservatism and computation time is not necessarily adapted. The purpose is then to obtain a trade-off suited to large-scale systems, an interesting feature being that the trade-off can be set by the user. The approach is used on a PLL network example and illustrates the new trade-off achieved.

Keywords Large-scale system, hierarchical analysis, convex optimization.

1 Introduction

Since the beginning, robustness to uncertainties is a major issue. The sensitivity transfer function was introduced by Bode as a measure of the impact of uncertainties on the closed loop. Some design methods were developed with robustness at their core. It is the case for Quantitative Feedback Theory [10] or H_∞ [28]. In the 80's-90's, μ-analysis [6, 18] was developed to investigate the (H_∞) performances of Linear Time Invariant systems in the presence of structured uncertainties. This approach is based on the computation of the structured singular value μ of frequency dependent matrices, which was proved to be NP-hard [3]. Fortunately, lower and upper bounds on μ can be efficiently computed; the μ upper bounds [7] allow to guarantee a certain level of performances with some conservatism. By efficient, it is understood that the computation time is bounded by a power function of the problem size [9]. A major interest of the use of μ upper bounds is to obtain a satisfying trade-off between the computation time and the conservatism of the obtained result.

Nevertheless, even if the computation of μ upper bound is efficient, its computation time can be important in the case of uncertain large-scale systems. The purpose of this paper is to propose a robustness analysis (of performances) method with a trade-off between computation time and conservatism adapted to large-scale systems. Our motivating example is the robustness analysis of Phase-Locked-Loop (PLL) networks [13], a challenging problem in Microelectronics.

In the case of large-scale systems with uncertainties defined by conic sector properties, that is unstructured uncertainties, Safonov approached the problem via a hierarchical analysis using recursively a propagation of conic sector properties result [19]. To the authors' best knowledge, the work of [19] is the only one devoted to the problem. Basically, a hierarchical system can be described as a tree with layers, each of them is composed of systems. Each system is modeled as the interconnection of sub-systems. These sub-systems are in fact the systems of the preceding layer. Assuming that the conic properties of the sub-systems are known, the propagation problem is to find 'propagated' conic properties for the system that are not trivially connected to the ones of the sub-systems. If one is able to find these 'propagated' properties, then the hierarchical analysis boils down to a recursive application of the propagation.

Unfortunately, its direct application to a (large-scale) system with structured uncertainties can lead to an overly conservative result. While keeping the same overall approach, we propose in this paper a new hierarchical analysis method which overcomes the disadvantages of [19]: the method is efficient and adapted to structured uncertainties. Our method is based on a generalization of the propagation of conic properties to quadratic ones. Preliminary results of the present work can be found in [5] where there was only one 'propagated' quadratic property (the conic sector one) with the uncertainties already described by quadratic properties. We present here other quadratic 'propagated' properties. Combining them (their 'intersection'), a finer description can be made of the system. The user can then set the trade-off by choosing the 'propagated' properties used, the main contributor to the computation time being their number.

As in [5], this propagation can be viewed as an embedding problem: find (simpler) sets that includes the uncertain system. Beyond the hierarchical analysis, this embedding has many interesting applications. For instance, in the Quantitative Feedback Theory [10], the aim is to design a controller that achieves some level of performances in front of all the uncertainties in the system. In the method, it is assumed that it is possible to know the propagation of the uncertainties through thet system. Another interesting application, is the use of the embedding to perform μ-synthesis in the integrated framework of [16]; the advantage being the computation time thanks to the embedding.

Our solution is based on a separation of graph theorem. First proposed in [17] as a general approach to feedback system analysis, specialized forms were proposed in *e.g.* [11, 23, 20] for (uncertain) LTI system analysis. Since the μ upper bound proposed in [7] can be interpreted as a particular application of the separation of graph theorem, this theorem was applied to extend μ-analysis to time-delay systems [21] or time-varying/nonlinear systems [15], to reduce the conservatism of the μ upper bound [24] to cite a few. We reveal here another interesting application of this powerful theorem.

Paper outline

Section 2 begins with definitions and fundamental properties of dissipativity properties which are used afterwards. It explicits then the uncertain large-scale system that is considered with the proposed approach. Section 3 proposed several dissipative properties that can be used practically. A numerical example on a PLL network is performed in Section 4. Section 5 concludes the paper.

Notations

$\mathbf{R}$ (respectively $\mathbf{C}$) denotes the set of real (resp. complex) numbers. $\bar{\mathbf{R}}$ denotes $\mathbf{R} \cup \{-\infty, +\infty\}$. $M_{\mathbb{R}}$ and $M_{\mathbb{I}}$ stands for the real and imaginary parts of M.

$\mathbf{R}^{m\times n}$ (respectively $\mathbf{C}^{m\times n}$) denotes the set of real (resp. complex) matrices of dimension $m \times n$. I_r and 0_r denote the identity and the zero matrices of size r. M^* (respectively M^T) stands for the transpose conjugate (resp. transpose) of M. For several matrices M_i, $i = 1, \ldots, n$, $\mathbf{bdiag}_i(M_i)$ denotes the matrix $\begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_n \end{bmatrix}$. $\mathbf{RH}_\infty$ (respectively $\mathbf{RL}_\infty$) denotes the set of matrices of stable (resp. non causally stable) rational transfer function.

Throughout the paper, when dimensions are omitted, they are assumed to be clear from the context. Moreover, we consistently denote uncertainties by Δ and interconnections by

M. The set $\mathbf{\Delta}$ is referred to as the uncertainty set. We denote by $\mathbf{\Delta}\star M$ the set $\{\Delta\star M,\ \Delta \in \mathbf{\Delta}\}$, with $\star$ standing for the Redheffer star product. This set is also referred to as an uncertain system. For the uncertain system $\mathbf{\Delta}\star M$, we further denotes the interconnection's partitioning of appropriate dimension by

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Finally, we denote

$$\mathcal{L}(M, \Phi_{11}, \Phi_{12}, \Phi_{22}, X, Y, Z) = \begin{bmatrix} M \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} -\Phi_{22} & 0 & -\Phi_{12}^* & 0 \\ 0 & X & 0 & Y \\ \hline -\Phi_{12} & 0 & -\Phi_{11} & 0 \\ 0 & Y^* & 0 & Z \end{array}\right] \begin{bmatrix} M \\ I \end{bmatrix}.$$

2 Approach for Hierarchical Analysis of Performances

2.1 Definitions and preliminaries

An uncertain system is modeled as an interconnection $\Delta \star M$ with $\Delta \in \mathbf{\Delta}$. Along with this definition, we assume:

Assumption 2.1 *$\mathbf{\Delta}$ is a bounded and connected subset of $\mathbf{RH}_\infty$ and M belongs to $\mathbf{RH}_\infty$.*

Introducing the internal signals and using the Fourier transform, we get:

$$\begin{aligned} p(j\omega) &= \Delta(j\omega) \quad q(j\omega) \\ \begin{bmatrix} q(j\omega) \\ z(j\omega) \end{bmatrix} &= M(j\omega) \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix} = \begin{bmatrix} A(j\omega) & B(j\omega) \\ C(j\omega) & D(j\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix}. \end{aligned} \tag{1}$$

The stability of an uncertain system is now defined.

Definition 2.1 *An uncertain system $\mathbf{\Delta} \star M$ is said to be stable (respectively non-causally stable) if for any $\Delta \in \mathbf{\Delta}$, the system $\Delta \star M$ is stable (respectively non-causally stable).*

In this section, dissipative properties are used. They are defined below.

Definition 2.2 *Let $X(j\omega)$, $Y(j\omega)$ and $Z(j\omega)$ be 3 transfer functions of $\mathbf{RL}_\infty$ such that $X(j\omega) = X(j\omega)^*$ and $Z(j\omega) = Z(j\omega)^*$. Then,*

1. *a system H is said to be $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative if for any $\omega \in \overline{\mathbf{R}}$ and for any non null $[z(j\omega)^* w(j\omega)^*]^*$ verifying $z(j\omega) = H(j\omega)w(j\omega)$:*

$$\left[\begin{array}{c} z(j\omega) \\ w(j\omega) \end{array}\right]^* \left[\begin{array}{cc} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & Z(j\omega) \end{array}\right] \left[\begin{array}{c} z(j\omega) \\ w(j\omega) \end{array}\right] > 0;$$

2. *an uncertainty set $\mathbf{\Delta}$ is said to be $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative if for any $\Delta \in \mathbf{\Delta}$, Δ is $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative;*

3. *more generally, an uncertain system $\mathbf{\Delta} \star M$ is said to be $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative if for any $\Delta \in \mathbf{\Delta}$, the system $\Delta \star M$ is $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative.*

Note that when H is stable and

$$\left[\begin{array}{cc} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & Z(j\omega) \end{array}\right] = \left[\begin{array}{cc} -I & 0 \\ 0 & \gamma^2 I \end{array}\right],$$

H has an H_∞ norm strictly less than γ. More generally, dissipative properties can model performance indices. A dissipativity property can also be viewed as an inclusion of sets (of systems or signals). With the preceding example, the set composed of H is included in the ball of systems with an H_∞ norm strictly less than γ. This set of systems is convex.

Lemma 2.1 *Let $X(j\omega)$, $Y(j\omega)$ and $Z(j\omega)$ be 3 transfer functions of $\mathbf{RL}_\infty$ such that $X(j\omega) = X(j\omega)^*$ and $Z(j\omega) = Z(j\omega)^*$. Assume that $X(j\omega) < 0$, then the set of systems*

$$\{H \mid H \textit{ is } \{X(j\omega), Y(j\omega), Z(j\omega)\} \textit{ dissipative}\}$$

is convex, and thus connected.

Due to its great interest in terms of geometrical interpretation, the proof is detailed hereafter.

Proof Let us define

$$H_c = -X^{-1}Y \text{ and } R^*R = Z - Y^*X^{-1}Y.$$

The dissipativity property of a system H writes then

$$(z - z_c)^*(-X)(z - z_c) < w^*R^*Rw \text{ with } z = Hw \text{ and } z_c = H_c w.$$

or equivalently

$$\|(-X)^{-1/2}(H - H_c)w\|_2 < \|Rw\|_2.$$

It is also equivalent to

$$\bar{\sigma}((-X)^{-1/2}(H - C)R^{-1}) < 1 \text{ with } \bar{\sigma} \text{ the maximum singular value.} \tag{2}$$

This set corresponds to a ball centered around C with a weighted norm. It is thus convex and bounded. $\square$

Fundamental properties are now given with a direct corollary. They states that new dissipativity properties can be generated from original ones. For ease of notation, they are stated for certain systems. The extension to uncertain systems is straightforward.

Lemma 2.2 *Let H_i be $\{X_i(j\omega), Y_i(j\omega), Z_i(j\omega)\}$ dissipative, $i = 1, \ldots, m$. Then $H = \boldsymbol{bdiag}_i(H_i)$ is $\{\boldsymbol{bdiag}_i(X_i(j\omega)), \boldsymbol{bdiag}_i(Y_i(j\omega)), \boldsymbol{bdiag}_i(Z_i(j\omega))\}$ dissipative.*

Lemma 2.3 *Let H be $\{X_k(j\omega), Y_k(j\omega), Z_k(j\omega)\}$ dissipative, $k = 1, \ldots, n$. Then for any $\tau_k(j\omega) > 0$, $k = 1, \ldots, n$, H is $\{\sum_k \tau_k(j\omega) X_k(j\omega), \sum_k \tau_k(j\omega) Y_k(j\omega), \sum_k \tau_k(j\omega) Z_k(j\omega)\}$ dissipative.*

Corollary 2.1 *Let H_i be $\{X_{ik}(j\omega), Y_{ik}(j\omega), Z_{ik}(j\omega)\}$ dissipative, $i = 1, \ldots, m$, $k = 1, \ldots, n$. Then for any $\tau_{ik}(j\omega) > 0$, $i = 1, \ldots, m$, $k = 1, \ldots, n$, $H = \boldsymbol{bdiag}_i(H_i)$ is*

$$\{\boldsymbol{bdiag}_i(\textstyle\sum_k \tau_{ik}(j\omega) X_{ik}(j\omega)), \boldsymbol{bdiag}_i(\sum_k \tau_{ik}(j\omega) Y_{ik}(j\omega)), \boldsymbol{bdiag}_k(\sum_k \tau_{ik}(j\omega) Z_{ik}(j\omega))\}$$

dissipative.

This corollary also defines a set of linearly parameterized dissipative properties. Let us denote it $\boldsymbol{\Phi}(j\omega)$.

2.2 Hierarchical system description and proposed approach

From [19], a large-scale system is described by a tree as illustrated in Figure 1 where a hierarchical structure arises naturally. Each branch of the tree is assigned an index. A branch, say i, is a two-way channel through which a signal w_i (the input) ascends and another signal z_i (the output) descends. The tree obtained by cutting branch i and retaining everything connected above is an uncertain system called $\boldsymbol{T_i}$ with input w_i and output z_i. If a tree $\boldsymbol{T_i}$ has other branches besides branch i then there is a single node denoted M_i from which other branches ascend. If branch i is the only branch in the tree, then $\boldsymbol{T_i}$ is called a leaf and is denoted by $\boldsymbol{\Delta}_i$. Each M_i and Δ_i is an LTI system. Furthermore, each leaf $\boldsymbol{\Delta}_i$ is uncertain but its dissipative properties are *a priori* known.

Assumption 2.2 *Each $\boldsymbol{\Delta}_i$ is a bounded and connected subset of $\mathbf{RH}_\infty$ and each M_i belongs to $\mathbf{RH}_\infty$.*

This is the counterpart of Assumption 2.1.

Assumption 2.3 *For any i, $\boldsymbol{\Delta}_i$ is an elementary uncertainty set: there exists* a priori *known $X_{ik}(j\omega)$, $Y_{ik}(j\omega)$ and $Z_{ik}(j\omega)$ such that $\boldsymbol{\Delta}_i$ is $\{X_{ik}(j\omega), Y_{ik}(j\omega), Z_{ik}(j\omega)\}$ dissipative.*

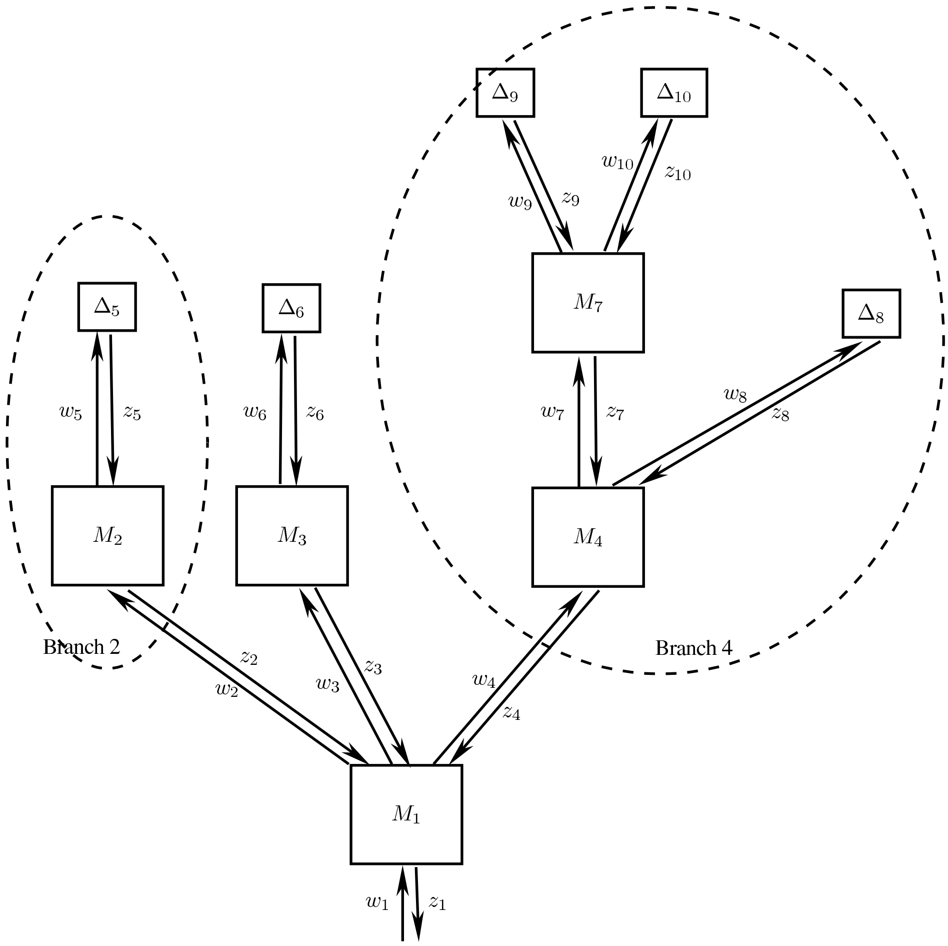

Figure 1: Uncertain linear large scale system

Examples of elementary uncertainty sets along with their dissipative properties can be found in [7, 22, 24]. The interest of this assumption will be made clearer in the discussion of the propagation theorem.

The proposed approach for the performance analysis of a hierarchical system as described in the introduction is based on a recursive application of the following propagation[1] of dissipativity properties of uncertainties through an interconnection theorem.

Theorem 2.1 *The uncertain system* $\mathbf{\Delta}\star M$ *is stable and* $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ *dissipative if and only if:*

1. *there exists* $\Delta_0 \in \mathbf{\Delta}$ *such that the system* $\Delta_0 \star M$ *is stable;*
2. *there exists 3 transfer functions* $\Phi_{11}(j\omega)$, $\Phi_{12}(j\omega)$ *and* $\Phi_{22}(j\omega)$ *of* $\mathbf{RL}_\infty$, *with* $\Phi_{11}(j\omega) = \Phi_{11}(j\omega)^*$ *and* $\Phi_{22}(j\omega) = \Phi_{22}(j\omega)^*$, *such that:*

   $$\text{the uncertainty set } \mathbf{\Delta} \text{ is } \{\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)\} \text{ dissipative}$$

   and

   $$\forall \omega \in \overline{\mathbf{R}},\ \mathcal{L}(M(j\omega), \Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega), X(j\omega), Y(j\omega), Z(j\omega)) > 0. \quad (3)$$

If the condition 1 *does not hold then the uncertain system is non-causally stable.*

Proof See Appendix A. □

The theorem allows to analyze the stability and performance of the uncertain large-scale system in one shot so that there is no need to analyze the stability separately.

Condition 1 is generally viewed as an assumption that is verified beforehand on the nominal system. This assumption is hopefully very mild and is extremely similar (even weaker in fact) to the one of μ-analysis [25].

The fact that the uncertainty set $\mathbf{\Delta}$ is $\{\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)\}$ dissipative in *condition 2* can also be verified *a priori* using Assumption 2.3 and the set of linearly parameterized dissipativity properties $\mathbf{\Phi}(j\omega)$ as defined by Corollary 2.1. Note however that this leads to sufficient conditions only. Note also that it is not compulsory as an assumption as it is not used in the proof. This assumption allows to increase the efficiency of the approach. It is possible to find them directly, and thus suppress it, as in [8] for instance.

As a consequence, Theorem 2.1 boils down to verify condition (3) in practise.

[1]The term propagation is kept in reference to [19] even if the meaning is slightly different.

Corollary 2.2 *Let $\mathbf{\Phi}(j\omega)$ be a set such that for any $(\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)) \in \mathbf{\Phi}(j\omega)$, the uncertainty set $\mathbf{\Delta}$ is $\{\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)\}$ dissipative.*

Then the uncertain system $\mathbf{\Delta} \star M$ is non causally stable and $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative if there exists $(\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)) \in \mathbf{\Phi}(j\omega)$ such that

$$\forall \omega \in \overline{\mathbf{R}},\ \mathcal{L}(M(j\omega), \Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega), X(j\omega), Y(j\omega), Z(j\omega)) > 0.$$

Moreover, $\mathbf{\Delta}$ needs to be a bounded and connected set only: it does not need to contain 0 as usually assumed. In our case it is an important fact from a practical point of view as the theorem is recursively applied so that the uncertainty set (the previous branches) does not necessarily contain 0.

When two uncertain systems $\mathbf{\Delta}_1 \star M_1$ and $\mathbf{\Delta}_2 \star M_2$ are homogenous, then they share the same dissipative properties: if $M_1 = M_2$ and $\mathbf{\Delta}_1 = \mathbf{\Delta}_2$, then $\mathbf{\Delta}_1 \star M_1$ is $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative if and only if $\mathbf{\Delta}_2 \star M_2$ is $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative. This is the case of the PLL network example of Section 4.

For computational purposes, it may also be interesting to replace

$$\begin{bmatrix} M(j\omega) \\ I \end{bmatrix} \text{ by } \begin{bmatrix} N(j\omega) \\ D(j\omega) \end{bmatrix}$$

with $M(j\omega) = N(j\omega)D(j\omega)^{-1}$.

Finally, the theorem also states that the analysis can be performed by a frequency by frequency approach. This fact will be used in the next section to drop the dependency on $j\omega$.

Let us now exemplify the use of Theorem 2.1 with the system illustrated in Figure 1. First, from the dissipative properties of $\mathbf{\Delta}_9$ and $\mathbf{\Delta}_{10}$, find some dissipative properties of the branch $\boldsymbol{T_7}$ using Theorem 2.1 with $\Delta = \mathbf{bdiag}(\Delta_9, \Delta_{10})$ and $M = M_7$. From these several dissipative properties and from the ones of $\mathbf{\Delta}_8$, use again Theorem 2.1 (and Corollary 2.1) with $\Delta = \mathbf{bdiag}(T_7, \Delta_8)$ and $M = M_4$ to find dissipative properties of the branch $\boldsymbol{T_4}$. However, it is possible if $\boldsymbol{T_7}$ is a bounded set (the connected part of the assumption is ensured by Lemma 2.1). This is the case if its dissipativity properties were well chosen: typically, it is needed a conic sector property, see Section 3.1. And so on until branch $\boldsymbol{T_1}$ where the dissipativity property is a performance index. The overall trade-off between conservatism and computation time then depends on the number of dissipative properties that are searched for at each step. The user can thus set this trade-off by setting the number of dissipative properties.

3 Practical Formulation of Dissipativity Propagation

In this section, we show how to find dissipative properties (referred to as 'propagated' in the introduction) for the uncertain system $\mathbf{\Delta} \star M$ from the ones of $\mathbf{\Delta}$: it is the propagation of dissipativity properties. For the sake of simplicity, we set the value of the frequency without loss of generality as the system is linear time-invariant, so that the dependency on $j\omega$ is dropped. The problem can be stated as follows.

Problem 3.1 *Let $\mathbf{\Phi}$ be a set such that for any $(\Phi_{11}, \Phi_{12}, \Phi_{22}) \in \mathbf{\Phi}$, the uncertainty set $\mathbf{\Delta}$ is $\{\Phi_{11}, \Phi_{12}, \Phi_{22}\}$ dissipative.*

From the set $\mathbf{\Phi}$, find X, Y and Z such that the uncertain system $\mathbf{\Delta} \star M$ is $\{X, Y, Z\}$ dissipative.

Corollary 3.1 *Problem 3.1 is solved by the following optimization problem: find $(\Phi_{11}, \Phi_{12}, \Phi_{22}) \in \mathbf{\Phi}$ and X, Y and Z such that*

$$\mathcal{L}(M, \Phi_{11}, \Phi_{12}, \Phi_{22}, X, Y, Z) > 0.$$

Proof It is a direct consequence of Corollary 2.2. □

Note that the optimization problem defined in Corollary 3.1 parameterizes all the possible propagated properties from the ones of $\mathbf{\Delta}$ in $\mathbf{\Phi}$: it is non conservative from a propagation perspective.

In the way the propagation is used, $\mathbf{\Delta}$ is either a leaf or a branch. In both case, either due to Assumption 2.3 or Corollary 2.1, the set $\mathbf{\Phi}$ is of the form

$$\left\{ \sum_i \tau_i \left(\Phi_{11i}, \Phi_{12i}, \Phi_{22i} \right) \right\}$$

with *a priori* known $(\Phi_{11i}, \Phi_{12i}, \Phi_{22i})$. The optimization problem defined in Corollary 3.1 boils down to find τ_i and X, Y and Z such that $\mathcal{L}(M, \Phi_{11}, \Phi_{12}, \Phi_{22}, X, Y, Z) > 0$. It is thus an LMI optimization problem, is convex and can be solved efficiently.

Corollary 3.1 defines a optimization problem with complex LMI constraints. For computational purpose, they can readily be converted as real LMI constraints [2]: a complex matrix M is positive definite if and only if the real matrix

$$\begin{bmatrix} M_{\mathbb{R}} & M_{\mathbb{I}} \\ -M_{\mathbb{I}} & M_{\mathbb{R}} \end{bmatrix}$$

is positive definite.

The three preceding remarks hold for all the optimization problems involved in this section.

To improve the overall conservatism of the hierarchical analysis, it is interesting to obtain the 'tightest' propagated dissipativity property. It is performed by interpreting the property in geometrical terms. For each geometrical interpretation, a notion of size is defined and one is interested in minimizing this size.

3.1 Conic sector: $X < 0$

In the case $X < 0$, let us define

$$H_c = -X^{-1}Y \text{ and } R^*R = Z - Y^*X^{-1}Y.$$

The dissipativity property of a system H writes then

$$(z - z_c)^*(-X)(z - z_c) < w^*R^*Rw \text{ with } z = Hw \text{ and } z_c = H_c w. \tag{4}$$

which defines the same set as a conic sector [19] in which a system H is said to be in the conic sector (C, P, Q), with C the cone center, whenever

$$\|Q^{-1/2}(z - Cw)\|_2 < \|P^{1/2}w\|_2 \text{ with } z = Hw$$

The link is provided by

$$-X = Q^{-1}, \; H_c = C \text{ and } R^*R = P.$$

For a SISO system, the inequality (4) defines a disk of center z_c and radius $w^*(X/(R^*R))w$. More generally, it is an ellipsoid. Indeed, the inequality can be rewritten as

$$\begin{bmatrix} z_{\mathbb{R}} - z_{c\mathbb{R}} \\ z_{\mathbb{I}} - z_{c\mathbb{I}} \end{bmatrix}^T \mathcal{P} \begin{bmatrix} z_{\mathbb{R}} - z_{c\mathbb{R}} \\ z_{\mathbb{I}} - z_{c\mathbb{I}} \end{bmatrix} < 1$$

with

$$\mathcal{P} = \frac{1}{\begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}^T \begin{bmatrix} (R^*R)_{\mathbb{R}} & (R^*R)_{\mathbb{I}} \\ -(R^*R)_{\mathbb{I}} & (R^*R)_{\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}} \begin{bmatrix} -X_{\mathbb{R}} & -X_{\mathbb{I}} \\ X_{\mathbb{I}} & -X_{\mathbb{R}} \end{bmatrix}. \tag{5}$$

Thus, for a given non null input w, the corresponding output signal $\tilde{z} = \begin{bmatrix} z_{\mathbb{R}}^T & z_{\mathbb{I}}^T \end{bmatrix}^T$ belongs to the ellipsoid

$$\epsilon_{\mathcal{P}} = \{\tilde{z} \,|\, (\tilde{z} - \tilde{z}_c)^T \mathcal{P}(\tilde{z} - \tilde{z}_c) < 1\}. \tag{6}$$

The volume is here evaluated as [1].

Definition 3.1 *The volume of the ellipsoid $\epsilon_{\mathcal{P}}$ defined by (6) and (5) is defined as*

$$\mathbf{vol}(\epsilon_{\mathcal{P}}) = \beta \det(\mathcal{P}^{-1})$$

where β is a positive scalar which depends on the size n_z of the vector $\tilde{z} - \tilde{z}_c$.

We are interested in finding the smallest one for all inputs such that $\|w\| = 1$ xxx ref Peaucelle xxx.

Problem 3.2 *Let* $\mathbf{\Phi}$ *be a set such that for any* $(\Phi_{11}, \Phi_{12}, \Phi_{22}) \in \mathbf{\Phi}$*, the uncertainty set* $\mathbf{\Delta}$ *is* $\{\Phi_{11}, \Phi_{12}, \Phi_{22}\}$ *dissipative.*

From the set $\mathbf{\Phi}$*, find* X*,* Y *and* Z *such that:*

1. *the uncertain system* $\mathbf{\Delta} \star M$ *is* $\{X, Y, Z\}$ *dissipative;*
2. *they minimize* $\max_{\Delta \in \mathbf{\Delta}} \max_{\|w\|=1} \mathbf{vol}(\epsilon_{\mathcal{P}})$.

Theorem 3.1 *Problem 3.2 is solved by the following optimization problem: find* $(\Phi_{11}, \Phi_{12}, \Phi_{22}) \in \mathbf{\Phi}$ *and* X*,* Y *and* Z *that minimize* $\log\left(\det\left(\begin{bmatrix} -X_{\mathbb{R}} & -X_{\mathbb{I}} \\ X_{\mathbb{I}} & -X_{\mathbb{R}} \end{bmatrix}^{-1}\right)\right)$ *and such that*

1. $\mathcal{L}(M, \Phi_{11}, \Phi_{12}, \Phi_{22}, X, Y, Z) > 0$ *holds;*
2. $\left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \geq \left[\begin{array}{cc|cc} Z_{\mathbb{R}} & Y_{\mathbb{R}}^* & Z_{\mathbb{I}} & Y_{\mathbb{I}}^* \\ Y_{\mathbb{R}} & X_{\mathbb{R}} & Y_{\mathbb{I}} & X_{\mathbb{I}} \\ \hline -Z_{\mathbb{I}} & -Y_{\mathbb{I}}^* & Z_{\mathbb{R}} & Y_{\mathbb{R}}^* \\ -Y_{\mathbb{I}} & -X_{\mathbb{I}} & Y_{\mathbb{R}} & X_{\mathbb{R}} \end{array}\right]$ *holds.*

This optimization problem is a determinant maximization under linear matrix inequality constraints [27] and is convex.

Proof Problem 3.2 writes

$$\begin{array}{llll} \text{minimize} & \text{maximize} & \text{maximize} & \mathbf{vol}(\epsilon_{\mathcal{P}}) \\ \text{over } X, Y, Z & \text{over } \Delta \in \mathbf{\Delta} & \text{over } \|w\| = 1 & \\ \text{subject to} & \mathbf{\Delta} \star M \text{ is } \{X, Y, Z\} \text{ dissipative} & & \end{array}$$

As the logarithm function is strictly increasing and as β is constant, the optimization problem is equivalent to

$$\begin{array}{llll} \text{minimize} & \text{maximize} & \text{maximize} & \log(\det(\mathcal{P}^{-1})) \\ \text{over } X, Y, Z & \text{over } \Delta \in \mathbf{\Delta} & \text{over } \|w\| = 1 & \\ \text{subject to} & \mathbf{\Delta} \star M \text{ is } \{X, Y, Z\} \text{ dissipative} & & \end{array}$$

Now $\max_{\|w\|=1} \log(\det(\mathcal{P}^{-1}))$ is equal to $\log\left(\det\left(\lambda_{max}\begin{bmatrix} -X_{\mathbb{R}} & -X_{\mathbb{I}} \\ X_{\mathbb{I}} & -X_{\mathbb{R}} \end{bmatrix}^{-1}\right)\right)$ with λ_{max} the minimal value verifying $\lambda_{max} I \geq \begin{bmatrix} (R^*R)_{\mathbb{R}} & (R^*R)_{\mathbb{I}} \\ -(R^*R)_{\mathbb{I}} & (R^*R)_{\mathbb{R}} \end{bmatrix}$. As a dissipativity property is defined up to a strictly positive multiplicative coefficient and as $\{X, Y, Z\}$ dissipativity defines the same ellipsoid as $\{\tau X, \tau Y, \tau Z\}$ dissipativity for any $\tau > 0$ since

$$\frac{1}{\begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}^T \begin{bmatrix} \tau(R^*R)_{\mathbb{R}} & \tau(R^*R)_{\mathbb{I}} \\ -\tau(R^*R)_{\mathbb{I}} & \tau(R^*R)_{\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}} \begin{bmatrix} -\tau X_{\mathbb{R}} & -\tau X_{\mathbb{I}} \\ \tau X_{\mathbb{I}} & -\tau X_{\mathbb{R}} \end{bmatrix} = \mathcal{P},$$

one can search for X, Y and Z such that $\lambda_{max} = 1$ without loss of generality. Thus, the optimization problem is equivalent to

$$\begin{array}{lll} \text{minimize} & \text{maximize} & \log\left(\det\left(\begin{bmatrix} -X_{\mathbb{R}} & -X_{\mathbb{I}} \\ X_{\mathbb{I}} & -X_{\mathbb{R}} \end{bmatrix}^{-1}\right)\right) \\ \text{over } X, Y, Z & \text{over } \Delta \in \mathbf{\Delta} & \\ \text{subject to} & \mathbf{\Delta} \star M \text{ is } \{X, Y, Z\} \text{ dissipative} & \\ & I \geq \begin{bmatrix} (R^*R)_{\mathbb{R}} & (R^*R)_{\mathbb{I}} \\ -(R^*R)_{\mathbb{I}} & (R^*R)_{\mathbb{R}} \end{bmatrix} & \end{array}$$

Finally, condition 1 of Theorem 3.1 is obtained by applying Corollary 3.1 and condition 2 is obtained by applying Schur's lemma [1]. □

3.2 Half Planes: $X = 0$

Half plane A dissipativity property with $X = 0$ rewrites

$$\xi^T \begin{bmatrix} z_{\mathbb{R}} \\ z_{\mathbb{I}} \end{bmatrix} - \eta > 0$$

with

$$\begin{aligned} \xi &= \begin{bmatrix} Y_{\mathbb{R}} & Y_{\mathbb{I}} \\ -Y_{\mathbb{I}} & Y_{\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}, \\ \eta &= -2 \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}^T \begin{bmatrix} Z_{\mathbb{R}} & Z_{\mathbb{I}} \\ -Z_{\mathbb{I}} & Z_{\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}. \end{aligned}$$

This last inequality express that the output signal belongs to a half plane defined by the hyperplane

$$\left\{ \begin{bmatrix} z_{\mathbb{R}} \\ z_{\mathbb{I}} \end{bmatrix} \middle| \; \xi^T \begin{bmatrix} z_{\mathbb{R}} \\ z_{\mathbb{I}} \end{bmatrix} = \eta \right\}.$$

ξ is a vector normal to the hyperplane and η the 'signed distance' of the hyperplane to the origin (the dot product of any point of the hyperplane with ξ).

Remark: the passitivity kind of performance is a specific half plane with

$$\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Band A band is the intersection of two half planes with the same normal direction but opposite way. As for the conic sector, we are interested in the smallest band for a given direction, that is for a given Y. The size of a band is defined by the distance between the two parallel hyperplanes.

Definition 3.2 *Let ξ, η_1 and η_2 be of appropriate dimensions and define the two half planes $\xi^T \begin{bmatrix} z_{\mathbb{R}} \\ z_{\mathbb{I}} \end{bmatrix} - \eta_1 > 0$ and $-\xi^T \begin{bmatrix} z_{\mathbb{R}} \\ z_{\mathbb{I}} \end{bmatrix} - \eta_2 > 0$. The size of the corresponding band is defined by $d_Y = |\eta_1 + \eta_2|$.*

We are interested in finding the smallest band for all inputs such that $\|w\| = 1$.

Problem 3.3 *Let $\mathbf{\Phi}$ be a set such that for any $(\Phi_{11}, \Phi_{12}, \Phi_{22}) \in \mathbf{\Phi}$, the uncertainty set $\mathbf{\Delta}$ is $\{\Phi_{11}, \Phi_{12}, \Phi_{22}\}$ dissipative. Let Y be a matrix of appropriate dimension.*

From the set $\mathbf{\Phi}$ and Y, find Z_1 and Z_2 such that:

1. *the uncertain system $\mathbf{\Delta} \star M$ is $\{0, Y, Z_1\}$ dissipative;*
2. *the uncertain system $\mathbf{\Delta} \star M$ is $\{0, -Y, Z_2\}$ dissipative;*
3. *they minimize $\max_{\Delta \in \mathbf{\Delta}} \max_{\|w\|=1} d_Y$.*

Theorem 3.2 *Problem 3.3 is solved by the following optimization problem: find $(\Phi_{111}, \Phi_{121}, \Phi_{221}) \in \mathbf{\Phi}$, Z_1, λ_1, $(\Phi_{112}, \Phi_{122}, \Phi_{222}) \in \mathbf{\Phi}$, Z_2, λ_2 and d that minimize d such that*

1. *$\mathcal{L}(M, \Phi_{111}, \Phi_{112}, \Phi_{122}, 0, Y, Z_1) > 0$ holds;*
2. *$\lambda_1 I \geq Z_1$ holds;*
3. *$\mathcal{L}(M, \Phi_{211}, \Phi_{212}, \Phi_{222}, 0, -Y, Z_2) > 0$ holds;*
4. *$\lambda_2 I \geq Z_2$ holds;*
5. *$\begin{bmatrix} d & \lambda_1 + \lambda_2 \\ \lambda_1 + \lambda_2 & 1 \end{bmatrix} \geq 0$ holds.*

This optimization problem is a minimization of a linear cost under linear matrix inequality constraints [1] and is convex.

It is then possible to generate several bands with different normals as performed in the example of Section 4. It is also possible to search for the direction of the band by letting Y to be free.

Proof Problem 3.3 writes

$$\begin{array}{llll} \text{minimize} & \text{maximize} & \text{maximize} & |\eta_1 + \eta_2| \\ \text{over } X, Y, Z & \text{over } \Delta \in \mathbf{\Delta} & \text{over } \|w\| = 1 & \\ \text{subject to} & \mathbf{\Delta} \star M \text{ is } \{0, Y, Z_1\} \text{ dissipative} & & \\ & \mathbf{\Delta} \star M \text{ is } \{0, -Y, Z_2\} \text{ dissipative} & & \end{array}$$

with $\eta_1 = -2 \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}^T \begin{bmatrix} Z_{1\mathbb{R}} & Z_{1\mathbb{I}} \\ -Z_{1\mathbb{I}} & Z_{1\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}$ and $\eta_2 = -2 \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}^T \begin{bmatrix} Z_{2\mathbb{R}} & Z_{2\mathbb{I}} \\ -Z_{2\mathbb{I}} & Z_{2\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}$.
This equivalent to the optimization problem

$$\begin{array}{lll} \text{minimize} & \text{maximize} & \text{maximize} \quad (\tilde{\eta}_1 + \tilde{\eta}_2)^2 \\ \text{over } Z_1, Z_2 & \text{over } \Delta \in \mathbf{\Delta} & \text{over } \|w\| = 1 \\ \text{subject to} & \mathbf{\Delta} \star M \text{ is } \{0, Y, Z_1\} \text{ dissipative} & \\ & \mathbf{\Delta} \star M \text{ is } \{0, -Y, Z_2\} \text{ dissipative} & \end{array}$$

with $\tilde{\eta}_1 = \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}^T \begin{bmatrix} Z_{1\mathbb{R}} & Z_{1\mathbb{I}} \\ -Z_{1\mathbb{I}} & Z_{1\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}$ and $\tilde{\eta}_2 = \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}^T \begin{bmatrix} Z_{2\mathbb{R}} & Z_{2\mathbb{I}} \\ -Z_{2\mathbb{I}} & Z_{2\mathbb{R}} \end{bmatrix} \begin{bmatrix} w_{\mathbb{R}} \\ w_{\mathbb{I}} \end{bmatrix}$.

Now $\max_{\|w\|=1} (\tilde{\eta}_1 + \tilde{\eta}_2)^2$ is equivalent to $\min d$ constrained by

$$d \geq (\lambda_1 + \lambda_2)^2$$

$$\lambda_1 I \geq \begin{bmatrix} Z_{1\mathbb{R}} & Z_{1\mathbb{I}} \\ -Z_{1\mathbb{I}} & Z_{1\mathbb{R}} \end{bmatrix}$$

$$\lambda_2 I \geq \begin{bmatrix} Z_{2\mathbb{R}} & Z_{2\mathbb{I}} \\ -Z_{2\mathbb{I}} & Z_{2\mathbb{R}} \end{bmatrix}.$$

After rewriting the LMI in complex form, the optimization is thus equivalent to

$$\begin{array}{lll} \text{minimize} & \text{maximize} & d \\ \text{over } Z_1, Z_2, \lambda_1, \lambda_2, d & \text{over } \Delta \in \mathbf{\Delta} & \\ \text{subject to} & \mathbf{\Delta} \star M \text{ is } \{0, Y, Z_1\} \text{ dissipative} & \\ & \lambda_1 I \geq Z_1 & \\ & \mathbf{\Delta} \star M \text{ is } \{0, -Y, Z_2\} \text{ dissipative} & \\ & \lambda_2 I \geq Z_2 & \\ & d \geq (\lambda_1 + \lambda_2)^2 & \end{array}$$

Finally, conditions 1 and 3 of Theorem 3.2 are obtained by applying Corollary 3.1 and condition 5 is obtained by applying Schur's lemma [1]. □

4 PLL network Example

Let us consider now a numerical example of hierarchical performance analysis of an uncertain large-scale system. One takes as an example the performance analysis of the active clock distribution network from [13] subject to technological dispersions. An active clock distribution network is composed of $N = 16$ mutually synchronized Phase-Locked-Loops (they constitute branches of the tree) delivering the clock signals to the chip. To be able to synchronize the PLLs exchange the information on their relative phase through the interconnection network and the phase detectors. This example is particularly well adapted as the performance is measured in frequency domain with homogeneous PLLs.

4.1 PLL network description

Since the principal aim of the system is the synchronization, the PLLs are homogeneous, that is have a common interconnection and the same uncertainty set. Of course, during the manufacturing process, there are inevitable technological dispersions which can be represented in the form of parametric uncertainties belonging to the same set. We have thus $\forall i \in \{1, \dots, N\}$:

$$T_i\left(j\omega\right) = \frac{k_i\left(j\omega + a_i\right)}{-\omega^2 + k_i j\omega + k_i a_i} \tag{7}$$

where k_i, a_i are the real uncertain parameters defined as $k_i \in (0.76 \cdot 10^4, 6.84 \cdot 10^4)$ and $a_i \in (91.1, 273.3)$. ω_0 is the current frequency defined by gridding.

The exchange of information between the PLLs is modelled by an interconnection matrix M defined in (8).

$$M_{net} = \left[\begin{array}{cccccccccccccccc|c}
0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{array}\right] \tag{8}$$

In this example, the transfer function between external signals w and z expresses the performance of the global PLL network and namely its ability to synchronize with periodic reference signal w. This reference signal is represented by its phase so that the PLL network has to track a ramp. More details can be found in [14].

4.2 Hierarchical analysis set up

The proposed hierarchical analysis approach is applied in two steps for this PLL network:

1. obtain dissipativity properties of each individual PLL, each PLL being a branch. Here note that the PLL are homogeneous so that the dissipativity properties obtained for one PLL is valid for the others as well;

2. obtain the performance of the overall network throught the interconnection of the 16 PLL branches and the matrix M_{net}.

Individual PLL Each PLL can be readily written in the form of an interconnection, which leads after normalization of the uncertainties to:

$$T_i(j\omega) = \Delta_i \star M_{PLL},\ \ \Delta_i \in \mathbf{\Delta}$$

with $\mathbf{\Delta}$ of the form

$$\left\{ \begin{bmatrix} \delta k_i I_2 & 0 \\ 0 & \delta a_i I_2 \end{bmatrix} \ , \text{ with } \delta = \begin{bmatrix} \delta k_i \\ \delta a_i \end{bmatrix} \in \mathbb{R}^2 \text{ such that } \|\delta\|_\infty \leq 1 \right\}.$$

It is a standard elementary uncertainty set (the leaves) of the form

$$\{ \ \mathbf{bdiag}_i\,(\delta_i I_{n_i})\ \ , \text{ with } \delta = [\delta_i] \in \mathbb{R}^r \text{ such that } \|\delta\|_\infty \leq 1 \ \}$$

representing parametric uncertainties. The dissipativity property of the uncertainty set can then be chosen of the form

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^* & \Phi_{22} \end{bmatrix} = \begin{bmatrix} -D & G \\ G^* & D \end{bmatrix}$$

where $D = \mathbf{bdiag}_i\,(D_i)$, with $D_i = D_i^* > 0$, and $G = \mathbf{bdiag}_i\,(G_i)$, with $G_i = -G_i^*$. This corresponds to the usual D-G scalings of the μ-analysis ([7]). The L scaling was introduced in [24] to reduce the conservatism in the case of $r \geq 2$. Indeed, it is possible to represent the branches T_i with a non standard uncertainty set $\mathbf{\Delta}$ (and the appropriate interconnection matrix M) of the form

$$\{ \ \delta \otimes I \ , \text{ with } \delta = [\delta_i] \in \mathbb{R}^r \text{ such that } \|\delta\|_\infty \leq 1 \ \}.$$

The dissipativity property of the uncertainty set can then be chosen of the form

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^* & \Phi_{22} \end{bmatrix} = \begin{bmatrix} -D + jL & G \\ G^* & \sum D_i \end{bmatrix} \text{ with } L = \begin{bmatrix} 0 & V_{1,2} & \dots & V_{1,r} \\ -V_{1,2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{r-1,r} \\ -V_{1,r} & \dots & -V_{r-1,r} & 0 \end{bmatrix} \tag{9}$$

where $D = \mathbf{bdiag}_i\,(D_i)$, with $D_i = D_i^* > 0$, $G = [\dots G_i \dots]$, with $G_i = -G_i^*$, and $V_{i,j} = V_{i,j}$ are real matrices. The elementary dissipativity properties of the uncertainty set are thus chosen of the form (9).

As for the dissipativity properties of the PLL itself, we chose:

- a conic sector alone (for comparison with the result obtained in [5]) or with;
- a conic sector and 4 bands (vertical, horizontal, and with a slope of +/-45 deg): $Y \in \{1, j, 1+j, 1-j\}$. This choice has been made *a priori*, without particular knowledge on a PLL frequency response.

Network performance The network performance is measured by its frequency response. The dissipativity property is thus chosen of the form (**??**).

4.3 Results

Individual PLL For illustration purpose, Figure 2 displays the obtained dissipativity properies of a PLL viewed as system embeddings for different frequencies. The red circle (the red star is its center) and lines represent the embeddings where as the green stars and purple circles represents the systems for some values of the uncertainties.

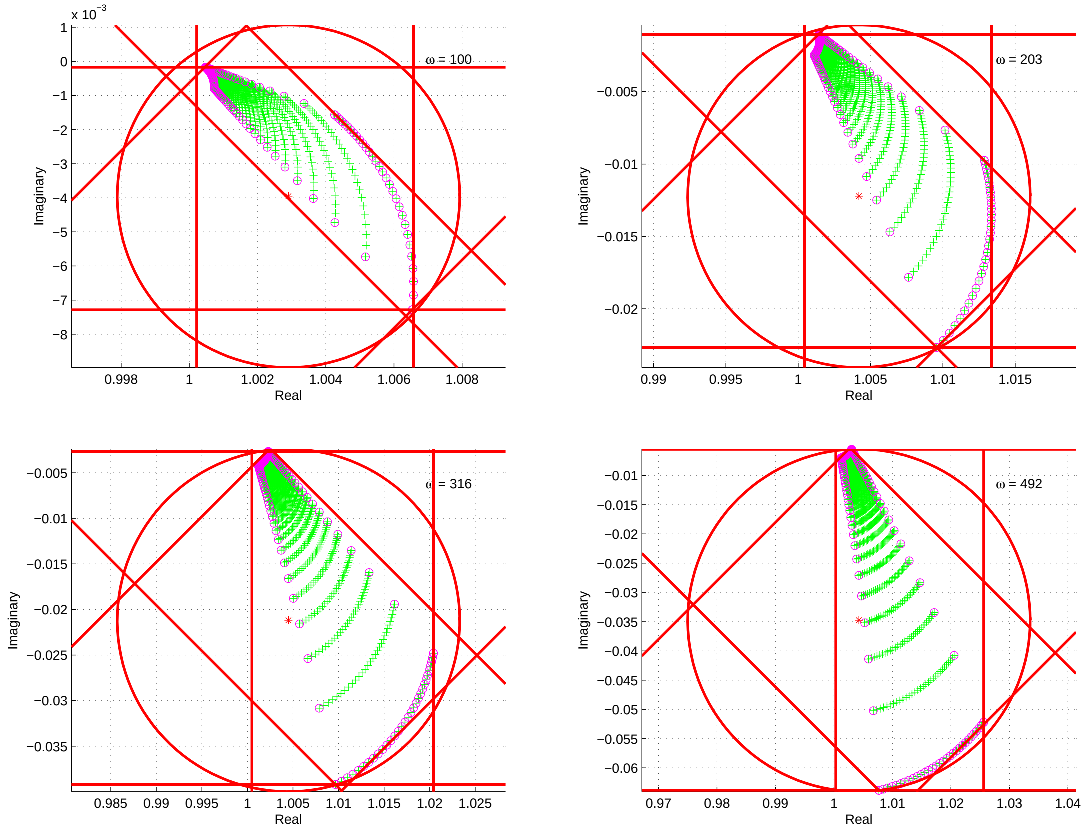

Figure 2: Dissipativity properties of a PLL viewed as embeddings

	Hierarchical analysis		μ-analysis
	Conic sector alone	Conic sector and 4 bands	
Maximum peak	13.5 dB (+7.4)	6.2 dB (+0.1)	6.1 dB
Computation time	72 sec (6 %)	767 sec (60 %)	1279 sec

Table 1: Characteristics of the analyses

Network performance We are now interested in the performance of the PLLs network displayed in Figure 3 while Table 4.3 displays the charateristics of the different analyses (the number in () for the hierarchical analysis columns is a comparison with the μ-analysis results).

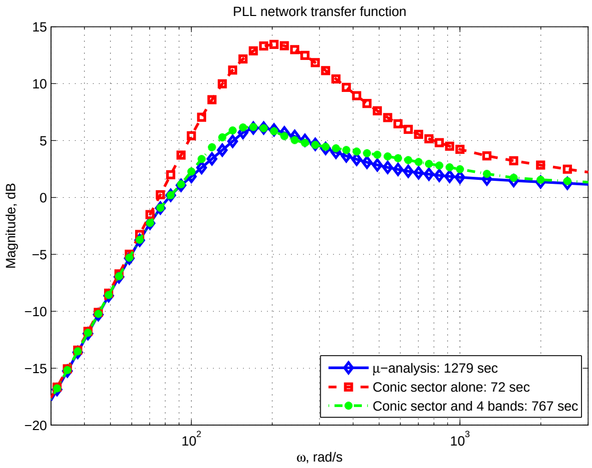

Figure 3: Performance of the PLLs network

All analyses show that the PLL network is able to follow a ramp as the slope at low frequencies is 40 dB/dec. Table 4.3 illustrates the trade-off between conservatism and computation time that can be set by the user with the hierarchical analysis approach: when using the conic sector alone, the result is conservative but is obtained really quickly; when using the conic sector with the bands, the result is much less conservative but is obtained in much more time. For this last hierarchical set up, the difference in the maximal peak value with μ-analysis is +0.1 dB, that corresponds to 1.2 % of ratio, which is negligible; the result was obtained in 60 % of the time needed for μ-analysis.

This difference is displayed in Figure 4 and shows that the peak is not attained at the same frequency. For most frequencies, μ-analysis performs better in terms of conservatism. Surprisingly, hierarchical analysis performs better in the frequency range (160,300) rad/s with a less demanding computation load at the same time.

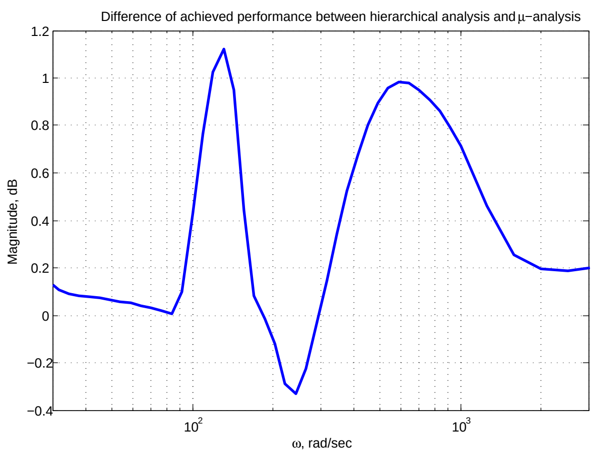

Figure 4: Difference in achieved performance between hierarchical analysis and μ-analysis

5 Conclusion

In this paper, a hierarchical analysis approach has been proposed for the performance of uncertain large-scale systems. It relies on the propagation of dissipativity properties of sub-systems through an interconnection; this propagation result is recursively applied leading to a multi steps analysis. The aim is to propose a trade-off adapted to these large-scale systems when a one-step approach as μ-analysis can lead to a large computation time. A numerical example on a PLL network illustrated the new achieved trade-off.

Further work directions are:

- find other dissipative properties that can be used. We think to a cone as proposed in [26] for instance;
- further assess the achieved trade-off for other examples, especially MIMO ones;
- assess the evolution of the achieved trade-off in function of the dissipativity properties used.

Another direction is to use differently the propagation result. It is used here in a multi steps approach; it could also be used to lead to a one step approach. This leads to a less conservative approach but with more demanding computation. This trade-off can also be assessed.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*, volume 15 of *Studies in Appllied Mathematics*. SIAM, Philadelphia, USA, June 1994.

[2] S. Boyd and L. El Ghaoui. Method of centers for minimizing generalized eigenvalues. *Linear Algebra and Applications, special issue on Numerical Linear Algebra Methods in Control*, 188:63–111, July 1993.

[3] R. D. Braatz, P. M. Young, J. C. Doyle, and M. Morari. Computational complexity of μ calculation. *IEEE Trans. Aut. Control*, AC-39(5):1000–1002, May 1994.

[4] Y.-S. Chou, A.L. Tits, and V. Balakrishnan. Stability multipliers and μ upper bounds: connections and implications for numerical verification of frequency domain conditions. *IEEE Trans. Aut. Control*, 44(5):906–913, 1999.

[5] M. Dinh, A. Korniienko, and G. Scorletti. Embedding of uncertainty propagation: Application to hierarchical performance analysis. In *IFAC Joint Conference, 5th Symposium on System Structure and Control*, pages 190–195, Grenoble, France, February 2013.

[6] J.C. Doyle. Analysis of feedback systems with structured uncertainties. *IEE Proc.*, 129-D(6):242–250, November 1982.

[7] M. K. H. Fan, A. L. Tits, and J. C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Control*, 36(1):25–38, January 1991.

[8] E. Feron, P. Apkarian, and P. Gahinet. Analysis and synthesis of robust control systems via parameter-dependent lyapunov functions. *IEEE Trans. Aut. Control*, 41(7):1041–1046, July 1996.

[9] M. R. Garey and D. S. Johnson. *Computers and Intractability: A guide to the theory of NP-Completeness.* W. H. Freeman, 1979.

[10] C.H. Houpis and S.J. Rasmussen. *Quantitative feedback theory: fundamentals and applications.* Control engineering. Marcel Dekker, 1999.

[11] T. Iwasaki and S. Hara. Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations. *IEEE Trans. Aut. Control*, 43:619–630, 1998.

[12] V.A. Jakubovič. The $\mathcal{S}$-procedure in nonlinear control theory. *Vestnik Leningrad Univ. (russian) Vestnik Leningrad Univ. Math. (amer.)*, 4 (amer.)(1 (russian)), 1971 (russian) 1977 (amer.).

[13] A. Korniienko, G. Scorletti, E. Colinet, and E. Blanco. Control law design for distributed multi-agent systems. Technical report, Laboratoire Ampère, Ecole Centrale de Lyon, 2011.

[14] A. Korniienko, G. Scorletti, E. Colinet, E. Blanco, J. Juillard, and D. Galayko. Control law synthesis for distributed multi-agent systems: Application to active clock distribution networks. In *IEEE American Control Conference*, pages 4691– 4696, San Francisco, June 2011.

[15] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Aut. Control*, pages 819–830, June 1997.

[16] Y.C. Paw and G.J. Balas. Development and application of an integrated framework for small uav flight control development. *Mechatronics*, 21(5):789–802, August 2011.

[17] M. G. Safonov. *Stability and Robustness of Multivariable Feedback Systems*. MIT Press, Cambridge, 1980.

[18] M. G. Safonov. Stability margin of diagonaly perturbed multivariable feedback systems. *IEE Proc., Part D*, 129(6):251–256, 1982.

[19] M. G. Safonov. Propagation of conic model uncertainty in hierarchical systems. *IEEE Transactions on Circuits and Systems*, pages 388–396, June 1983.

[20] C.W. Scherer. LPV control and full block multipliers. *Automatica*, 73:361–375, 2001.

[21] G. Scorletti. *Approche Unifiée de l'analyse et la commande des systèmes par formulation LMI*. PhD thesis, Université d'Orsay, Paris, France, 1997. In French.

[22] G. Scorletti. Robustness analysis with time delays. In IEEE, editor, *Proc. IEEE Conf. on Decision and Control*, pages 3824–3829, San Diego, California, December 1997.

[23] G. Scorletti. A more praticle formulation for robustness analysis. In *IFAC Conference on System Structure and Control*, Nantes, France, July 1998.

[24] G. Scorletti, X. Bombois, M. Barenthin, and V. Fromion. Improved efficient analysis for systems with uncertain parameters. In *IEEE Conf. Decision and Control*, pages 5038–5043, New Orleans, dec. 2007.

[25] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and design.* John Wiley & Sons, Chicester, England, 2005.

[26] A.L. Tits, V. Balakrishnan, and L. Lee. Robustness under bounded uncertainty with phase information. *IEEE Trans. Aut. Control*, 44(1), 1999.

[27] L. Vandenberghe, S. Boyd, and S.P. Wu. Determinant maximization with linear matrix inequality constraints. submitted to SIMAX, March 1996.

[28] G. Zames. Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Aut. Control*, AC-26(2):301–320, April 1981.

A Proof of Theorem 2.1

It is a corollary of Lemma A.1 for stability and Lemma A.2 for dissipativity by noticing that condition (13) implies condition (11).

Lemma A.1 *The uncertain system* $\boldsymbol{\Delta} \star M$ *is stable if and only if:*

1. *there exists $\Delta_0 \in \mathbf{\Delta}$ such that the system $\Delta_0 \star M$ is stable;*

2. *there exists 3 transfer functions $\Phi_{11}(j\omega)$, $\Phi_{12}(j\omega)$ and $\Phi_{22}(j\omega)$ of $\mathbf{RL}_\infty$, with $\Phi_{11}(j\omega) = \Phi_{11}(j\omega)^*$ and $\Phi_{22}(j\omega) = \Phi_{22}(j\omega)^*$, such that:*

$$\text{the uncertainty set } \mathbf{\Delta} \text{ is } \{\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)\} \text{ dissipative} \tag{10}$$

and for any $\omega \in \overline{\mathbf{R}}$

$$\begin{bmatrix} A(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} -\Phi_{22}(j\omega) & -\Phi_{12}(j\omega)^* \\ -\Phi_{12}(j\omega) & -\Phi_{11}(j\omega) \end{bmatrix} \begin{bmatrix} A(j\omega) \\ I \end{bmatrix} > 0. \tag{11}$$

If the condition 1 does not hold then the uncertain system is non-causally stable.

Lemma A.2 *The uncertain system $\mathbf{\Delta} \star M$ is $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative if and only if there exists 3 transfer functions $\Phi_{11}(j\omega)$, $\Phi_{12}(j\omega)$ and $\Phi_{22}(j\omega)$ of $\mathbf{RL}_\infty$, with $\Phi_{11}(j\omega) = \Phi_{11}(j\omega)^*$ and $\Phi_{22}(j\omega) = \Phi_{22}(j\omega)^*$, such that:*

$$\text{the uncertainty set } \mathbf{\Delta} \text{ is } \{\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)\} \text{ dissipative} \tag{12}$$

and for any $\omega \in \overline{\mathbf{R}}$

$$\begin{bmatrix} M(j\omega) \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} -\Phi_{22}(j\omega) & 0 & -\Phi_{12}(j\omega)^* & 0 \\ 0 & X(j\omega) & 0 & Y(j\omega) \\ \hline -\Phi_{12}(j\omega) & 0 & -\Phi_{11}(j\omega) & 0 \\ 0 & Y(j\omega)^* & 0 & Z(j\omega) \end{array}\right] \begin{bmatrix} M(j\omega) \\ I \end{bmatrix} > 0. \tag{13}$$

B Proof of Lemma A.1

Non-causal stability We begin by proving that the non-causal stability of $\mathbf{\Delta} \star M$ is equivalent to *condition 2*. As M is stable, we only need to prove the non-causal stability of the feedback $\mathbf{\Delta} \star A$. By definition, we have: for any Δ and ω

$$\det\left(I - A(j\omega)\Delta(j\omega)\right) \neq 0.$$

We prove it by contradiction. So let us assume that *condition 2* is verified but there exists Δ_c and ω_c such that

$$\det\left(I - A(j\omega_c)\Delta_c(j\omega_c)\right) = 0.$$

Equivalently, there exists a non null $q_c(j\omega_c)$ such that

$$\left(I - A(j\omega_c)\Delta_c(j\omega_c)\right) q_c(j\omega_c) = 0.$$

Let $p_c(j\omega_c) = \Delta_c(j\omega_c)q_c(j\omega_c)$, then $q_c(j\omega_c) = A(j\omega_c)p_c(j\omega_c)$. By definition of dissipativity of $\mathbf{\Delta}$, we have:

$$\begin{bmatrix} p_c(j\omega_c) \\ q_c(j\omega_c) \end{bmatrix}^* \begin{bmatrix} \Phi_{11}(j\omega_c) & \Phi_{12}(j\omega_c) \\ \Phi_{12}(j\omega_c)^* & \Phi_{22}(j\omega_c) \end{bmatrix} \begin{bmatrix} p_c(j\omega_c) \\ q_c(j\omega_c) \end{bmatrix} > 0.$$

Condition (13) implies

$$\begin{bmatrix} A(j\omega_c) \\ I \end{bmatrix}^* \begin{bmatrix} -\Phi_{22}(j\omega_c) & -\Phi_{12}(j\omega_c)^* \\ -\Phi_{12}(j\omega_c) & -\Phi_{11}(j\omega_c) \end{bmatrix} \begin{bmatrix} A(j\omega_c) \\ I \end{bmatrix} > 0.$$

Post et pre multiplying by $q_c(j\omega_c)$ yields

$$\begin{bmatrix} q_c(j\omega_c) \\ p_c(j\omega_c) \end{bmatrix}^* \begin{bmatrix} -\Phi_{22}(j\omega_c) & -\Phi_{12}(j\omega_c)^* \\ -\Phi_{12}(j\omega_c) & -\Phi_{11}(j\omega_c) \end{bmatrix} \begin{bmatrix} q_c(j\omega_c) \\ p_c(j\omega_c) \end{bmatrix} > 0$$

which is a contradiction.

Necessity It is evident from the non-causal stability equivalence.

Sufficiency As M is stable, we only need to prove the stability of the feedback $\mathbf{\Delta} \star A$. By contradiction, we show that this assumption with *condition 1* leads to the existence of Δ_c and ω_c such that

$$\det\left(I - A(j\omega_c)\Delta_c(j\omega_c)\right) = 0.$$

Which is a contradiction from the non-causal stability proof. So let us assume that there exists $\Delta_u \in \mathbf{\Delta}$ such that $\Delta_u \star A$ is unstable while *conditions 1* and *2* are met.

As Δ_u and A are stable, applying the Nyquist criterion, it shows that the Nyquist curve defined by

$$\det\left(I - A(j\omega)\Delta_u(j\omega)\right), \ \omega \in \overline{\mathbf{R}}$$

encircles the origin in the complex plan. At the same time, as $\Delta_0 \star A$ is stable, the curve

$$\det\left(I - A(j\omega)\Delta_0(j\omega)\right), \ \omega \in \overline{\mathbf{R}}$$

does not encircle the origin. Now, as $\mathbf{\Delta}$ is a connected set, it is possible to find a continuous path inside the set that link $\Delta_0(j\omega)$ to $\Delta_u(j\omega)$, that is, a continuous function Ψ_ω: $[0,\ 1] \longmapsto \mathbf{\Delta}(j\omega)$ such that $\Psi_\omega(0) = \Delta_0(j\omega)$ and $\Psi_\omega(1) = \Delta_u(j\omega)$. As the determinant is a continuous function, the function which at $\Delta(j\omega)$ associates $\det\left(I - M(j\omega)\Delta(j\omega)\right)$ composed with Ψ_ω is a continuous function of $\lambda \in [0, 1]$. Thus there exists Δ_c and ω_c such that

$$\det\left(I - A(j\omega_c)\Delta_c(j\omega_c)\right) = 0.$$

C Proof of Lemma A.2

Necessity The necessity is proved by construction of

$$\Phi(j\omega) = \begin{bmatrix} \Phi_{11}(j\omega) & \Phi_{12}(j\omega) \\ \Phi_{12}(j\omega)^* & \Phi_{22}(j\omega) \end{bmatrix}$$

satisfying constraints 12 and 13. For convenience of writing, we drop the dependency to $j\omega$ in this part of the proof.

By definition: for any $\Delta \in \boldsymbol{\Delta}$,

$$\begin{bmatrix} p \\ w \end{bmatrix}^* \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} > 0 \tag{14}$$

such that

$$\begin{aligned} p &= \Delta \quad q \\ q &= \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} \end{aligned}$$

This last equality can be rewritten as

$$\begin{bmatrix} I & -\Delta \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} = 0. \tag{15}$$

From Finsler's lemma [1], condition (14) holds for $\begin{bmatrix} p^* & w^* \end{bmatrix}^*$ defined by (15) if and only if there exists τ such that[2]:

$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \tau \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^* \begin{bmatrix} I \\ -\Delta^* \end{bmatrix} \begin{bmatrix} I & -\Delta \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} > 0.$$

That is: there exists τ such that for any $\Delta \in \boldsymbol{\Delta}$,

$$\left[\begin{array}{cc} C & D \\ 0 & I \\ \hline I & 0 \\ A & B \end{array}\right]^* \left[\begin{array}{c|c} \begin{matrix} X & Y \\ Y^* & Z \end{matrix} & 0 \\ \hline 0 & \tau \begin{bmatrix} I \\ -\Delta^* \end{bmatrix} \begin{bmatrix} I & -\Delta \end{bmatrix} \end{array}\right] \left[\begin{array}{cc} C & D \\ 0 & I \\ \hline I & 0 \\ A & B \end{array}\right] > 0.$$

Let $\begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix}$ be such that

$$\begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix}_\perp = \left[\begin{array}{cc} C & D \\ 0 & I \\ \hline I & 0 \\ A & B \end{array}\right],$$

[2]In fact, τ should depend on Δ, that is τ_Δ. As shown in [4], it can be used a continuous function $\tau(\Delta)$ on the closure of $\boldsymbol{\Delta}$. But as $\boldsymbol{\Delta}$ is bounded, it can be selected independent of Δ (take the maximum on the closure of $\boldsymbol{\Delta}$). This fact will be used several times.

then, by applying Finsler's lemma, we get the equivalent condition: there exists τ and η such that for any $\Delta \in \mathbf{\Delta}$,

$$\left[\begin{array}{c|c} \begin{matrix} X & Y \\ Y^* & Z \end{matrix} & 0 \\ \hline 0 & \tau \begin{bmatrix} I \\ -\Delta^* \end{bmatrix} \begin{bmatrix} I & -\Delta \end{bmatrix} \end{array}\right] + \eta \begin{bmatrix} \mu_1^* \\ \mu_2^* \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} > 0$$

It is equivalent by Schur's lemma to: there exists τ and η such that for any $\Delta \in \mathbf{\Delta}$,

$$\begin{cases} \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} + \eta \mu_1^* \mu_1 > 0 \\ \tau \begin{bmatrix} I \\ -\Delta^* \end{bmatrix} \begin{bmatrix} I & -\Delta \end{bmatrix} + \eta \mu_2^* \mu_2 - \eta \mu_2^* \mu_1 \left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} + \eta \mu_1^* \mu_1 \right)^{-1} \eta \mu_1^* \mu_2 > 0 \end{cases}$$

Thus there exists τ, η and $\epsilon > 0$ such that for any $\Delta \in \mathbf{\Delta}$,

$$\tau \begin{bmatrix} I \\ -\Delta^* \end{bmatrix} \begin{bmatrix} I & -\Delta \end{bmatrix} + \eta \mu_2^* \mu_2 - \eta \mu_2^* \mu_1 \left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} + \eta \mu_1^* \mu_1 \right)^{-1} \eta \mu_1^* \mu_2 - \epsilon I > 0$$

Let us define

$$\Phi = \eta \mu_2^* \mu_2 - \eta \mu_2^* \mu_1 \left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} + \eta \mu_1^* \mu_1 \right)^{-1} \eta \mu_1^* \mu_2 - \epsilon I.$$

Then, using Finsler's lemma, the condition there exists τ such that for any $\Delta \in \mathbf{\Delta}$,

$$\tau \begin{bmatrix} I \\ -\Delta^* \end{bmatrix} \begin{bmatrix} I & -\Delta \end{bmatrix} + \Phi > 0$$

is equivalent to for any $\Delta \in \mathbf{\Delta}$,

$$\begin{bmatrix} \Delta \\ I \end{bmatrix}^* \Phi \begin{bmatrix} \Delta \\ I \end{bmatrix} > 0.$$

That is $\mathbf{\Delta}$ is $\{\Phi_{11}, \Phi_{12}, \Phi_{22}\}$ dissipative.

For the remaining part, let us notice that

$$-\Phi + \eta \mu_2^* \mu_2 - \eta \mu_2^* \mu_1 \left(\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} + \eta \mu_1^* \mu_1 \right)^{-1} \eta \mu_1^* \mu_2 > 0.$$

Then by Schur's lemma, it is equivalent to

$$\left[\begin{array}{c|c} \begin{matrix} X & Y \\ Y^* & Z \end{matrix} & 0 \\ \hline 0 & -\Phi \end{array}\right] + \eta \begin{bmatrix} \mu_1^* \\ \mu_2^* \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} > 0.$$

Applying Finsler's lemma, it is equivalent to

$$\begin{bmatrix} C & D \\ 0 & I \\ \hline I & 0 \\ A & B \end{bmatrix}^* \left[\begin{array}{cc|c} X & Y & \\ Y^* & Z & 0 \\ \hline \multicolumn{2}{c|}{0} & -\Phi \end{array}\right] \begin{bmatrix} C & D \\ 0 & I \\ \hline I & 0 \\ A & B \end{bmatrix} > 0$$

which is exactly constraint (13) after rearrangement.

Sufficiency It follows from the interpretation of a dissipativity property in terms of inclusion of sets. The $\{\Phi_{11}(j\omega), \Phi_{12}(j\omega), \Phi_{22}(j\omega)\}$ dissipativity of $\mathbf{\Delta}(j\omega)$ is equivalent to

$$\left\{ \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} \neq 0 \;\middle|\; \exists \Delta(j\omega),\ p(j\omega) = \Delta(j\omega) q(j\omega) \right\}$$
$$\subseteq$$
$$\left\{ \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} \neq 0 \;\middle|\; \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix}^* \Phi(j\omega) \begin{bmatrix} p(j\omega) \\ q(j\omega) \end{bmatrix} > 0 \right\}$$

which implies

$$\Lambda \subseteq \Gamma$$

with

$$\Lambda = \left\{ \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix} \neq 0 \;\middle|\; \exists \Delta(j\omega),\ p(j\omega) = \Delta(j\omega) q(j\omega),\ q(j\omega) = \begin{bmatrix} A(j\omega) & B(j\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix} \right\}$$

$$\Gamma = \left\{ \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix} \neq 0 \;\middle|\; \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix}^* \begin{bmatrix} I & 0 \\ A(j\omega) & B(j\omega) \end{bmatrix}^* \Phi(j\omega) \begin{bmatrix} I & 0 \\ A(j\omega) & B(j\omega) \end{bmatrix} \begin{bmatrix} p(j\omega) \\ w(j\omega) \end{bmatrix} > 0 \right\}$$

Now, the uncertain system $\mathbf{\Delta} \star M$ is $\{X(j\omega), Y(j\omega), Z(j\omega)\}$ dissipative if and only if condition (14) is verified for the set Λ. Due to the inclusion, this is implied by condition (14) being verified for the set Γ. Using S-procedure [12, 1], it is the case if

$$\begin{bmatrix} C(j\omega) & D(j\omega) \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & Z(j\omega) \end{bmatrix} \begin{bmatrix} C(j\omega) & D(j\omega) \\ 0 & I \end{bmatrix} + \dots$$
$$\begin{bmatrix} I & 0 \\ A(j\omega) & B(j\omega) \end{bmatrix}^* (-\Phi(j\omega)) \begin{bmatrix} I & 0 \\ A(j\omega) & B(j\omega) \end{bmatrix} > 0.$$

This is exactly condition (13) after factorization.

D Direct Additive Uncertainty Embedding

There exist:

- a matrix $G_{nom} \in \mathbb{C}^{n_z \times n_w}$;
- invertible matrices $W_o \in \mathbb{C}^{n_z \times n_z}$ and $W_i \in \mathbb{C}^{n_w \times n_w}$;

such that the uncertain system $\mathbf{\Delta} \star M$ is included in the set G_u defined

$$G_u = \{ \ G_{nom} + W_o \Delta_u W_i \ , \ \bar{\sigma}(\Delta_u) < 1 \ \}$$

if and only if there exists a solution to one of the two following problems:

1. First: there exist
   - a matrix $P_{nom} \in \mathbb{C}^{n_z \times n_w}$;
   - a positive definite matrix $P_o \in \mathbb{C}^{n_z \times n_z}$;
   - a positive definite matrix $P_i \in \mathbb{C}^{n_w \times n_w}$;
   - a matrix $\Phi \in \mathbb{C}^{(n_o+n_i)\times(n_o+n_i)}$;

   such that (16) and (17) are satisfied

$$\forall \Delta \in \mathbf{\Delta}, \ \begin{bmatrix} \Delta \\ I \end{bmatrix}^* \Phi \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \tag{16}$$

$$\left[\begin{array}{ccc} I & 0 & 0 \\ 0 & C & D \\ \hline 0 & I & 0 \\ 0 & A & B \\ \hline 0 & 0 & I \end{array}\right]^* \left[\begin{array}{cc|c|c} -P_o & I & 0 & P_{nom} \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \Phi & 0 \\ \hline -P_{nom}^* & 0 & 0 & -P_i \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & C & D \\ \hline 0 & I & 0 \\ 0 & A & B \\ \hline 0 & 0 & I \end{array}\right] < 0 \tag{17}$$

   with $P_o = W_o W_o^*$, $P_i = W_i^* W_i$ and $P_{nom} = G_{nom}$.

2. Second: there exist
   - a matrix $\widetilde{P}_{nom} \in \mathbb{C}^{n_z \times n_w}$;
   - a positive definite matrix $\widetilde{P}_o \in \mathbb{C}^{n_z \times n_z}$;
   - a positive definite matrix $P_i \in \mathbb{C}^{n_w \times n_w}$;
   - a matrix $\Phi \in \mathbb{C}^{(n_o+n_i)\times(n_o+n_i)}$;

   such that (16) and (18) are satisfied

$$\left[\begin{array}{ccc} I & 0 & 0 \\ 0 & C & D \\ \hline 0 & I & 0 \\ 0 & A & B \\ \hline 0 & 0 & I \end{array}\right]^* \left[\begin{array}{cc|c|c} -\widetilde{P}_o & \widetilde{P}_o & 0 & \widetilde{P}_{nom} \\ \widetilde{P}_o & 0 & 0 & 0 \\ \hline 0 & 0 & \Phi & 0 \\ \hline -\widetilde{P}_{nom}^* & 0 & 0 & -P_i \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & C & D \\ \hline 0 & I & 0 \\ 0 & A & B \\ \hline 0 & 0 & I \end{array}\right] < 0 \tag{18}$$

   with $\widetilde{P}_o = W_o^{-*} W_o^{-1}$, $P_i = W_i^* W_i$ and $\widetilde{P}_{nom} = \widetilde{P}_o G_{nom}$.