Measuring Functions Smoothness with Local Fractional Derivatives

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Abstract

We study a notion of local fractional differentiation, obtained by localizing the classical fractional derivative. We show that it is strongly related with the local Hölder exponent, and give an interpretation of this result in terms of 2-microlocal analysis.

1 Introduction

Measuring the local smoothness of functions proves to be an important task for many applications in such diverse fields as mathematical analysis, signal and image processing or geophysics. Depending on the situation, various definitions of local regularity have been proposed. The most often used is probably the one based on Hölder spaces in their various versions. Such a characterization is for instance central in multifractal analysis, and is an instrumental tool for image segmentation or denoising, and Internet traffic characterization. Other important measures of local regularity include (local) fractional dimensions (e.g. box, Hausdorff or regularization dimension), which have been used in various contexts, such as tribology or image classification. In this paper, we are interested in comparing the classical Hölder characterizations (and their refinements, see below), with yet another measure, based on the degree of local fractional differentiability (LFD). This notion was introduced in [11] as an attempt to localize the classical fractional derivative [18]. In [11], it is for instance proved that Weierstrass function W is locally fractionally differentiable at any point up to an order which is precisely the pointwise Hölder exponent of W. In this work, we provide further results in this direction (correcting along the way some inaccuracies of [11]). In particular, we prove that, for all functions belonging to a large functional space, the degree of LFD coincides with the *local* Hölder exponent. Furthermore, we give a precise interpretation of the fractional derivative in terms of 2-microlocal analysis.

Other works dealing with different aspects of the local properties of fractional integrodifferentiation include [5, 6, 10, 16, 17], and we refer the interested reader to these papers.

The rest of this paper is organized as follows: In section 2, we recall the definitions of the local and pointwise Hölder exponents and of the LFD. Section 3 proves the equality between the local Hölder exponent and the degree of LFD. In section 4, we extend this result to a more precise one using 2-microlocal analysis. Finally, section 5 contains examples on simple functions, which allow to understand in a concrete way how LFD acts on signals.

2 Measures of local regularity

2.1 Pointwise Hölder exponent

Definition 1 *Let α be a positive real number which is not an integer, and $x_0 \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is in $C^\alpha_{x_0}$ if there exists a polynomial P_{x_0} of degree less than α such that:*

$$|f(x) - P_{x_0}(x)| \leq c|x - x_0|^\alpha. \tag{1}$$

When $\alpha \in]0, 1[$, this reduces to:

$$|f(x) - f(x_0)| \leq c|x - x_0|^\alpha \tag{2}$$

The pointwise Hölder exponent of f at x_0, denoted $\alpha_p(x_0)$, is the supremum of the α-s for which (1) holds. Extension to higher dimensions is straightforward, but will not be considered here.

As said above, this regularity characterization is widely used because it has direct interpretations both mathematically and in applications. It has been for instance used for speech synthesis [3] and image analysis [14]. However, the pointwise Hölder exponent has also a number of drawbacks, a major one being that it is not stable under the action of (pseudo) differential operators. Thus, for instance, knowing the pointwise Hölder exponent of a function at a point x_0 is not sufficient to predict the Hölder exponent of its derivative at the same point, and the same happens for the Weyl fractional derivative (see below).

2.2 Local Hölder exponent

The local Hölder exponent α_l measures slightly different features as compared to α_p. It is defined as follows: Let $\alpha \in]0, 1[$, $\Omega \subset \mathbb{R}$. One classically says that $f \in C^\alpha_l(\Omega)$ if:

$$\exists C : \forall x, y \in \Omega : \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq C$$

Let now:

$$\alpha_l\left(f, x_0, \rho\right) = \sup\left\{\alpha : f \in C^\alpha_l\left(B\left(x_0, \rho\right)\right)\right\}$$

Note that $\alpha_l(f, x_0, \rho)$ is non increasing as a function of ρ. We may now give the definition of the local Hölder exponent:

Definition 2 *Let f be a continuous function. The local Hölder exponent of f at x_0 is the real number:*

$$\alpha_l(f, x_0) = \lim_{\rho \to 0} \alpha_l(f, x_0, \rho)$$

This exponent is stable under pseudo-differentiation or integration. Moreover, it is easier to estimate than the pointwise Hölder exponent. Its main drawback is that it is not as precise as the pointwise one [7].

An important difference between α_p and α_l is well illustrated on the example of the chirp $f(x) = |x|^\beta cos(1/|x|^\gamma)$, $f(0) = 0$, with $\beta, \gamma > 0$. In this case, at $x = 0$, $\alpha_p = \beta$ while $\alpha_l = \frac{\beta}{1+\gamma}$. Thus, while α_p is sensitive only to what happens "at" 0, α_l measures also the local oscillatory behavior of the signal "around" 0.

We shall need the following characterization of the local Hölder exponent in terms of wavelet coefficients:

Proposition 1 *Let $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$ be an orthonormal basis of $L^2(\mathbb{R})$ and denote the discrete wavelet coefficients of f by $c_{j,k}$, i.e.*

$$c_{j,k} = 2^j \int f(x)\psi(2^j x - k)dx$$

Then, the local Hölder exponent of f at x is

$$\alpha_l = \lim_{\eta \to 0} (\sup\{s /\ \exists C,\ \forall c_{j,k} \subset B(x, \eta),\ |c_{j,k}| \leq C 2^{-sj}\}) \tag{3}$$

Note that, while neither α_p nor α_l yield complete characterization, it is possible to combine the nice properties of each exponent: this is the topic of 2-microlocal analysis, which we shall use in section 4.

2.3 Local fractional derivative

In this section, we recall the definition of LFD introduced in [11] and make precise the notion of degree of LFD. We start by briefly recalling the definition of the classical fractional derivative in the case where the order is between 0 and 1:

Definition 3 *[18] The (Riemann-Liouville) fractional derivative of a function f of order q $(0 < q < 1)$ is defined as:*

$$\begin{aligned} D_x^q f(x') &= \begin{cases} D_{x+}^q f(x'), & x' > x, \\ D_{x-}^q f(x'), & x' < x. \end{cases} \\ &= \frac{1}{\Gamma(1-q)} \begin{cases} \frac{d}{dx'} \int_x^{x'} f(t)(x'-t)^{-q} dt, & x' > x, \\ -\frac{d}{dx'} \int_{x'}^{x} f(t)(t-x')^{-q} dt, & x' < x. \end{cases} \end{aligned} \tag{4}$$

These fractional derivatives exists almost everywhere as soon as f is absolutely continuous ([18], page 35). When $x = \pm\infty$, i.e. the integration is performed on a semi-infinite domain, the corresponding derivatives $D^q_{\pm\infty}f$ are called the Weyl fractional derivatives.

In [18], the effect of fractional *integration* on *global* Hölder spaces is investigated in full detail. Our aim here is to obtain results for fractional derivatives and local/pointwise exponents.

Note for further use the following classical property of the Weyl derivative ([18], theorem 7.1):

Proposition 2 *Let f belong to $L^1(\mathbb{R})$. Then, provided f is sufficiently smooth:*

$$\widehat{D^q_{-\infty}f}(\omega) = (i\omega)^q \widehat{f}(\omega)$$

where $\widehat{f}$ denotes the Fourier transform of f.

The same type of property holds for wavelet coefficients:

Proposition 3 *[15] Let $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$ be an orthonormal basis of $L^2(\mathbb{R})$ with ψ in the Schwartz class, and denote the discrete wavelet coefficients of f by $c_{j,k}$. Then, the wavelet of coefficients of $D^q_{-\infty}f$ (in another wavelet basis) are*

$$d_{j,k} = 2^{-jq} c_{j,k}$$

It is an easy consequence of this Proposition and Proposition 1 that the Weyl derivative of order q decreases the local Hölder exponent by exactly q. In contrast, no such property holds for α_p.

The main motivation for introducing local fractional derivatives is to try and remedy to two sometimes undesirable properties of fractional derivatives: Non locality and the behaviour with respect to constants. As for the first point, it is clear from the definition that the fractional derivative of a function f depends on the values of f on the whole interval $[x', x]$. The second feature is also well-known. For instance, the fractional derivative of order q from the right of the function $f(x) = x^p$ $(x > 0,\ p > -1)$ is:

$$D^q_{0+}x^p = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q} \tag{5}$$

Substituting $p = 0$ for a constant function in the above formula, one gets $D^q_{0+}1 = 1/\Gamma(1-q)x^{-q}$, i.e. the fractional derivative of a constant is not zero in general. In particular, the fractional derivative of a function changes if one adds a constant to this function. Thus, the fractional derivative of a function depends on the choice of the origin, whereas the usual notion of differentiability is a local concept independent of the origin. The aim of the local fractional derivative is to modify in a simple way the usual fractional derivative to obtain locality and translation invariance.

The basic idea is straightforward: Let x be the point at which one wants to study the differentiability of f. One first subtracts the value of f at x. This washes out the effect of a constant term. Second, one introduces a limit, as shown below, to obtain a local quantity.

Definition 4 *The local fractional derivative of order q $(0 < q < 1)$ of a function $f \in C^0 : \mathbb{R} \to \mathbb{R}$ is defined as*

$$\mathcal{D}^q f(x) = \lim_{x' \to x} D_x^q (f(x') - f(x)) \tag{6}$$

if the limit exists in $\mathbb{R} \cup \{\infty\}$.

As an example, take again $f(x) = x^p$ $(x > 0,\ p > -1)$. One computes easily that $\mathcal{D}^q f(0)$ equals 0 if $q < p$, ∞ if $q > p$, and $\Gamma(q+1)$ if $q = p$. Although in this case, the limits all exist and there is a q where the limit is finite and non zero, this is not the general situation. Thus, as emphasized in the remark below, we are not in general interested in the value of $\mathcal{D}^q f$, but in the critical q.

This concept of local fractional derivative has been used to study the local fractional differentiability of nowhere differentiable functions [11]. Equations involving these local fractional derivatives have been studied [2, 12] and have found to be useful in studying phenomena in fractal space or time.

With this notion of LFD, it is natural to define the critical order of local fractional differentiability, or degree of LFD, as the largest value for which the LFD exists. The next Proposition shows that this is a well defined notion.

Definition and Proposition 1 *The degree of LFD of the continuous function f at x is defined as:*

$$q_c(x) = \sup\{q \in [0,1] : \mathcal{D}^q f(x) \text{ exists at } x \text{ and is finite}\}.$$

Proof:
Let E be the set:

$$E = \{q \in [0,1] : \mathcal{D}^q f(x) \text{ exists at } x \text{ and is finite}\}.$$

All we need to prove is that E is non empty, so that $q_c(x)$ is well defined as the supremum of a subset of $[0,1]$. Note that $D_x^0 f = f$. Since we are dealing with continuous functions, we get that $0 \in E$. Thus $q_c(x)$ exists and is non negative. Remark finally that, if $q > 0$ belongs to E, then clearly all $q' \in [0,q]$ also belong to E, as is easily seen from definition 4. Thus E is always a segment.

■

Remark: In general, $\mathcal{D}^q f(x)$ will be zero for $q < q_c$ and infinite for $q > q_c$ when the limit exists. Thus, q_c may be understood as a cut-off value, much in the same way as Hölder exponents or fractional dimensions. At the cut-off, $\mathcal{D}^q f(x)$ may or not be finite non zero.
Remark: The definition of q_c can easily be extended to functions in L^2, and to some classes distributions, as for instance homogeneous ones.

3 Relation between the degree of LFD and Hölder exponents

It is intuitively clear that the notions of degree of LFD and Hölder exponents must be related in some way. The aim of this section is to prove, via elementary means, that, for a large class of functions, q_c indeed coincides with α_l. From an intuitive point of view, the fact that it is the local exponent that comes into play rather than the pointwise one stems from the fact that LFD starts by integrating the function around the point of interest, so that the behavior in a whole neighborhood is important. Thus, for instance, if f has a strong oscillatory behavior around 0, like the chirp, this will have consequences on $\mathcal{D}_0^q f$ through the integration in (4). Also, α_l behaves well under pseudo-differentiation, while α_p does not.

We start by proving a simple proposition about local Hölder exponents. Let $g : \Omega \to \mathbb{R}$ be in $C^\alpha(\Omega)$, $\alpha > 0$. where $\Omega \subset \mathbb{R}$ is open and $x_0 \in \Omega$. For $x \in \Omega$ define

$$g_+(x) = \begin{cases} g(x) - g(x_0) & x > x_0 \\ 0 & x \le x_0 \end{cases} \tag{7}$$

and

$$g_-(x) = \begin{cases} g(x) - g(x_0) & x < x_0 \\ 0 & x \ge x_0 \end{cases} . \tag{8}$$

Let $\Omega^+ = \{x \in \Omega : x \ge x_0\}$ and $\Omega^- = \{x \in \Omega : x \le x_0\}$ Also define $g_R : \Omega^+ \to \mathbb{R}$ to be the restriction of g on Ω^+ and $g_L : \Omega^- \to \mathbb{R}$ to be the restriction of g on Ω^-. The proposition states that the local Hölder exponent of g is exactly the minimum of the exponents of g_+ and g_-. This will result from two basic lemmas. The first one is lemma 1.1 from [18]. In our notation, it reads:

Lemma 1 $\alpha_l(g, x_0, \rho) \ge \min\{\alpha_l(g_R, x_0, \rho), \alpha_l(g_L, x_0, \rho)\}$ $\forall \rho$ *s. t.* $B(x_0, \rho) \subset \Omega$.

Since this is true for all ρ, it implies that

$$\alpha_l(g, x_0) \ge \min\{\alpha_l(g_R, x_0), \alpha_l(g_L, x_0)\} \tag{9}$$

Lemma 2 $\alpha_l(g_+, x_0) = \alpha_l(g_R, x_0)$ *and* $\alpha_l(g_-, x_0) = \alpha_l(g_L, x_0)$.

Proof: We consider only the first case, i.e., $\alpha_l(g_+, x_0) = \alpha_l(g_R, x_0)$. The second follows similarly. The inequality

$$\sup \frac{|g_R(x) - g_R(y)|}{|x-y|^\alpha} \le \sup \frac{|g_+(x) - g_+(y)|}{|x-y|^\alpha}$$

holds because the supremum on the left is taken on a subinterval of the domain for the supremum on the right. This implies $\alpha_l(g_+, x_0) \le \alpha_l(g_R, x_0)$. For the reverse inequality, note that, for all x, y with $x < x_0, y > x_0$, we have that $g_+(x) = 0$, $g_+(y) = g(y) - g(x_0)$ and $|x - y| > |x_0 - y|$. As a consequence:

$$\frac{|g_+(x) - g_+(y)|}{|x - y|^\alpha} \leq \frac{|g(x_0) - g(y)|}{|x_0 - y|^\alpha}.$$

This in turn implies $\alpha_l(g_+, x_0) \geq \alpha_l(g_R, x_0)$.

Proposition 4 $\alpha_l(g, x_0) = \min\{\alpha_l(g_+, x_0), \alpha_l(g_-, x_0)\}$.

Proof:
From inequality 9 and lemma 2 it follows that $\alpha_l(g, x_0) \geq \min\{\alpha_l(g_+, x_0), \alpha_l(g_-, x_0)\}$. In order to prove the converse inequality, we prove $\alpha_l(g_+, x_0) \geq \alpha_l(g, x_0)$.

$$\begin{aligned}
\sup_{B(x_0,\rho)} \frac{|g_+(x) - g_+(y)|}{|x - y|^\alpha} &= \sup_{[x_0,x_0+\rho)} \frac{|g_+(x) - g_+(y)|}{|x - y|^\alpha} \\
&= \sup_{[x_0,x_0+\rho)} \frac{|g(x) - g(y)|}{|x - y|^\alpha} \\
&\leq \sup_{B(x_0,\rho)} \frac{|g(x) - g(y)|}{|x - y|^\alpha}.
\end{aligned}$$

This implies that $\alpha_l(g_+, x_0) \geq \alpha_l(g, x_0)$. Similarly, $\alpha_l(g_-, x_0) \geq \alpha_l(g, x_0)$, hence the result. ■

Theorem 1 *Let f be a continuous function in L^2. Then $q_c(f, x_0) = \alpha_l(f, x_0)$.*

Proof: Defining $f_\pm$ as in (7),(8), we write

$$\begin{aligned}
D^q_{x_0}(f(x) - f(x_0)) &= D^q_{x_0}(f_+ + f_-) \\
&= \begin{cases} D^q_{x_0+}(f_+(x) + f_-(x)), & x \geq x_0, \\ D^q_{x_0-}(f_+(x) + f_-(x)), & x \leq x_0. \end{cases} \\
&= \frac{1}{\Gamma(1-q)} \begin{cases} \frac{d}{dx} \int_{x_0}^x f_+(t)(x-t)^{-q} dt, & x \geq x_0, \\ -\frac{d}{dx} \int_x^{x_0} f_-(t)(t-x)^{-q} dt, & x \leq x_0, \end{cases} \\
&= \frac{1}{\Gamma(1-q)} \begin{cases} \frac{d}{dx} \int_{-\infty}^x f_+(t)(x-t)^{-q} dt, & x \geq x_0, \\ -\frac{d}{dx} \int_x^{\infty} f_-(t)(t-x)^{-q} dt, & x \leq x_0, \end{cases} \\
&\equiv \begin{cases} D^q_{-\infty} f_+ & x \geq x_0, \\ D^q_{+\infty} f_- & x \leq x_0, \end{cases}
\end{aligned}$$

From the definitions of f_+ and f_-, it is clear that $D^q_{-\infty} f_+ = 0$ when $x < x_0$ and $D^q_{+\infty} f_- = 0$ when $x > x_0$. Therefore, if we write $g(x) = D^q_{x_0}(f(x) - f(x_0))$, we have that $g_\pm$, as given by definitions (7) and (8), are also equal to $D^q_{\mp\infty} f_\pm$. We have thus replaced our derivatives by Weyl ones, for which we know that order q differentiation simply decreases the local Hölder exponent by q. Using Proposition 4,

$$\begin{aligned}\alpha_l(g,x_0) &= \min\{\alpha_l(g_+,x_0),\alpha_l(g_-,x_0)\}\\ &= \min\{\alpha_l(f_+,x_0)-q,\alpha_l(f_-,x_0)-q\}\\ &= \min\{\alpha_l(f_+,x_0),\alpha_l(f_-,x_0)\}-q\\ &= \alpha_l(f,x_0)-q.\end{aligned}$$

Thus the local Hölder exponent of the function $x \to D^q_{x_0}(f(x)-f(x_0))$ is non negative iff $q < \alpha_l(f,x_0)$. In consequence the critical value $q_c(x_0)$ such that the limit $\mathcal{D}^q f(x_0)$ exists and is finite for $q < q_c(x_0)$ but not for $q > q_c(x_0)$ is exactly $\alpha_l(f,x_0)$. ∎

4 Fractional Differentiation and 2 Microlocal Analysis

In this section, we provide a new interpretation of fractional derivative in terms of 2-microlocal analysis. This will allow in particular to understand the result of the previous section in a more transparent way. We first recall some basic facts about 2-microlocal analysis.

4.1 2-microlocal spaces

The definition of 2-microlocal spaces [1] is based on a Littlewood Paley analysis. A Littlewood Paley analysis is a spatially localized filter bank. One may also understand it as an intermediate between a discrete and a continuous wavelet analysis. More precisely, let $\mathcal{S}(\mathbb{R})$ be the Schwartz space and define:

$$\varphi \in \mathcal{S}(\mathbb{R}) = \begin{cases} \widehat{\varphi}(\xi) = 1, \parallel \xi \parallel < \frac{1}{2} \\ \widehat{\varphi}(\xi) = 0, \parallel \xi \parallel > 1. \end{cases}$$

and

$$\varphi_j(x) = 2^j \varphi(2^j x).$$

One has

$$\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi).$$

The $\{\varphi_j\}$ set acts as low pass filter bank, which leads naturally to the associated band pass filter bank:

$$\psi_j = \varphi_{j+1} - \varphi_j.$$

Definition 5 *Let $u \in \mathcal{S}'(\mathbb{R})$. The Littlewood Paley Analysis of u is the set of distributions:*

$$\begin{cases} S_0 u &= \varphi * u \\ \Delta_j u &= \psi_j * u \end{cases}$$

One has:

$$u = S_0 u + \sum_{j=0}^{\infty} \Delta_j u.$$

We can now define the two microlocal spaces $C_{x_0}^{s,s'}$.

Definition 6 *A distribution $u \in \mathcal{S}'(\mathbb{R})$ belongs to the 2-microlocal space $C_{x_0}^{s,s'}$ if there exists a positive constant c such that, for all j:*

$$\begin{cases} |S_0 u(x)| \leq c(1+|x-x_0|)^{-s'} \\ |\Delta_j u(x)| \leq c 2^{-js}\ (1+2^j|x-x_0|)^{-s'} \end{cases}$$

The 2-microlocal spaces are related to the pointwise Hölder spaces through:

Theorem 2 *[9] $\forall x_0 \in \mathbb{R}$, $\forall s > 0$:*

- $C_{x_0}^{s} \subset C_{x_0}^{s,-s}$
- $C_{x_0}^{s,s'} \subset C_{x_0}^{s}, \forall s + s' > 0$

For a given f, we may associate to each point x_0 its *2-microlocal domain*, i.e. the subset of $\mathbb{R} \times \mathbb{R}$ of couples (s, s') such that $f \in C_{x_0}^{s,s'}$. It is easy to show that $f \in C_{x_0}^{s,s'}$ implies that $f \in C_{x_0}^{s-\epsilon,s'+\epsilon}$ for all positive ϵ. This induces a particular shape for the frontier of the 2-microlocal domain:

Definition and Proposition 2 *2-microlocal frontier parameterization [7]*

Let $f : \mathbb{R} \to \mathbb{R}$, and

$$S(s', x) = \sup \left\{ s : f \in C_x^{s,s'} \right\}$$

The 2-microlocal frontier is the set of points

$$\Gamma(f, x_0) = \{(S(s'), s'))\}$$

The function $S(., x_0)$ is decreasing and convex. Moreover, one has, for all positive τ:

$$S(\sigma + \tau, x_0) \geq S(\sigma, x_0) - \tau$$

By slight abuse of notation, we shall call $S(\sigma, x_0) = S(\sigma)$ the 2-microlocal frontier. As said in section 2, the 2-microlocal spaces generalize the Hölder spaces and allow to re-interpret both α_l and α_p:

Proposition 5 *[7] For all x, we have :*

$$\alpha_l(x) = S(0, x)$$

and, provided $\sup_{\epsilon>0} S(\epsilon) > 0$,

$$\alpha_p(x) = \sigma_0(x)$$

where $\sigma_0(x)$ is the unique value for which

$$S(-\sigma_0, x) = \sigma_0$$

In other words, α_l is obtained as the intersection between the 2-microlocal frontier and the s-axis, while α_p is the intersection between the 2-microlocal frontier and the line $s' = -s$, provided $\sup_{\epsilon>0} S(\epsilon) > 0$. This last relation holds if f has some minimum overall regularity, i.e. for instance f belongs to the global Hölder space C^ω for some positive ω.

Finally, we mention the following crucial property of 2-microlocal spaces.

Proposition 6

$$f \in C_{x_0}^{s,s'} \quad \textit{iff} \quad \frac{df}{dx} \in C_{x_0}^{s-1,s'}$$

In fact, more is true, as pseudo-differential operators may be considered instead of plain differentials. We shall deal with a version of this result below.

4.2 Fractional Derivative as 2-microlocal frontier shifting

It is well-known that the Weyl fractional derivative, being a pseudo-differential operator, amounts to a horizontal translation of the 2-microlocal domain. In this section we show that this also holds under certain conditions for the Riemann-Liouville fractional derivative. This allows to understand the results of the previous section in a more general frame. We start by a Lemma (recall the definitions of $f_\pm$ from previous section).

Lemma 3 *Let f be a continuous nowhere differentiable function. Denote $S(\sigma)$ (resp. $S^+(\sigma)$, $S^-(\sigma)$) the frontier of the 2-microlocal domain of f (resp. f_+, f_-) at x. Then:*

$$\forall \sigma, \quad S(\sigma) = min(S^+(\sigma), S^-(\sigma))$$

Proof:

Since $f = f_+ + f_-$, we have that $S(\sigma) \geq min(S^+(\sigma), S^-(\sigma))$. For the reverse inequality, note that, for an arbitrary function g, cutting g into g_+ and g_- will at most introduce a discontinuity in the derivative of g. Since we are dealing here with a nowhere differentiable f, lumping together f_+ and f_- at x cannot increase the regularity.

■

Theorem 3 *Let f be a continuous nowhere differentiable function in $L^2(\mathbb{R})$. Then $f \in C_{x_0}^{s,s'}$ iff $D_{x_0}^q f(x) \in C_{x_0}^{s-q,s'}$.*

Proof: Write

$$D_{x_0}^q f(x) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} (I_+^{q-1} f_+ + I_-^{q-1} f_-), \tag{10}$$

where

$$I_+^{q-1} f_+ = \int_{-\infty}^{x} f_+(t)(x-t)^{-q} dt$$

and

$$I_-^{q-1} f_- = \int_{x}^{\infty} f_-(t)(t-x)^{-q} dt.$$

$I_\pm^{q-1}$ are by definition the Weyl fractional integral operators. It is well-known and easy to see that the Weyl fractional integral of order $q-1$ shifts the 2-microlocal frontier by $1-q$ towards the right along the s-axis. For a proof, note for instance that, for $g \in L^2$, $\widehat{I_+^{q-1} g}(\omega) = (i\omega)^{1-q} \hat{g}$ (see theorem 7.1 in [18]). As a consequence, $|\Delta_j I_+^{q-1} g| \leq c 2^{-j(q-1)} \, |\Delta_j g|$.

Since the derivative of first order shifts the frontier to the left by 1, we get that the operator $(d/dx) I_\pm^{q-1}$ shifts the frontier by q towards the left. From this it is clear that $f_\pm \in C_{x_0}^{s_\pm, s'_\pm}$ iff $(d/dx) I_\pm^q f_\pm(x) \in C_{x_0}^{s_\pm - q, s'_\pm}$. Hence the result follows from Lemma 3. ■

5 Examples

In this section we consider two examples and show that the critical order is equal to the local Hölder exponent in these cases. Of course, this is simply a consequence of theorem 1, but making the direct computation is enlightening and allows to understand more concretely the mechanisms of LFD.

5.1 Chirp-like triangle function

Our first example is the function defined by:

$$f(x) = \begin{cases} a_n x + b_n & \frac{1}{n} - \epsilon_n \le x \le \frac{1}{n} \\ -a_n x + c_n & \frac{1}{n} \le x \le \frac{1}{n} + \epsilon_n \\ 0 & \text{otherwise} \end{cases}, \tag{11}$$

where $n \in \mathbb{N}$, $\epsilon_n = n^{-\beta}$, $a_n = n^{\beta-\gamma}$ $b_n = n^{-\gamma} - n^{\beta-\gamma-1}$ and $c_n = n^{-\gamma} + n^{\beta-\gamma-1}$, with $\beta > 2$ and $\gamma \in (0,1)$. We have to evaluate

$$\frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^q} dt = \frac{1}{\Gamma(1-q)} \left(\frac{d}{dx} G(x) + \frac{d}{dx} H(x)\right)$$

where, for $\frac{1}{n+1} + (n+1)^{-\beta} < x \le \frac{1}{n} + n^{-\beta}$,

$$G(x) = \int_0^{\frac{1}{n+1}+(n+1)^{-\beta}} \frac{f(t)}{(x-t)^q} dt \tag{12}$$

and

$$H(x) = \begin{cases} \int_{\frac{1}{n}-n^{-\beta}}^{x} \frac{f(t)}{(x-t)^q} dt & \frac{1}{n} - n^{-\beta} < x \le \frac{1}{n} + n^{-\beta} \\ 0 & \frac{1}{n+1} + (n+1)^{-\beta} < x \le \frac{1}{n} - n^{-\beta} \end{cases} \tag{13}$$

Consider

$$G(x) = \sum_{j=n+1}^{\infty} \int_{\frac{1}{j}-j^{-\beta}}^{\frac{1}{j}+j^{-\beta}} \frac{f(t)}{(x-t)^q} dt.$$

$$\begin{aligned} \int_{\frac{1}{j}-j^{-\beta}}^{\frac{1}{j}+j^{-\beta}} \frac{f(t)}{(x-t)^q} dt &= \int_{\frac{1}{j}-j^{-\beta}}^{\frac{1}{j}} \frac{a_j t + b_j}{(x-t)^q} dt + \int_{\frac{1}{j}}^{\frac{1}{j}+j^{-\beta}} \frac{-a_j t + c_j}{(x-t)^q} dt \\ &= \frac{j^{\beta-\gamma}}{(1-q)(2-q)} \left((x - \frac{1}{j} + j^{-\beta})^{2-q} + (x - \frac{1}{j} - j^{-\beta})^{2-q} \right. \\ &\qquad \left. -2(x - \frac{1}{j})^{2-q}\right) \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{dG}{dx} &= \sum_{j=n+1}^{\infty} \frac{j^{\beta-\gamma}}{1-q} \left((x - \frac{1}{j} + j^{-\beta})^{1-q} + (x - \frac{1}{j} - j^{-\beta})^{1-q} - 2(x - \frac{1}{j})^{1-q}\right) \\ &= -q \sum_{j=n+1}^{\infty} j^{\beta-\gamma} (x - \frac{1}{j} + \eta j^{-\beta})^{-1-q} (\eta j^{-\beta})^2, \end{aligned} \tag{14}$$

where $\eta \in [-1,1]$ depends on x and j. As can be checked easily each term in the above sum is bounded by $j^{-\gamma-\beta+1+q}$ and therefore the sum converges uniformly and goes to zero with x at least when $q < \gamma+\beta-2$. In order to evaluate $H(x)$ we have to consider two cases: $1/n - n^{-\beta} \leq x \leq 1/n$ and $1/n \leq x < 1/n + n^{-\beta}$. In the first case we have

$$\begin{aligned} H(x) &= \int_{\frac{1}{n}-n^{-\beta}}^{x} \frac{a_n t + b_n}{(x-t)^q} dt \\ &= \frac{n^{\beta-\gamma}(x-\frac{1}{n}+n^{-\beta})^{2-q}}{(1-q)(2-q)}. \end{aligned}$$

Therefore in this case we have

$$\frac{dH}{dx} = \frac{n^{\beta-\gamma}(x-\frac{1}{n}+n^{-\beta})^{1-q}}{(1-q)}. \tag{15}$$

In the second case we get

$$\begin{aligned} H(x) &= \int_{\frac{1}{n}-n^{-\beta}}^{\frac{1}{n}} \frac{a_n t + b_n}{(x-t)^q} dt + \int_{\frac{1}{n}}^{x} \frac{-a_n t + c_n}{(x-t)^q} dt \\ &= \frac{n^{\beta-\gamma}}{(1-q)(2-q)} \left((x-\frac{1}{n}+n^{-\beta})^{2-q} - 2(x-\frac{1}{n})^{2-q} \right). \end{aligned}$$

In this case we have

$$\frac{dH}{dx} = \frac{n^{\beta-\gamma}}{(1-q)} \left((x-\frac{1}{n}+n^{-\beta})^{1-q} - 2(x-\frac{1}{n})^{1-q} \right). \tag{16}$$

Now we can substitute $x = 1/n + \eta n^{-\beta}$ ($\eta \in [-1,1]$) in equations (15) and (16). and check for the behavior as x approaches zero ($n \to \infty$). This shows that dH/dx is of the order of $n^{\beta q-\gamma}$, giving γ/β as a critical order.

On the other hand, it is not hard to prove that the pointwise exponent is γ, while the local one is γ/β, as expected.

5.2 IFS

Self-similar functions, as considered in [8], or Fractal Interpolations Functions (see [4]) provide a whole class of functions for which the equality between the degree of LFD and the local Hölder exponent is interesting to check. Contrarily to the case above, the graphs of such functions are, under some assumptions, nowhere differentiable, and possess a multifractal structure. Without entering into details, let us recall that their pointwise Hölder exponent varies discontinuously everywhere, while α_l is constant with, for all x, $\alpha_l(x) = \min_y \alpha_p(y)$. Furthermore, the level sets of the function $x \to \alpha_p(x)$ are all dense or empty, so that, in every neighbourhood of any x, there is a y where $\alpha_p(y) = \alpha_l$. This is precisely the mechanism that makes q_c and α_l coincide in this case: indeed, simple but tedious computations show that, at each point x, q_c is obtained as the lim inf of $\alpha_p(y)$ when y tends to x.

6 Conclusion

We have elucidated the meaning of the degree of local fractional differentiability as an equivalent to the local Hölder exponent, and given an interpretation of fractional differentiation in terms of 2-microlocal analysis. This provides yet another link between the fields of fractional integration and Hölder regularity analysis (see also [19]). An easy extension of our work is to the case where the order of differentiation is larger than one. Finally, this paper has dealt exclusively with theoretical considerations. In [13], we study the question of estimating some local regularity measures on sample data.

We thank the referees for pointing out some references and improving the manuscript.

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