# Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains

**Moshe Marcus**
Department of Mathematics,
Technion, Haifa, ISRAEL

**Laurent Veron**
Department of Mathematics,
Univ. of Tours, FRANCE

**Abstract**

We study the generalized boundary value problem for nonnegative solutions of of $-\Delta u + g(u) = 0$ in a bounded Lipschitz domain $\Omega$, when $g$ is continuous and nondecreasing. Using the harmonic measure of $\Omega$, we define a trace in the class of outer regular Borel measures. We amphasize the case where $g(u) = |u|^{q-1}u$, $q > 1$. When $\Omega$ is (locally) a cone with vertex $y$, we prove sharp results of removability and characterization of singular behavior. In the general case, assuming that $\Omega$ possesses a tangent cone at every boundary point and $q$ is subcritical, we prove an existence and uniqueness result for positive solutions with arbitrary boundary trace. We obtain sharp results involving Besov spaces with negative index on k-dimensional edges and apply our results to the characterization of removable sets and good measures on the boundary of a polyhedron.

*1991 Mathematics Subject Classification.* 35K60; 31A20; 31C15; 44A25; 46E35.

*Key words.* Laplacian; Poisson potential; harmonic measure; singularities; Borel measures; Bessel capacities; Harnack inequalities; Besov spaces; singular integrals.

# Contents



# 1 Introduction

In this article we study boundary value problems with measure data on the boundary, for equations of the form

$$-\Delta u + g(u) = 0 \ \text{ in } \ \Omega \tag{1.1}$$

where $\Omega$ is a bounded *Lipschitz domain* in $\mathbb{R}^N$ and $g$ is a continuous nondecreasing function vanishing at 0. A function $u$ is a solution of the equation if $u$ and $g(u)$ belong to $L^1_{\text{loc}}(\Omega)$ and the equation holds in the distribution sense. The definition of a solution satisfying a prescribed boundary condition is more complex and will be described later on.

Boundary value problems for (1.1) with measure boundary data in smooth domains (or, more precisely, in $C^2$ domains) have been studied intensively in the last 20 years. Much of this work concentrated on the case of power nonlinearities, namely, $g(u) = |u|^{q-1}u$ with $q > 1$. For details we address the reader to the following papers and the references therein: Le Gall [1-2], Dynkin and Kuznetsov [1-3], Mselati [1] (employing in an essential way probabilistic tools) and Marcus and Veron [1-4] (employing purely analytic methods).

The study of the corresponding linear boundary value problem in Lipschitz domains is classical. This study shows that, with a proper interpretation, the basic results known for smooth domains remain valid in the Lipschitz case. Of course there are important differences too: in the Poisson integral formula the Poisson kernel must be replaced by the Martin kernel and, when the boundary data is given by a function in $L^1$, the standard surface measure must be replaced by the harmonic measure. The Hopf principle does not hold anymore, but it is partially replaced by the Carleson lemma and the boundary Harnack principle due to Dahlberg [7]. A summary of the basic results for the linear case, to the extent needed in the present work, is presented in Section 2.

One might expect that in the nonlinear case the results valid for smooth domains extend to Lipschitz domains in a similar way. This is indeed the case as long as the boundary data is in $L^1$. However, in problems with measure boundary data, we encounter essentially new phenomena.

Following is an overview of our main results on boundary value problems for (1.1).

**A.** *General nonlinearity and finite measure data.*

We start with the weak $L^1$ formulation of the boundary value problem

$$-\Delta u + g(u) = 0 \text{ in } \Omega, \;\; u = \mu \text{ on } \partial\Omega \;\;, \tag{1.2}$$

where $\mu \in \mathfrak{M}(\partial\Omega)$.

Let $x_0$ be a point in $\Omega$, to be kept fixed, and let $\rho = \rho_\Omega$ denote the first eigenfunction of $-\Delta$ in $\Omega$ normalized by $\rho(x_0) = 1$. It turns out that the family of test functions appropriate for the boundary value problem is

$$X(\Omega) = \left\{ \eta \in W_0^{1,2}(\Omega) : \rho^{-1}\Delta\eta \in L^\infty(\Omega) \right\}. \tag{1.3}$$

If $\eta \in \Omega$ then $\sup |\eta|/\rho < \infty$.

Let $\mathbb{K}[\mu]$ denote the harmonic function in $\Omega$ with boundary trace $\mu$. Then $u$ is an $L^1$-*weak solution* of (1.2) if

$$u \in L^1_\rho(\Omega), \quad g(u) \in L^1_\rho(\Omega) \tag{1.4}$$

and

$$\int_\Omega (-u\Delta\eta + g(u)\eta)\, dx = -\int_\Omega (\mathbb{K}[\mu]\Delta\eta)\, dx \quad \forall \eta \in X(\Omega). \tag{1.5}$$

Note that in (1.5) the boundary data appears only in an implicit form. In the next result we present a more explicit link between the solution and its boundary trace.

A sequence of domains $\{\Omega_n\}$ is called a *Lipschitz exhaustion* of $\Omega$ if, for every $n$, $\Omega_n$ is Lipschitz and

$$\Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega = \cup \Omega_n, \quad \mathbb{H}_{N-1}(\partial\Omega_n) \to \mathbb{H}_{N-1}(\partial\Omega). \tag{1.6}$$

**Proposition 1.1** *Let $\{\Omega_n\}$ be an exhaustion of $\Omega$, let $x_0 \in \Omega_1$ and denote by $\omega_n$ (respectively $\omega$) the harmonic measure on $\partial\Omega_n$ (respectively $\partial\Omega$) relative to $x_0$. If $u$ is an $L^1$-weak solution of* (1.2) *then, for every $Z \in C(\bar{\Omega})$,*

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Zu\, d\omega_n = \int_{\partial\Omega} Z\, d\mu. \tag{1.7}$$

We note that any solution of (1.1) is in $W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p > 1$ and consequently possesses an integrable trace on $\partial\Omega_n$.

In general problem (1.2) does not possess a solution for every $\mu$. We denote by $\mathfrak{M}^g(\partial\Omega)$ the set of measures $\mu \in \mathfrak{M}(\partial\Omega)$ for which such a solution exists. The following statements are established in the same way as in the case of smooth domains:

(i) If a solution exists it is unique. Furthermore the solution depends monotonically on the boundary data.
(ii) If $u$ is an $L^1$-weak solution of (1.2) then $|u|$ (resp. $u_+$) is a subsolution of this problem with $\mu$ replaced by $|\mu|$ (resp. $\mu_+$).

A measure $\mu \in \mathfrak{M}(\partial\Omega)$ is *$g$-admissible* if $g(\mathbb{K}[|\mu|]) \in L^1_\rho(\Omega)$. When there is no risk of confusion we shall simply write 'admissible' instead of '$g$-admissible'. The following provides a sufficient condition for existence.

**Theorem 1.2** *If $\mu$ is $g$-admissible then problem* (1.2) *possesses a unique solution.*

**B.** *The boundary trace of positive solutions of* (1.1)*; general nonlinearity.*

We say that $u \in L^1_{\mathrm{loc}}(\Omega)$ is a *regular solution* of the equation (1.1) if $g(u) \in L^1_\rho(\Omega)$.

**Proposition 1.3** *Let $u$ be a positive solution of the equation* (1.1)*. If $u$ is regular then $u \in L^1_\rho(\Omega)$ and it possesses a boundary trace $\mu \in \mathfrak{M}(\partial\Omega)$. Thus $u$ is the solution of the boundary value problem* (1.2) *with this measure $\mu$.*

As in the case of smooth domains, a positive solution possesses a boundary trace even if the solution is not regular. The boundary trace may be defined in several ways; in every case it is expressed by an *unbounded measure.* A definition of trace is 'good' if the trace uniquely determines the solution. A discussion of the various definitions of boundary trace, for boundary value problems in $C^2$ domains, with power nonlinearities, can be found in [27], [8] and the references therein. In [22] the authors introduced a definition of trace – later referred to as the 'rough trace' by Dynkin [8] – which proved to be 'good' in the subcritical case, but not in the supercritical case (see [23]). Mselati [28] obtained a 'good' definition of trace for the problem with $g(u) = u^2$ and $N \geq 4$, in which case this non-linearity is supercritical. His approach employed probabilistic methods developed by Le Gall in a series of papers. For a presentation of these methods we refer the reader to his book [21]. Following this work the authors introduced in [27] a notion of trace, called 'the precise trace', defined in the framework of the fine topology associated with the capacity $C_{2/q,q'}$ on $\partial\Omega$. This definition of trace turned out to be 'good' for all power nonlinearities $g(u) = u^q$, $q > 1$, at least in the class of $\sigma$-moderate solutions. In the subcritical case, the precise trace reduces to the rough trace. At the same time Dynkin [9] extended Mselati's result to the case $(N+1)/(N-1) \leq q \leq 2$.

In the present paper we confine ourselves to boundary value problems with rough trace data. (See the definition below.) However we develop a framework for the study of existence and uniqueness (see Theorem 1.10 below) which can be applied to a large class of nonlinearities and can be adapted to other notions of trace as well. In particular, it can be adapted to the 'precise trace' for power nonlinearities (in smooth domains) and to a related notion of trace for Lipschitz domains. This issue will be addressed in a subsequent paper.

Here are the main results in this part of the paper, including the relevant definitions.

**Definition 1.4** *Let $u$ be a positive supersolution, respectively subsolution, of* (1.1). *A point $y \in \partial\Omega$ is a* regular boundary point *relative to $u$ if there exists an open neighborhood $D$ of $y$ such that $g \circ u \in L^1_\rho(\Omega \cap D)$. If no such neighborhood exists we say that $y$ is a* singular boundary point *relative to $u$.*

*The set of regular boundary points of $u$ is denoted by $\mathcal{R}(u)$; its complement on the boundary is denoted by $\mathcal{S}(u)$. Evidently $\mathcal{R}(u)$ is relatively open.*

**Theorem 1.5** *Let $u$ be a positive solution of* (1.1) *in $\Omega$. Then $u$ possesses a trace on $\mathcal{R}(u)$, given by a Radon measure $\nu$.*

*Furthermore, for every compact set $F \subset \mathcal{R}(u)$,*

$$\int_\Omega (-u\Delta\eta + g(u)\eta)\,dx = -\int_\Omega (\mathbb{K}[\nu\chi_F]\Delta\eta)\,dx \tag{1.8}$$

*for every $\eta \in X(\Omega)$ such that $\operatorname{supp}\eta \cap \partial\Omega \subset F$ and $\nu\chi_F \in \mathfrak{M}^g(\partial\Omega)$.*

**Definition 1.6** *Let $g \in \mathcal{G}$. Let $u$ be a positive solution of* (1.1) *with regular boundary set $\mathcal{R}(u)$ and singular boundary set $\mathcal{S}(u)$. The Radon measure $\nu$ in $\mathcal{R}(u)$ associated with $u$ as in Theorem 1.5 is called the* regular part of the trace of $u$. *The couple $(\nu, \mathcal{S}(u))$ is called the* boundary trace *of $u$ on $\partial\Omega$. This trace is also represented by the (possibly unbounded) Borel measure $\bar\nu$ given by*

$$\bar\nu(E) = \begin{cases} \nu(E), & \text{if } E \subset \mathcal{R}(u) \\ \infty, & \text{otherwise.} \end{cases} \tag{1.9}$$

*The boundary trace of $u$ in the sense of this definition will be denoted by* $\mathrm{tr}_{\partial\Omega} u$.

*Let*

$$V_\nu := \sup\{u_{\nu\chi_F} :\ F \subset \mathcal{R}(u),\ F \text{ compact}\} \tag{1.10}$$

*where $u_{\nu\chi_F}$ denotes the solution of* (1.2) *with $\mu = \nu\chi_F$. Then $V_\nu$ is called the* semi-regular component *of $u$.*

**Definition 1.7** *A compact set $F \subset \partial\Omega$ is* removable *relative to* (1.1) *if the only non-negative solution $u \in C(\bar\Omega \setminus F)$ which vanishes on $\bar\Omega \setminus F$ is the trivial solution $u = 0$.*

**Lemma 1.8** *Let $g \in \mathcal{G}$ and assume that $g$ satisfies the Keller-Osserman condition. Let $F \subset \partial\Omega$ be a compact set and denote by $\mathcal{U}_F$ the class of solutions $u$ of* (1.1) *which satisfy the condition,*

$$u \in C(\bar\Omega \setminus F), \quad u = 0 \text{ on } \partial\Omega \setminus F. \tag{1.11}$$

*Then there exists a function* $U_F \in \mathcal{U}_F$ *such that*

$$u \leq U_F \quad \forall u \in \mathcal{U}_F.$$

*Furthermore,* $\mathcal{S}(U_F) =: F' \subset F$*;* $F'$ *need not be equal to* $F$*.*

**Definition 1.9** $U_F$ *is called the* maximal solution *associated with* $F$*. The set* $F' = \mathcal{S}(U_F)$ *is called the* $g$*-kernel of* $F$ *and denoted by* $k_g(F)$*.*

**Theorem 1.10** *Let* $g \in \mathcal{G}$ *and assume that* $g$ *is convex and satisfies the Keller-Osserman condition.*
Existence. *The following set of conditions is necessary and sufficient for existence of a solution* $u$ *of the generalized boundary value problem*

$$-\Delta u + g(u) = 0 \ \ in \ \ \Omega, \quad \mathrm{tr}_{\partial\Omega} u = (\nu, F), \tag{1.12}$$

*where* $F \subset \partial\Omega$ *is a compact set and* $\nu$ *is a Radon measure on* $\partial\Omega \setminus F$*.*
*(i) For every compact set* $E \subset \partial\Omega \setminus F$*,* $\nu\chi_E \in \mathfrak{M}^g(\partial\Omega)$*.*
*(ii) If* $k_g(F) = F'$*, then* $F \setminus F' \subset \mathcal{S}(V_\nu)$*.*
*When this holds,*

$$V_\nu \leq u \leq V_\nu + U_F. \tag{1.13}$$

*Furthermore if* $F$ *is a removable set then* (1.2) *possesses exactly one solution.*

Uniqueness. *Given a compact set* $F \subset \partial\Omega$*, assume that*

$$U_E \text{ is the unique solution with trace } (0, k_g(E)) \tag{1.14}$$

*for every compact* $E \subset F$*. Under this assumption:*
(a) *If* $u$ *is a solution of* (1.12) *then*

$$\max(V_\nu, U_F) \leq u \leq V_\nu + U_F. \tag{1.15}$$

(b) *Equation* (1.1) *possesses at most one solution satisfying* (1.15)*.*
(c) *Condition* (1.14) *is necessary and sufficient in order that* (1.12) *possess at most one solution.*

Monotonicity.
(d) *Let* $u_1, u_2$ *be two positive solutions of* (1.1) *with boundary traces* $(\nu_1, F_1)$ *and* $(\nu_2, F_2)$ *respectively. Suppose that* $F_1 \subset F_2$ *and that* $\nu_1 \leq \nu_2\chi_{F_1} =: \nu_2'$*. If* (1.14) *holds for* $F = F_2$ *then* $u_1 \leq u_2$*.*

In the remaining part of this paper we consider equation (1.1) with power nonlinearity:

$$- \Delta u + |u|^{q-1}u = 0 \tag{1.16}$$

with $q > 1$.

**C.** *Classification of positive solutions in a conical domain possessing an isolated singularity at the vertex.*

Let $C_S$ be a cone with vertex 0 and opening $S \subset S^{N-1}$, where $S$ is a Lipschitz domain. Put $\Omega = C_S \cap B_1(0)$. Denote by $\lambda_S$ the first eigenvalue and by $\phi_S$ the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S)$ normalized by $\max \phi_S = 1$. Put

$$\alpha_S = \frac{1}{2}(N - 2 + \sqrt{(N-2)^2 + 4\lambda_S}$$

and

$$\Phi_1 = \frac{1}{\gamma} x^{-\alpha_S} \phi_S(x/|x|)$$

where $\gamma_S$ is a positive number. $\Phi_1$ is a harmonic function in $C_S$ vanishing on $\partial C_S \setminus \{0\}$ and $\gamma$ is chosen so that the boundary trace of $\Phi_1$ is $\delta_0$ (=Dirac measure on $\partial C_S$ with mass 1 at the origin). Further denote $\Omega_S = C_S \cap B_1(0)$.

It was shown in [11] that, if $q \geq 1 + \frac{2}{\alpha_S}$ there is no solution of (1.16) in $\Omega$ with isolated singularity at 0. We obtain the following result.

**Theorem 1.11** *Assume that $1 < q < 1 + \frac{2}{\alpha_S}$. Then $\delta_0$ is admissible for $\Omega$ and consequently, for every real $k$, there exists a unique solution of this equation in $\Omega$ with boundary trace $k\delta_0$. This solution, denoted by $u_k$ satisfies*

$$u_k(x) = k\Phi_1(x)(1 + o(1)) \quad \text{as } x \to 0. \tag{1.17}$$

*The function*

$$u_\infty = \lim_{k\to\infty} u_k$$

*is the unique positive solution of (5.1) in $\Omega_S$ which vanishes on $\partial\Omega \setminus \{0\}$ and is strongly singular at 0, i.e.,*

$$\int_\Omega u_\infty^q \rho \, dx = \infty \tag{1.18}$$

*where $\rho$ is the first eigenfunction of $-\Delta$ in $\Omega$ normalized by $\rho(x_0) = 1$ for some (fixed) $x_0 \in \Omega$. Its asymptotic behavior at 0 is given by,*

$$u_\infty(x) = |x|^{-\frac{2}{q-1}} \omega_S(x/|x|)(1 + o(1)) \qquad \text{as } x \to 0 \tag{1.19}$$

*where $\omega$ is the (unique) positive solution of*

$$-\Delta'\omega - \lambda_{N,q}\omega + |\omega|^{q-1}\omega = 0 \tag{1.20}$$

*on $S^{N-1}$ with*

$$\lambda_{N,q} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right). \tag{1.21}$$

As a consequence one can state the following classification result.

**Theorem 1.12** *Assume that $1 < q < q_S = 1 + 2/\alpha_S$ and denote*

$$\tilde{\alpha}_S = \frac{1}{2}\big(2 - N + \sqrt{(N-2)^2 + 4\lambda_S}\,\big).$$

*If $u \in C(\bar{\Omega}_S \setminus \{0\})$ is a positive solution of (1.16) vanishing on $(\partial C_S \cap B_{r_0}(0)) \setminus \{0\}$, the following alternative holds:*

*Either*

$$\limsup_{x\to 0} |x|^{-\tilde{\alpha}_S}\, u(x) < \infty,$$

*or*

*there exists $k > 0$ such that (1.17) holds,*

*or*

*(1.19) holds.*

In the first case $u \in C(\bar{\Omega}$; in the second, $u$ possesses a *weak singularity* at the vertex while in the last case $u$ has a *strong singularity* there.

**D.** *Criticality in Lipschitz domains.*

Let $\Omega$ be a Lipschitz domain and let $\xi \in \partial\Omega$. We say that $q_\xi$ is the *critical value for* (1.16) *at* $\xi$ if, for $1 < q < q_\xi$, the equation possesses a solution with boundary trace $\delta_\xi$ while, for $q > q_\xi$ no such solution exists. We say that $q^\sharp_\xi$ is the *secondary critical value at* $\xi$ if for $1 < q < q^\sharp_\xi$ there exists a non-trivial solution of (1.16) which vanishes on $\partial\Omega \setminus \{\xi\}$ but for $q > q^\sharp_\xi$ no such solution exists.

In the case of smooth domains, $q_\xi = q^\sharp_\xi$ and $q_\xi = (N+1)/(N-1)$ for every boundary point $\xi$. Furthermore, if $q = q_\xi$ there is no solution with isolated singularity at $\xi$, i.e., an isolated singularity at $\xi$ is removable.

In Lipschitz domains *the critical value depends on the point.* Clearly $q_\xi \leq q^\sharp_\xi$, but the question whether, in general, $q_\xi = q^\sharp_\xi$ remains open. However we prove that, if $\Omega$ is a polyhedron, $q_\xi = q^\sharp_\xi$ at every point and the function

$\xi \longrightarrow q_\xi$ obtains only a finite number of values. In fact it is constant on each open face and each open edge, of any dimension. In addition, if $q = q_\xi$, an isolated singularity at $\xi$ is removable. The same holds true in a piecewise $C^2$ domain $\Omega$ except that $\xi \longrightarrow q_\xi$ is not constant on edges but it is continuous on every relatively open edge.

For general Lipschitz domains, we can provide only a partial answer to the question posed above.

We say that $\Omega$ possesses a *tangent cone* at a point $\xi \in \partial\Omega$ if the limiting inner cone with vertex at $\xi$ is the same as the limiting outer cone at $\xi$.

**Theorem 1.13** *Suppose that $\Omega$ possesses a tangent cone $C_\xi^\Omega$ at a point $\xi \in \partial\Omega$ and denote by $q_{c,\xi}$ the critical value for this cone at the vertex $\xi$. Then*

$$q_\xi = q_\xi^\sharp = q_{c,\xi}.$$

*Furthermore, if $1 < q < q_\xi$ then $\delta_\xi$ is admissible, i.e.,*

$$M_\xi := \int_\Omega K(x,\xi)^q \rho(x)dx < \infty.$$

We do not know if, under the assumptions of this theorem, an isolated singularity at $\xi$ is removable when $q = q_{c,\xi}$. It would be useful to resolve this question.

**E.** *The generalized boundary value problem in Lipschitz domains: the subcritical case.*

In the case of smooth domains, a boundary value problem for equation (1.16) is either subcritical or supercritical. This is no longer the case when the domain is merely Lipschitz since the criticality varies from point to point. In this part of the paper we discuss the generalized boundary value problem in the strictly subcritical case. Later we discuss the mixed case (partly subcritical and partly supercritical) when $\Omega$ is a polyhedron and the boundary data is given by a bounded measure.

Under the conditions of Theorem 1.13 we know that, if $\xi \in \partial\Omega$ and $1 < q < q_\xi$ then $K(\cdot,\xi) \in L^1_\rho(\Omega)$. In the next result, we derive, under an additional restriction on $q$, *uniform* estimates of the norm $\|K(\cdot,\xi)\|_{L^1_\rho(\Omega)}$. Such estimates are needed in the study of existence and uniqueness. For its statement we need the following notation:

If $z \in \partial\Omega$, we denote by $S_{z,r}$ the opening of the largest cone $C_S$ with vertex at $z$ such that $C_S \cap B_r(z) \subset \Omega \cup \{z\}$. If $E$ is a compact subset of $\partial\Omega$ we denote:

$$q_E^* = \lim_{r\to 0} \inf \left\{ q_{S_{z,r}} : z \in \partial\Omega,\ \mathrm{dist}\,(z,E) < r \right\}.$$

We observe that

$$q_E^* \leq \inf\{q_{c,z} : z \in E\}$$

but this number also measures, in a sense, the rate of convergence of interior cones to the limiting cones. If $\Omega$ is convex then $q_E^* \leq (N+1)/(N-1)$ for every non-empty set $E$. On the other hand if $\Omega$ is the *complement* of a bounded convex set then $q_E^* = (N+1)/(N-1)$.

**Theorem 1.14** *If $E$ is a compact subset of $\partial\Omega$ and $1 < q < q_E^*$ then, there exists $M > 0$ such that,*

$$\int_\Omega K^q(x,y)\rho(x)dx \leq M \quad \forall y \in E. \tag{1.22}$$

Using this theorem we obtain,

**Theorem 1.15** *Assume that $\Omega$ is a bounded Lipschitz domain and $u$ is a positive solution of* (1.16). *If $y \in \mathcal{S}(u)$ (i.e. $y \in \partial\Omega$ is a singular point of $u$) and $1 < q < q_{\{y\}}^*$ then, for every $k > 0$, the measure $k\delta_y$ is admissible and*

$$u \geq u_{k\delta_y} = \text{ solution with boundary trace } k\delta_y. \tag{1.23}$$

*Remark.* It can be shown that, if $q > q_{\{y\}}^*$, (1.23) may not hold. For instance, such solutions exist if $\Omega$ is a smooth, obtuse cone and $y$ is the vertex of the cone. Therefore the condition $q < q_{\{y\}}^*$ for every $y \in \partial\Omega$ is, in some sense necessary for uniqueness in the subcritical case.

As a consequence we first obtain the existence and uniqueness result in the context of bounded measures.

**Theorem 1.16** *Let $E \subset \partial\Omega$ be a closed set and assume that $1 < q < q_E^*$. Then, for every $\mu \in \mathfrak{M}(\Omega)$ such that $\operatorname{supp}\mu \subset E$ there exists a (unique) solution $u_\mu$ of* (5.1) *in $\Omega$ with boundary trace $\mu$.*

Further, using Theorems 1.10, 1.11 and 1.14, we establish the existence and uniqueness result for generalized boundary value problems.

**Theorem 1.17** *Let $\Omega$ be a bounded Lipschitz domain which possesses a tangent cone at every boundary point. If*

$$1 < q < q_{\partial\Omega}^*$$

*then, for every positive, outer regular Borel measure $\bar\nu$ on $\partial\Omega$, there exists a unique solution $u$ of (1.16) such that $tr_{\partial\Omega}(u) = \bar\nu$.*

**F.** *On the action of Poisson type kernels with fractional dimension.*

In preparation for the study of supercritical boundary value problems we establish an harmonic analytic result, extending a well known result on the action of Poisson kernels on Besov spaces with negative index (see [31, 1.14.4.] and [26]). We first quote the classical result for comparison purposes.

**Proposition 1.18** *Let $1 < q < \infty$ and $s > 0$. Then, for any bounded Radon measure $\mu$ in $\mathbb{R}^{n-1}$,*

$$I(\mu) = \int_{\mathbb{R}^n_+} |\mathbb{K}_n[\mu](y)|^q \, e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|^q_{B^{-s,q}(\mathbb{R}^{n-1})} \,. \tag{1.24}$$

Here $\mathbb{K}_n[\mu]$ denotes the Poisson potential of $\mu$ in $\mathbb{R}^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, namely,

$$\mathbb{K}_n[\mu](y) = \gamma_n y_1 \int_{\mathbb{R}^{n-1}} \frac{d\mu(z)}{\left(y_1^2 + |\zeta - z|^2\right)^{n/2}} \quad \forall y = (y_1, \zeta) \in \mathbb{R}^n_+ \tag{1.25}$$

where $\gamma_n$ is a constant depending only on $n$. The notation $I \approx J$ means that $c^{-1} I \leq J \leq cI$ for some $c > 0$.

In this paper we prove,

**Theorem 1.19** *Let $1 < q$, $m$ a positive integer and $\nu \in \mathbb{R}$ such that $m+1 \leq \nu$. For every $s \in (0, m/q')$ there exists a positive constant $c$ such that, for every bounded positive measure $\mu$ supported in $\mathbb{R}^m \cap B_{R/2}(0)$, $R > 1$,*

$$\begin{aligned} \frac{1}{c} \|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} &\leq \int_0^R \Big( \int_{|\zeta|<R} |\mathbb{K}_{\nu,m}[\mu](\tau,\zeta)|^q \, d\zeta \Big) \tau^{sq-1} \, d\tau \\ &\leq cR^{(s+\nu-m)q+1} \|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} \,. \end{aligned} \tag{1.26}$$

*Here*

$$\mathbb{K}_{\nu,m}[\mu](\tau,\zeta) = \int_{\mathbb{R}^m} \frac{\tau^{\nu-m} d\mu(z)}{(\tau^2 + |\zeta - z|^2)^{\nu/2}} \quad \forall \tau \in [0, \infty). \tag{1.27}$$

*This also holds when $s = m/q'$, provided that the diameter of* $\operatorname{supp}\mu$ *is sufficiently small.*

This is proved in Section 7 (see Theorem 7.8) using a slightly different notation. Note that

$$\mathbb{K}_n[\mu] = \gamma_n \mathbb{K}_{n,n-1}[\mu].$$

**G.** *The admissibility condition and the critical value in a k-wedge.*

The next step towards the study of boundary value problems in a polyhedron is the treatment of such problems in a k-wedge (or k-dihedron) i.e., the domain defined by the intersection of $k$ hyperplanes in $\mathbb{R}^N$, $1 < k < N$. The edge is an $(N-k)$ dimensional space. We note that the case $k = N$ (which corresponds to a cone) has been studied previously in this paper while the case $N = 1$ (i.e. a half space) is classical.

We denote by $D_A$ a k-wedge such that, its edge $d_A$ is identified with $\mathbb{R}^{N-k}$ and the 'opening' of the wedge is $A = D_A \cap S^{k-1}$. If $S_A$ denotes the spherical domain

$$S_A = \{x \in \mathbb{R}^N : |x| = 1,\, x \in A \times \prod_{j=k}^{N-1} [0, \pi]\} \subset S^{N-1}\} \tag{1.28}$$

then

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}, \quad D_{A,R} = D_A \cap \Gamma_R$$

where

$$\Gamma_R = \{x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k} : \, |x'| < r,\; |x''| < R\}.$$

Let $\lambda_A$ be the first eigenvalue of $-\Delta_{_{S^{N-1}}}$ in $W_0^{1,2}(S_A)$ and denote

$$\begin{aligned} \kappa_+ &= \frac{1}{2}\left(2 - N + \sqrt{(N-2)^2 + 4\lambda_A}\right) \\ \kappa_- &= \frac{1}{2}\left(2 - N - \sqrt{(N-2)^2 + 4\lambda_A}\right). \end{aligned} \tag{1.29}$$

One can show that the Martin kernel $K_A$ in $D_A$ relative to points $z \in d_A$ is given by

$$K_A(x, z) = c_{_A} \frac{|x'|^{\kappa_+} \omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x - z|^{(N-2+2\kappa_+)}}, \tag{1.30}$$

where $\omega^{\{N-k+1\}}$ is a related eigenfunction in $A$ and $x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Using this formula we obtain the admissibility condition for a measure $\mu \in \mathfrak{M}(d_A)$ such that $\operatorname{supp} \mu \subset B_R(0)$:

$$\int_{\Gamma_R} \Big( \int_{\mathbb{R}^{N-k}} \frac{|x'|^{\kappa_+} d|\mu|(z)}{(|x'|^2 + |x''|^2)^{(N-2+2\kappa_+)/2}} \Big)^q |x'|^{\kappa_+} dx < \infty \tag{1.31}$$

where $\Gamma_R := \{x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k} : |x'| < R, |x''| < R\}$.

Using this expression we show that the condition

$$1 < q < q_c := \frac{\kappa_+ + N}{\kappa_+ + N - 2}. \tag{1.32}$$

is necessary and sufficient in order that the Dirac measure $\mu = \delta_P$, supported at a point $P \in d_A$, be admissible.

In addition we show that the condition

$$1 < q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+} \tag{1.33}$$

is necessary and sufficient for the existence of a non-trivial solution $u$ of (1.16) in $D_A$ which vanishes on $\partial D_A \setminus d_A$. Furthermore, when this condition holds, there exist admissible non-trivial positive *bounded measures* $\mu$ on $d_A$, i.e., measures such that $\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A)$.

Finally we have the following removability result:

**Theorem 1.20** *Assume that $q_c \leq q < q_c^*$. A measure $\mu \in \mathfrak{M}(\partial D_A)$, with compact support contained in $d_A$, is good relative to* (1.16) *in $D_A$ if and only if $\mu$ vanishes on every Borel set $E \subset d_A$ such that $C^{N-k}_{s,q'}(E) = 0$, where $s = 2 - \frac{k+\kappa_+}{q'}$ and $C^{N-k}_{s,q'}$ is the Bessel capacity with the indicated indices in $\mathbb{R}^{N-k}$ (which we identify with the edge $d_A$).*

Recall that $\mu$ is 'good' if the specified equation possesses a solution with boundary data $\mu$. The above result implies in particular that sets with $C^{N-k}_{s,q'}$-capacity zero are conditionally removable. However we obtain a much stronger result later on.

**H.** *Boundary value problems in a polyhedron: the supercritical case.*

In the final part of the paper (Sections 8) we study boundary value problems in the supercritical case in polyhedrons, with trace given by bounded measures. For such domains $\Omega$ we provide a complete characterization of 'good measures', i.e., measures $\mu$ on $\partial\Omega$ such that (1.16) possesses a (unique) solution with boundary trace $\mu$. We also provide a complete characterization of *removable sets.* These results, with rather obvious modifications, also apply to piecewise $C^2$ domains. The case of general Lipschitz domains and boundary trace given by unbounded Borel measures will be treated in a subsequent paper.

**Theorem 1.21** *Let $\Omega$ be an $N$-dimensional polyhedron. Let $L$ denote one of the faces, or edges, or vertices of $\Omega$ and let $Q_L$ denote the half space with* *boundary $L$, or the wedge with edge $L$, or the cone with vertex $L$ such that $\Omega \subset D_L$ and $\partial Q_L$ is determined by the faces of $\Omega$ adjacent to $L$. Thus $\partial\Omega$ is the union of the sets $\partial\Omega \cap \partial Q_L$.*

*Denote by $A_L$ the opening of $Q_L$ so that, in the notation of* **G**, *$Q_L = D_{A_L}$ and denote by $\kappa_+(L)$, $q_c(L)$ etc. the various notations introduced in* **G** *relative to $A_L$. In particular let $k(L)$ denote the co-dimension of the linear space spanned by $L$ and put*

$$s(L) = 2 - \frac{k(L) + \kappa_+(L)}{q'}.$$

*Let $\mu$ be a bounded measure on $\partial\Omega$, (possibly a signed measure). Then $\mu$ is a good measure relative to (1.16) in $\Omega$, if and only if, for every $L$ as above and every Borel set $E \subset L$ the following condition holds.*

*If $1 \leq k = \mathrm{codim} L < N$ then*

$$\begin{aligned} C^{N-k}_{s(L),q'}(E) = 0 \Longrightarrow \mu(E) = 0 \qquad &\text{if } q_c(L) \leq q < q^*_c(L) \\ q \geq q^*_c(L) \Longrightarrow \mu(L) = 0 \qquad &\text{if } q \geq q^*_c(L) \end{aligned} \tag{1.34}$$

*and if $k = N$ (i.e., $L$ is a vertex)*

$$q \geq q_c(L) = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_{A_L}}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_{A_L}}} \Longrightarrow \mu(L) = 0. \tag{1.35}$$

*In all cases, if $1 < q < q_c(L)$ then there is no restriction on $\mu\chi_L$.*

**I.** *Characterization of removable sets.*

Let $\Omega$ be an N-dimensional polyhedron.

Theorem 1.21 provides a necessary and sufficient condition for the removability of a singular set $E$ relative to the family of solutions $u$ such that

$$\int_\Omega |u|^q \rho \, dx < \infty.$$

The next result provides a necessary and sufficient condition for removability in the full sense, as defined in Definition 1.9.

**Theorem 1.22** *Let $\Omega$ be an N-dimensional polyhedron and let $E$ be a compact subset of $\partial\Omega$. A nonempty compact set $E \subset \partial\Omega$ is removable if and only if, for every $L$ as in* **G** *such that $1 \leq k = \mathrm{codim} L < N$ the following condition holds:*
*either*

$$q_c(L) \leq q < q_c^*(L) \;\; and \;\; C^{N-k}_{s(L),q'}(E) = 0$$

*or* $q \geq q_c^*(L)$. *In the case* $k = N$ *the condition is* $q \geq q_c^*(L) = q_c(L)$.

**Aknowledgements**. The authors are grateful to the High Council for Scientific and Technological Cooperation between France and Israel for its financial support.

# 2 Boundary value problems

## 2.1 Classical harmonic analysis in Lipschitz domains

A bounded domain $\Omega \subset \mathbb{R}^N$ is called a Lipschitz domain if there exist positive numbers $r_0, \lambda_0$ and a cylinder

$$O_{r_0} = \{\xi = (\xi_1, \xi') \in \mathbb{R}^N : |\xi'| < r_0, \; |\xi_1| < r_0\} \tag{2.1}$$

such that, for every $y \in \partial\Omega$ there exist:
(i) A Lipschitz function $\psi^y$ on the $(N-1)$-dimensional ball $B'_{r_0}(0)$ with Lipschitz constant $\geq \lambda_0$;
(ii) An isometry $T^y$ of $\mathbb{R}^N$ such that

$$\begin{aligned}
&T^y(y) = 0, \quad (T^y)^{-1}(O_{r_0}) := O^y_{r_0}, \\
&T^y(\partial\Omega \cap O^y_{r_0}) = \{(\psi^y(\xi'), \xi') : \xi' \in B'_{r_0}(0)\} \\
&T^y(\Omega \cap O^y_{r_0}) = \{(\xi_1, \xi') : \xi' \in B'_{r_0}(0), \; -r_0 < \xi_1 < \psi^y(\xi')\}
\end{aligned} \tag{2.2}$$

The constant $r_0$ is called a *localization constant* of $\Omega$; $\lambda_0$ is called a *Lipschitz constant* of $\Omega$. The pair $(r_0, \lambda_0)$ is called a *Lipshitz character* (or, briefly, L-character) of $\Omega$. Note that, if $\Omega$ has L-character $(r_0, \lambda_0)$ and $r' \in (0, r_0)$, $\lambda' \in (\lambda_0, \infty)$ then $(r', \lambda')$ is also an L-character of $\Omega$.

By the Rademacher theorem, the outward normal unit vector exists $\mathcal{H}^{N-1}$−a.e. on $\partial\Omega$, where $\mathcal{H}^{N-1}$ is the N-1 dimensional Hausdorff measure. The unit normal at a point $y \in \partial\Omega$ will be denoted by $\mathbf{n}_y$.

We list below some facts concerning the Dirichlet problem in Lipschitz domains.

**A.1**- Let $x_0 \in \Omega$, $h \in C(\partial\Omega)$ and denote $L_{x_0}(h) := v_h(x_0)$ where $v_h$ is the solution of the Dirichlet problem

$$\left\{ \begin{array}{rl} -\Delta v = 0 & \in \Omega \\ v = h & \text{on } \partial\Omega. \end{array} \right. \tag{2.3}$$

Then $L_{x_0}$ is a continuous linear functional on $C(\partial\Omega)$. Therefore there exists a unique Borel measure on $\partial\Omega$, called the harmonic measure in $\Omega$, denoted by $\omega_\Omega^{x_0}$ such that

$$v_h(x_0) = \int_{\partial\Omega} h d\omega_\Omega^{x_0} \quad \forall h \in C(\partial\Omega). \tag{2.4}$$

When there is no danger of confusion, the subscript $\Omega$ will be dropped. Because of Harnack's inequality the measures $\omega^{x_0}$ and $\omega^x$, $x_0$, $x \in \Omega$ are mutually absolutely continuous. For every fixed $x \in \Omega$ denote the Radon-Nikodym derivative by

$$K(x,y) := \frac{d\omega^x}{d\omega^{x_0}}(y) \quad \text{for } \omega^{x_0}\text{-a.e. } y \in \partial\Omega. \tag{2.5}$$

Then, for every $\bar{x} \in \Omega$, the function $y \mapsto K(\bar{x}, y)$ is positive and continuous on $\partial\Omega$ and, for every $\bar{y} \in \partial\Omega$, the function $x \mapsto K(x, \bar{y})$ is harmonic in $\Omega$ and satisfies

$$\lim_{x\to y} K(x,\bar{y}) = 0 \quad \forall y \in \partial\Omega \setminus \{\bar{y}\}.$$

By [15]

$$\lim_{z\to y} \frac{G(x,z)}{G(x_0,z)} = K(x,y) \quad \forall y \in \partial\Omega \tag{2.6}$$

Thus the kernel $K$ defined above is the *Martin kernel.*

The following is an equivalent definition of the harmonic measure [15]: For any closed set $E \subset \partial\Omega$

$$\begin{aligned}&\omega^{x_0}(E) := \\ &\inf\{\phi(x_0) : \ \phi \in C(\Omega)_+ \ \text{superharmonic in } \Omega, \ \liminf_{x\to E} \phi(x) \geq 1\}.\end{aligned} \tag{2.7}$$

The extension to open sets and then to arbitrary Borel sets is standard.

By (2.4), (2.5) and (2.7), the unique solution $v$ of (2.3) is given by

$$\begin{aligned}&v(x) = \int_{\partial\Omega} K(x,y)h(y)d\omega^{x_0}(y) = \\ &\inf\{\phi \in C(\Omega) : \phi \text{ superharmonic}, \ \liminf_{x\to y} \phi(x) \geq h(y), \ \forall y \in \partial\Omega\}.\end{aligned} \tag{2.8}$$

For details see [15].

**A.2**- Let $(x_0, y_0) \in \Omega \times \partial\Omega$. A function $v$ defined in $\Omega$ is called a kernel function at $y_0$ if it is positive and harmonic in $\Omega$ and verifies $v(x_0) = 1$ and $\lim_{x\to y} v(x) = 0$ for any $y \in \partial\Omega \setminus \{y_0\}$. It is proved in [15, Sec 3] that the kernel function at $y_0$ is unique. Clearly this unique function is $K(\cdot, y_0)$.

**A.3**- We denote by $G(x,y)$ the Green kernel for the Laplacian in $\Omega \times \Omega$. This means that the solution of the Dirichlet problem

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2.9}$$

with $f \in C^2(\overline{\Omega})$, is expressed by

$$u(x) = \int_{\Omega} G(x,y) f(y) dy \quad \forall y \in \overline{\Omega}. \tag{2.10}$$

We shall write (2.10) as $u = \mathbb{G}[f]$.

**A.4**- Let $\Lambda$ be the first eigenvalue of $-\Delta$ in $W_0^{1,2}(\Omega)$ and denote by $\rho$ the corresponding eigenfunction normalized by $\max_\Omega \rho = 1$.

Let $0 < \delta < \operatorname{dist}(x_0, \Omega)$ and put

$$C_{x_0,\delta} := \max_{|x - x_0| = \delta} G(x, x_0)/\rho(x).$$

Since $C_{x_0,\delta}\, \rho - G(\cdot, x_0)$ is superharmonic, the maximum principle implies that

$$0 \leq G(x, x_0) \leq C_{x_0,\delta}\, \rho(x) \quad \forall x \in \Omega \setminus B_\delta(x_0). \tag{2.11}$$

On the other hand, by [18, Lemma 3.4]: for any $x_0 \in \Omega$ there exists a constant $C_{x_0} > 0$ such that

$$0 \leq \rho(x) \leq C_{x_0} G(x, x_0) \quad \forall x \in \Omega. \tag{2.12}$$

**A.5**- For every bounded regular Borel measure $\mu$ on $\partial\Omega$ the function

$$v(x) = \int_{\partial\Omega} K(x,y) d\mu(y) \quad \forall x \in \Omega, \tag{2.13}$$

is harmonic in $\Omega$. We denote this relation by $v = \mathbb{K}[\mu]$.

**A.6**- Conversely, for every *positive* harmonic function $v$ in $\Omega$ there exists a unique positive bounded regular Borel measure $\mu$ on $\partial\Omega$ such that (2.13) holds. The measure $\mu$ is constructed as follows [15, Th 4.3].

Let $SP(\Omega)$ denote the set of continuous, non-negative superharmonic functions in $\Omega$. Let $v$ be a positive harmonic function in $\Omega$.

If $E$ denotes a relatively closed subset of $\Omega$, denote by $R_v^E$ the function defined in $\Omega$ by

$$R_v^E(x) = \inf\{\phi(x) : \phi \in SP(\Omega),\ \phi \geq v \text{ in } E\}.$$

Then $R_v^E$ is superharmonic in $\Omega$, $R_v^E$ decreases as $E$ decreases and, if $F$ is another relatively closed subset of $\Omega$, then

$$R_v^{E\cup F} \leq R_v^E + R_v^F.$$

Now, relative to a point $x \in \Omega$, the measure $\mu$ is defined by,

$$\mu_v^x(F) = \inf\{R_v^E(x) : E = \bar{D} \cap \Omega,\ D \text{ open in } \mathbb{R}^N,\ D \supset F\}, \tag{2.14}$$

for every compact set $F \subset \partial\Omega$. From here it is extended to open sets and then to arbitrary Borel sets in the usual way.

It is easy to see that, if $D$ contains $\partial\Omega$ then $R_v^{\bar{D}\cap\Omega} = v$. Therefore

$$\mu_v^x(\partial\Omega) = v(x). \tag{2.15}$$

In addition, if $F$ is a compact subset of the boundary, the function $x \mapsto \mu_v^x(F)$ is harmonic in $\Omega$ and vanishes on $\partial\Omega \setminus F$.

**A.7**- If $x, x_0$ are two points in $\Omega$, the Harnack inequality implies that $\mu_v^x$ is absolutely continuous with respect to $\mu_v^{x_0}$. Therefore, for $\mu_v^{x_0}$-a.e. point $y \in \partial\Omega$, the density function $d\mu_v^x/d\mu_v^{x_0}(y)$ is a kernel function at $y$. By the uniqueness of the kernel function it follows that

$$\frac{d\mu_v^x}{d\mu_v^{x_0}}(y) = K(x,y), \quad \mu_v^{x_0}\text{-a.e. } y \in \partial\Omega. \tag{2.16}$$

Therefore, using (2.15),

$$\begin{aligned} &(a) \quad \mu_v^x(F) = \int_F K(x,y)\,d\mu^{x_0}(y), \\ &(b) \quad v(x) = \int_{\partial\Omega} K(x,y)\,d\mu^{x_0}(y). \end{aligned} \tag{2.17}$$

**A.8**- By a result of Dahlberg [7, Theorem 3], the (interior) normal derivative of $G(\cdot, x_0)$ exists $\mathcal{H}_{N-1}$-a.e. on $\partial\Omega$ and is positive. In addition, for every Borel set $E \subset \partial\Omega$,

$$\omega^{x_0}(E) = \gamma_N \int_E \partial G(\xi, x_0)/\partial \mathbf{n}_\xi \, dS_\xi, \tag{2.18}$$

where $\gamma_N(N-2)$ is the surface area of the unit ball in $\mathbb{R}^N$ and $dS$ is surface measure on $\partial\Omega$. Thus, for each fixed $x \in \Omega$, the harmonic measure $\omega^x$ is absolutely continuous relative to $\mathcal{H}_{N-1}\big|_{\partial\Omega}$ with density function $P(x,\cdot)$ given by

$$P(x,\xi) = \partial G(\xi, x)/\partial \mathbf{n}_\xi \ \text{ for a.e. } \ \xi \in \partial\Omega. \tag{2.19}$$

In view of (2.8), the unique solution $v$ of (2.3) is given by

$$v(x) = \int_{\Omega} P(x,\xi)h(\xi)dS_{\xi} \tag{2.20}$$

for every $h \in C(\partial\Omega)$. Accordingly $P$ is the *Poisson kernel* for $\Omega$. The expression on the right hand side of (2.20) will be denoted by $\mathbb{P}[h]$. We observe that,

$$\mathbb{K}[h\omega^{x_0}] = \mathbb{P}[h] \quad \forall h \in C(\partial\Omega). \tag{2.21}$$

**A.9**- The *boundary Harnack principle* , first proved in [7], can be formulated as follows [16].

Let $D$ be a Lipschitz domain with L-character $(r_0, \lambda_0)$. Let $\xi \in \partial D$ and $\delta \in (0, r_0)$. Assume that $u, v$ are positive harmonic functions in $D$, vanishing on $\partial D \cap B_\delta(\xi)$. Then there exists a constant $C = C(N, r_0, \lambda_0)$ such that,

$$C^{-1}u(x)/v(x) \leq u(y)/v(y) \leq Cu(x)/v(x) \quad \forall x, y \in B_{\delta/2}(\xi). \tag{2.22}$$

**A.10**- Let $D, D'$ be two Lipschitz domains with L-character $(r_0, \lambda_0)$. Assume that $D' \subset D$ and $\partial D \cap \partial D'$ contains a relatively open set $\Gamma$. Let $x_0 \in D'$ and let $\omega, \omega'$ denote the harmonic measures of $D, D'$ respectively, relative to $x_0$. Then, for every compact set $F \subset \Gamma$, there exists a constant $c_F = C(F, N, r_0, \lambda_0, x_0)$ such that

$$\omega' \lfloor_F \leq \omega \lfloor_F \leq c_F \omega' \lfloor_F. \tag{2.23}$$

Indeed, if $G, G'$ denote the Green functions of $D, D'$ respectively then, by the boundary Harnack principle ,

$$\partial G'(\xi, x_0)/\partial \mathbf{n}_\xi \leq \partial G(\xi, x_0)/\partial \mathbf{n}_\xi \leq c_F \partial G(\xi, x_0)/\partial \mathbf{n}_\xi \quad \text{for a.e.} \quad \xi \in F. \tag{2.24}$$

Therefore (2.23) follows from (2.18).

**A.11**- By [18, Lemma 3.3], for every positive harmonic function $v$ in $\Omega$,

$$\int_{\Omega} v(x)G(x, x_0)dx < \infty. \tag{2.25}$$

In view of (2.12), it follows that $v \in L^1_\rho(\Omega)$.

## 2.2 The dynamic approach to boundary trace.

Let $\Omega$ be a bounded Lipschitz domain and $\{\Omega_n\}$ be a *Lipschitz exhaustion* of $\Omega$. This means that, for every $n$, $\Omega_n$ is Lipschitz and

$$\Omega_n \subset \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega = \cup \Omega_n, \quad \mathbb{H}_{N-1}(\partial\Omega_n) \to \mathbb{H}_{N-1}(\partial\Omega). \tag{2.26}$$

**Lemma 2.1** *Let $x_0 \in \Omega_1$ and denote by $\omega_n$ (respectively $\omega$) the harmonic measure in $\Omega_n$ (respectively $\Omega$) relative to $x_0$. Then, for every $Z \in C(\bar{\Omega})$,*

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Z\, d\omega_n = \int_{\partial\Omega} Z\, d\omega. \tag{2.27}$$

*Proof.* By the definition of harmonic measure

$$\int_{\partial\Omega_n} d\omega_n = 1.$$

We extend $\omega_n$ as a Borel measure on $\bar{\Omega}$ by setting $\omega_n(\bar{\Omega}\setminus\partial\Omega_n) = 0$, and keep the notation $\omega_n$ for the extension. Since the sequence $\{\omega_n\}$ is bounded, there exists a weakly convergent subsequence (still denoted by $\{\omega_n\}$). Evidently the limiting measure, say $\tilde{\omega}$ is supported in $\partial\Omega$ and $\tilde{\omega}(\partial\Omega) = 1$. It follows that for every $Z \in C(\bar{\Omega})$,

$$\int_{\partial\Omega_n} Z\, d\omega_n \to \int_{\partial\Omega} Z\, d\tilde{\omega}.$$

Let $\zeta := Z \mid_{\partial\Omega}$ and $z := \mathbb{K}^{\Omega}[\zeta]$. Again by the definition of harmonic measure,

$$\int_{\partial\Omega_n} z\, d\omega_n = \int_{\partial\Omega} \zeta\, d\omega = z(x_0).$$

It follows that

$$\int_{\partial\Omega} \zeta\, d\tilde{\omega} = \int_{\partial\Omega} \zeta\, d\omega,$$

for every $\zeta \in C(\partial\Omega)$. Consequently $\tilde{\omega} = \omega$. Since the limit does not depend on the subsequence it follows that the whole sequence $\{\omega_n\}$ converges weakly to $\omega$. This implies (2.27).

□

In the next lemma we continue to use the notation introduced above.

**Lemma 2.2** *Let $x_0 \in \Omega_1$, let $\mu$ be a bounded Borel measure on $\partial\Omega$ and put $v := \mathbb{K}^{\Omega}[\mu]$. Then, for every $Z \in C(\bar{\Omega})$,*

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Zv\, d\omega_n = \int_{\partial\Omega} Z\, d\mu. \tag{2.28}$$

*Proof.* It is sufficient to prove the result for positive $\mu$. Let $h_n := v \mid_{\partial\Omega_n}$. Evidently $v = \mathbb{K}^{\Omega_n}[h_n\omega_n]$ in $\Omega_n$. Therefore

$$v(x_0) = \int_{\partial\Omega_n} h_n d\omega_n = \mu(\partial\Omega).$$

Let $\mu_n$ denote the extension of $h_n\omega_n$ as a measure in $\bar{\Omega}$ such that $\mu_n(\bar{\Omega}\setminus\partial\Omega_n)=0$. Then $\{\mu_n\}$ is bounded and consequently there exists a weakly convergent subsequence $\{\mu_{n_j}\}$. The limiting measure, say $\tilde{\mu}$, is supported in $\partial\Omega$ and

$$\tilde{\mu}(\partial\Omega)=v(x_0)=\mu(\partial\Omega). \tag{2.29}$$

It follows that for every $Z\in C(\bar{\Omega})$,

$$\int_{\partial\Omega_{n_j}} Z\,d\mu_{n_j}\to\int_{\partial\Omega} Z\,d\tilde{\mu}.$$

To complete the proof, we have to show that $\tilde{\mu}=\mu$. Let $F$ be a closed subset of $\partial\Omega$ and put,

$$\mu^F=\mu\chi_F,\quad v^F=\mathbb{K}^\Omega[\mu^F].$$

Let $h_n^F:=v^F\mid_{\partial\Omega_n}$ and let $\mu_n^F$ denote the extension of $h_n^F\omega_n$ as a measure in $\bar{\Omega}$ such that $\mu_n^F(\bar{\Omega}\setminus\partial\Omega_n)=0$. As in the previous part of the proof, there exists a weakly convergent subsequence of $\{\mu_{n_j}^F\}$. The limiting measure $\tilde{\mu}^F$ is supported in $F$ and

$$\tilde{\mu}^F(F)=\tilde{\mu}^F(\partial\Omega)=v^F(x_0)=\mu^F(\partial\Omega)=\mu(F).$$

As $v^F\le v$, we have $\tilde{\mu}^F\le\tilde{\mu}$. Consequently

$$\mu(F)\le\tilde{\mu}(F). \tag{2.30}$$

Observe that $\tilde{\mu}$ depends on the first subsequence $\{\mu_{n_j}\}$, but not on the second subsequence. Therefore (2.30) holds for every closed set $F\subset\partial\Omega$, which implies that $\mu\le\tilde{\mu}$. On the other hand, $\mu$ and $\tilde{\mu}$ are positive measures which, by (2.29), have the same total mass. Therefore $\mu=\tilde{\mu}$.

□

**Lemma 2.3** *Let $\mu\in\mathfrak{M}(\partial\Omega)$ (= space of bounded Borel measures on $\partial\Omega$). Then $\mathbb{K}[\mu]\in L^1_\rho(\Omega)$ and there exists a constant $C=C(\Omega)$ such that*

$$\|\mathbb{K}[\mu]\|_{L^1_\rho(\Omega)}\le C\,\|\mu\|_{\mathfrak{M}(\partial\Omega)}\,. \tag{2.31}$$

*In particular if $h\in L^1(\partial\Omega;\omega)$ then*

$$\|\mathbb{P}[h]\|_{L^1_\rho(\Omega)}\le C\,\|h\|_{L^1(\partial\Omega;\omega)}\,. \tag{2.32}$$

*Proof.* Let $x_0$ be a point in $\Omega$ and let $K$ be defined as in (2.5). Put $\phi(\cdot) = G(\cdot, x_0)$ and $d_0 = \operatorname{dist}(x_0, \Omega)$. Let $(r_0, \lambda_0)$ denote the Lipschitz character of $\Omega$.

By [3, Theorem 1], there exist positive constants $c_1(N, r_0, \lambda_0, d_0)$ and $c_0(N, r_0, \lambda_0, d_0)$ such that for every $y \in \partial\Omega$,

$$c_1^{-1} \frac{\phi(x)}{\phi^2(x')} |x-y|^{2-N} \leq K(x,y) \leq c_1 \frac{\phi(x)}{\phi^2(x')} |x-y|^{2-N}, \tag{2.33}$$

for all $x, x' \in \Omega$ such that

$$c_0|x-y| < \operatorname{dist}(x', \partial\Omega) \leq |x'-y| < |x-y| < \frac{1}{4}\min(d_0, r_0/8). \tag{2.34}$$

Therefore, by (2.12) and (2.11), there exists a constant $c_2(N, r_0, \lambda_0, d_0)$ such that

$$c_2^{-1} \frac{\phi^2(x)}{\phi^2(x')} |x-y|^{2-N} \leq \rho(x)K(x,y) \leq c_2 \frac{\phi^2(x)}{\phi^2(x')} |x-y|^{2-N}$$

for $x, x'$ as above. There exists a constant $\bar{c}_0$, depending on $c_0, N$, such that, for every $x \in \Omega$ satisfying $|x-y| < \frac{1}{4}\min(d_0, r_0/8)$ there exists $x' \in \Omega$ which satisfies (2.34) and also

$$|x - x'| \leq \bar{c}_0 \min(\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(x', \partial\Omega)).$$

By the Harnack chain argument, $\phi(x)/\phi(x')$ is bounded by a constant depending on $N, \bar{c}_0$. Therefore

$$c_3^{-1}|x-y|^{2-N} \leq \rho(x)K(x,y) \leq c_3|x-y|^{2-N} \tag{2.35}$$

for some constant $c_3(N, r_0, \lambda_0, d_0)$ and all $x \in \Omega$ sufficiently close to the boundary.

Assuming that $\mu \geq 0$,

$$\int_\Omega \mathbb{K}[\mu](x)\rho(x)dx = \int_{\partial\Omega}\int_\Omega K(x,\xi)\rho(x)dx\, d\mu(\xi) \leq C\, \|\mu\|_{\mathfrak{M}(\partial\Omega)}.$$

In the general case we apply this estimate to $\mu_+$ and $\mu_-$. This implies (2.31). For the last statement of the theorem see (2.21).

□

**Proposition 2.4** *Let $v$ be a positive harmonic function in $\Omega$ with boundary trace $\mu$. Let $Z \in C^2(\overline{\Omega})$ and let $\tilde{G} \in C^\infty(\Omega)$ be a function that coincides* *with $x \mapsto G(x, x_0)$ in $Q \cap \Omega$ for some neighborhood $Q$ of $\partial\Omega$ and some fixed $x_0 \in \Omega$. In addition assume that there exists a constant $c > 0$ such that*

$$|\nabla Z \cdot \nabla \tilde{G}| \leq c\rho. \tag{2.36}$$

*Under these assumptions, if $\zeta := Z\tilde{G}$ then*

$$-\int_{\Omega} v\Delta\zeta \, dx = \int_{\partial\Omega} Z d\mu. \tag{2.37}$$

*Remark.* This result is useful in a k-dimensional dihedron in the case where $\mu$ is concentrated on the edge. In such a case one can find, for every smooth function on the edge, a lifting $Z$ such that condition (2.36) holds. See Section 8 for such an application.

*Proof.* Let $\{\Omega_n\}$ be a $C^1$ exhaustion of $\Omega$. We assume that $\partial\Omega_n \subset Q$ for all $n$ and $x_0 \in \Omega_1$. Let $\tilde{G}_n(x)$ be a function in $C^1(\Omega_n)$ such that $\tilde{G}_n$ coincides with $G^{\Omega_n}(\cdot, x_0)$ in $Q \cap \Omega_n$, $\tilde{G}_n(\cdot, x_0) \to \tilde{G}(\cdot, x_0)$ in $C^2(\Omega \setminus Q)$ and $\tilde{G}_n(\cdot, x_0) \to \tilde{G}(\cdot, x_0)$ in $\mathrm{Lip}\,(\Omega)$. If $\zeta_n = Z\tilde{G}_n$ we have,

$$\begin{aligned}
-\int_{\Omega_n} v\Delta\zeta_n \, dx &= \int_{\partial\Omega_n} v\partial_{\mathbf{n}}\zeta \, dS = \int_{\partial\Omega_n} vZ\partial_{\mathbf{n}}\tilde{G}_n(\xi, x_0) \, dS \\
&= \int_{\partial\Omega_n} vZP^{\Omega_n}(x_0, \xi) \, dS = \int_{\partial\Omega_n} vZ \, d\omega_n.
\end{aligned}$$

By Lemma 2.2,

$$\int_{\partial\Omega_n} vZ \, d\omega_n \to \int_{\partial\Omega} Z \, d\mu.$$

On the other hand, in view of (2.36), we have

$$\Delta\zeta_n = \tilde{G}_n\Delta Z + Z\Delta\tilde{G}_n + 2\nabla Z \cdot \nabla\tilde{G}_n \to \Delta Z$$

in $L^1_\rho(\Omega)$; therefore,

$$-\int_{\Omega_n} v\Delta\zeta_n \, dx \to -\int_{\Omega} v\Delta\zeta \, dx.$$

□

**Definition 2.5** *Let $D$ be a Lipschitz domain and let $\{D_n\}$ be a Lipschitz exhaustion of $D$. We say that $\{D_n\}$ is a* uniform Lipschitz exhaustion *if there exist positive numbers $\bar{r}, \bar{\lambda}$ such that $D_n$ has L-character $(\bar{r}, \bar{\lambda})$ for all $n \in \mathbb{N}$. The pair $(\bar{r}, \bar{\lambda})$ is an L-character of the exhaustion.*

**Lemma 2.6** *Assume $D, D'$ are two Lipschitz domains such that*

$$\Gamma \subset \partial D \cap \partial D' \subset \partial(D \cup D')$$

*where $\Gamma$ is a relatively open set. Suppose $D, D', D \cup D'$ have L-character $(r_0, \lambda_0)$. Let $x_0$ be a point in $D \cap D'$ and put*

$$d_0 = \min(\mathrm{dist}\,(x_0, \partial D), \mathrm{dist}\,(x_0, \partial D')).$$

*Let $u$ be a positive harmonic function in $D \cup D'$ and denote its boundary trace on $D$ (resp. $D'$) by $\mu$ (resp. $\mu'$). Then, for every compact set $F \subset \Gamma$, there exists a constant $c_F = c(F, r_0, \lambda_0, d_0, N)$ such that*

$$c_F^{-1} \mu' \lfloor_F \leq \mu \lfloor_F \leq c_F \mu' \lfloor_F. \tag{2.38}$$

*Proof.* We prove (2.38) in the case that $D' \subset D$. This implies (2.38) in the general case by comparison of the boundary trace on $\partial D$ or $\partial D'$ with the boundary trace on $\partial(D \cup D')$.

Let $Q$ be an open set such that $Q \cap D$ is Lipschitz and

$$F \subset Q, \quad \bar{Q} \cap D \subset D', \quad \bar{Q} \cap \partial D \subset \Gamma.$$

Then there exist uniform Lipschitz exhaustions of $D$ and $D'$, say $\{D_n\}$ and $\{D'_n\}$, possessing the following properties:

(i) $\bar{D}'_n \cap Q = \bar{D}_n \cap Q$.

(ii) $x_0 \in D'_1$ and $\mathrm{dist}\,(x_0, \partial D'_1) \geq \frac{1}{4} d_0$.

(iii) There exist $r_Q > 0$ and $\lambda_Q > 0$ such that both exhaustions have L-character $(r_Q, \lambda_Q)$.

Put $\Gamma_n := \partial D_n \cap Q = \partial D'_n \cap Q$ and let $\omega_n$ (resp. $\omega'_n$) denote the harmonic measure, relative to $x_0$, of $D_n$ (resp. $D'_n$). By Lemma 2.2,

$$\int_{\Gamma_n} \phi\, u(y)\, d\omega_n(y) \to \int_\Gamma \phi\, d\mu,$$

and

$$\int_{\Gamma_n} \phi\, u(y) d\omega'_n(y) \to \int_\Gamma \phi\, d\mu'$$

for every $\phi \in C_c(Q)$. By **A.10** there exists a constant $c_Q = c(Q, r_Q, \lambda_Q, d_0, N)$ such that

$$\omega'_n \lfloor_{\Gamma_n} \leq \omega_n \lfloor_{\Gamma_n} \leq c_Q \omega'_n \lfloor_{\Gamma_n}.$$

This implies (2.38).

$\square$

### 2.3 $L^1$ data

We denote by $X(\Omega)$ the space of test functions,

$$X(\Omega) = \left\{\eta \in W_0^{1,2}(\Omega) : \rho^{-1}\Delta\eta \in L^\infty(\Omega)\right\}. \tag{2.39}$$

Let $X_+(\Omega)$ denote its positive cone.

Let $f \in L^\infty(\Omega)$, and let $u$ be the weak $W_0^{1,2}$ solution of the Dirichlet problem

$$-\Delta u = f \text{ in } \Omega \quad , \quad u = 0 \text{ on } \partial\Omega \tag{2.40}$$

If $\Omega$ is a Lipschitz domain (as we assume here) then $u \in C(\bar{\Omega})$ (see [32]). Since $\mathbb{G}[f]$ is a weak $W_0^{1,2}$ solution, it follows that the solution of (2.40), which is unique in $C(\bar{\Omega})$, is given by $u = \mathbb{G}[f]$. If, in addition, $|f| \leq c_1\rho$ then, by the maximum principle,

$$|u| \leq (c_1/\Lambda)\rho, \tag{2.41}$$

where $\Lambda$ is the first eigenvalue of $-\Delta$ in $\Omega$.

In particular, if $\eta \in X(\Omega)$ then $\eta \in C(\bar{\Omega})$ and it satisfies

$$-\mathbb{G}[\Delta\eta] = \eta, \tag{2.42}$$

$$|\eta| \leq \Lambda^{-1}\left\|\rho^{-1}\Delta\eta\right\|_{L^\infty}\rho. \tag{2.43}$$

If, in addition, $\Omega$ is a $C^2$ domain then the solution of (2.40) is in $C^1(\bar{\Omega})$.

**Lemma 2.7** *Let $\Omega$ be a Lipschitz bounded domain. Then for any $f \in L^1_\rho(\Omega)$ there exists a unique $u \in L^1_\rho(\Omega)$ such that*

$$-\int_\Omega u\Delta\eta\,dx = \int_\Omega f\eta dx \quad \forall\eta \in X(\Omega). \tag{2.44}$$

*Furthermore $u = \mathbb{G}[f]$. Conversely, if $f \in L^1_{loc}(\Omega)$, $f \geq 0$ and there exists $x_0 \in \Omega$ such that $\mathbb{G}[f](x_0) < \infty$ then $f \in L^1_\rho(\Omega)$. Finally*

$$\|u\|_{L_\rho(\Omega)} \leq \Lambda^{-1}\|f\|_{L_\rho(\Omega)} \tag{2.45}$$

*Proof.* First assume that $f$ is bounded. We have already observed that, in this case, the weak $W_0^{1,2}$ solution $u$ of the Dirichlet problem (2.40) is in $C(\bar{\Omega})$ and $u = \mathbb{G}[f]$. Furthermore, it follows from [4] that

$$\int_\Omega \nabla\eta\cdot\nabla u dx = -\int_\Omega u\Delta\eta dx.$$

Thus $u = \mathbb{G}[f]$ is also a weak $L^1_\rho$ solution (in the sense of (2.44)).

Let $\eta_0$ be the weak $W_0^{1,2}$ solution of (2.40) when $f = \mathrm{sgn}(u)\rho$; evidently $\eta_0 \in X(\Omega)$. If $u \in L^1_\rho(\Omega)$ is a solution of (2.44) for some $f \in L^1_\rho(\Omega)$ then

$$\int_\Omega |u|\rho dx = \int_\Omega f\eta_0 dx \leq \Lambda^{-1}\int_\Omega |f|\rho dx. \tag{2.46}$$

The second inequality follows from (2.41). This proves (2.45) and implies uniqueness.

Now assume that $f \in L^1_\rho(\Omega)$ and let $\{f_n\}$ be a sequence of bounded functions such that $f_n \to f$ in this space. Let $u_n$ be the weak $W_0^{1,2}$ solution of (2.40) with $f$ replaced by $f_n$. Then $u_n$ satisfies (2.44) and $u_n = \mathbb{G}[f_n]$. By (2.45), $\{u_n\}$ converges in $L^1_\rho(\Omega)$, say $u_n \to u$. In view of (2.11) it follows that $u = \mathbb{G}[f]$ and that $u$ satisfies (2.44).

If $f \in L^1_{loc}(\Omega)$, $f \geq 0$ and $\mathbb{G}[f](x_0) < \infty$ then, by (2.12), $f \in L^1_\rho(\Omega)$.

□

**Lemma 2.8** *Let $\Omega$ be a Lipschitz bounded domain. If $f \in L^1_\rho(\Omega)$ and $h \in L^1(\partial\Omega;\omega)$, there exists a unique $u \in L^1_\rho(\Omega)$ satisfying*

$$\int_\Omega (-u\Delta\eta - f\eta)\, dx = -\int_\Omega \mathbb{P}[h]\Delta\eta dx \quad \forall \eta \in X(\Omega) \tag{2.47}$$

*or equivalently*

$$u = \mathbb{G}[f] - \mathbb{P}[h]. \tag{2.48}$$

*The following estimate holds*

$$\begin{aligned} \|u\|_{L^1_\rho(\Omega)} &\leq c\left(\|f\|_{L^1_\rho(\Omega)} + \|\mathbb{P}[h]\|_{L^1_\rho(\Omega)}\right) \\ &\leq c\left(\|f\|_{L^1_\rho(\Omega)} + \|h\|_{L^1(\partial\Omega,\omega)}\right). \end{aligned} \tag{2.49}$$

*Furthermore, for any nonnegative element $\eta \in X(\Omega)$, we have*

$$-\int_\Omega |u|\, \Delta\eta\, dx \leq -\int_\Omega \mathbb{P}[|h|]\Delta\eta dx + \int_\Omega \eta f \mathrm{sgn}(u)\, dx, \tag{2.50}$$

*and*

$$-\int_\Omega u_+\Delta\eta\, dx \leq -\int_\Omega \mathbb{P}[h_+]\Delta\eta dx + \int_\Omega \eta f \mathrm{sgn}_+(u)\, dx. \tag{2.51}$$

*Proof. Existence.* By Lemma 2.3, the assumption on $h$ implies that $\mathbb{P}[|h|] \in L^1_\rho(\Omega)$. If we denote by $v$ the unique function in $L^1_\rho(\Omega)$ which satifies

$$-\int_\Omega v\Delta\eta dx = -\int_\Omega f\eta dx \quad \forall \eta \in X(\Omega),$$

then $u = v - \mathbb{P}[h] \in L^1_\rho(\Omega)$ and (2.47) holds.

By Lemma 2.7, (2.48) is equivalent to (2.47).

*Estimate (2.49)* This inequality follows from (2.47) and (2.45).

*Estimate (2.51).* Let $\{\Omega_n\}$ be an exhaustion of $\Omega$ by *smooth domains.* If $u$ is the solution of (2.47) and $h_n := u\big|_{\partial\Omega_n}$ then, in $\Omega_n$,

$$u = \mathbb{G}^{\Omega_n}[f] - \mathbb{P}^{\Omega_n}[h_n] \;\text{ in } \Omega_n,$$

or equivalently,

$$\begin{aligned}\int_{\Omega_n} (-u\Delta\eta - f\eta)\, dx &= -\int_{\Omega_n} \mathbb{P}[h_n]\Delta\eta dx \\ &= -\int_{\partial\Omega_n} (\partial\eta/\partial\mathbf{n}) h_n dx \quad \forall \eta \in X(\Omega_n).\end{aligned} \tag{2.52}$$

We recall that, since $\Omega_n$ is smooth, $\eta \in X(\Omega_n)$ implies that $\eta \in C^1(\bar\Omega_n)$. In addition it is known that (see e.g. [33]), for every non-negative $\eta \in X(\Omega_n)$,

$$\int_{\Omega_n} (-|u|\Delta\eta - f\eta \,\mathrm{sign}\, u)\, dx \le -\int_{\partial\Omega_n} \partial\eta/\partial\mathbf{n}|h_n| dx \tag{2.53}$$

Let $\rho_n$ be the first eigenfunction of $-\Delta$ in $\Omega_n$, normalized by $\rho_n(\bar x) = 1$ for some $\bar x \in \Omega_1$. Let $\eta$ be a non-negative function in $X(\Omega)$ and let $\eta_n$ be the solution of the problem

$$\Delta z = (\Delta\eta)\rho_n/\rho \;\text{ in } \Omega_n, \;\; z = 0 \;\text{ on } \partial\Omega_n.$$

Then $\eta_n \in X(\Omega_n)$ and, since $\rho_n \to \rho$,

$$\Delta\eta_n \to \Delta\eta, \quad \eta_n \to \eta.$$

If $v := \mathbb{P}[|h|]$ then $v \ge |u|$ so that

$$\tilde h_n := v\big|_{\partial\Omega_n} \ge |h_n|.$$

Therefore

$$\begin{aligned}&-\int_{\partial\Omega_n} \partial\eta_n/\partial\mathbf{n}|h_n| dx \le -\int_{\partial\Omega_n} \partial\eta/\partial\mathbf{n}|\tilde h_n| dx = \\ &-\int_{\Omega_n} \mathbb{P}^{\Omega_n}[\tilde h_n]\Delta\eta_n dx = -\int_{\Omega_n} v\Delta\eta_n dx \to -\int_\Omega v\Delta\eta dx.\end{aligned} \tag{2.54}$$

Finally, (2.53) and (2.54) imply (2.50).

*Estimate* (2.51) This inequality is obtained by adding (2.47) and (2.50).

□

**Definition 2.9** *We shall say that a function $g : \mathbb{R} \mapsto \mathbb{R}$ belongs to $\mathcal{G}(\mathbb{R})$ if it is continuous, nondecreasing and $g(0) = 0$.*

**Lemma 2.10** *Let $\Omega$ be a Lipschitz bounded domain and $g \in \mathcal{G}(\mathbb{R})$. If $f \in L^1_\rho(\Omega)$ and $h \in L^1(\partial\Omega; \omega)$, there exists a unique $u \in L^1_\rho(\Omega)$ such that $g(u) \in L^1_\rho(\Omega)$ and*

$$\int_\Omega (-u\Delta\eta + (g(u) - f)\eta)\, dx = -\int_\Omega \mathbb{P}[h]\Delta\eta\, dx \quad \forall \eta \in X(\Omega). \tag{2.55}$$

*The correspondence $(f, h) \mapsto u$ is increasing.*

*If $u, u'$ are solutions of (2.55) corresponding to data $f, h$ and $f', h'$ respectively then the following estimate holds:*

$$\begin{aligned} &\left\|u - u'\right\|_{L^1_\rho(\Omega)} + \left\|g(u) - g(u')\right\|_{L^1_\rho(\Omega)} \\ &\leq c\left(\left\|f - f'\right\|_{L^1_\rho(\Omega)} + \left\|\mathbb{P}[h - h']\right\|_{L^1_\rho(\Omega)}\right) \\ &\leq c\left(\left\|f - f'\right\|_{L^1_\rho(\Omega)} + \left\|h - h'\right\|_{L^1(\partial\Omega,\omega)}\right). \end{aligned} \tag{2.56}$$

*Finally, for any nonnegative element $\eta \in X(\Omega)$, we have*

$$-\int_\Omega |u|\, \Delta\eta\, dx + \int_\Omega |g(u)|\eta\, dx \leq -\int_\Omega \mathbb{P}[|h|]\Delta\eta dx + \int_\Omega \eta f \operatorname{sgn}(u)\, dx, \tag{2.57}$$

*and*

$$-\int_\Omega u_+\Delta\eta\, dx + \int_\Omega g(u)_+\eta\, dx \leq -\int_\Omega \mathbb{P}[h_+]\Delta\eta dx + \int_\Omega \eta f \operatorname{sgn}_+(u)\, dx. \tag{2.58}$$

*Proof.* If $u, u'$ are two solutions as stated above then $v = u - u'$ satisfies

$$\int_\Omega (-v\Delta\eta + F\eta)\, dx = -\int_\Omega \mathbb{P}[h - h']\Delta h dx \quad \forall \eta \in X(\Omega) \tag{2.59}$$

where $F = g(u) - g(u') - (f - f') \in L^1_\rho(\Omega)$. Applying (2.50) to this equation and using the properties of $g$ described in Definition 2.9 we obtain (2.56). Similarly we obtain (2.57) and (2.58), using (2.50) and (2.51). These inequalities imply uniqueness and monotone dependence on data.

In the case that $f$ and $h$ are bounded, existence is obtained by the standard variational method. In general we approach $f$ in $L^1_\rho(\Omega)$ by functions in $C^\infty_c(\Omega)$ and $h$ in $L^1(\partial\Omega; \omega)$ by functions in $C(\partial\Omega)$ and employ (2.56).

□

## 3 Measure data

Denote by $\mathfrak{M}_\rho(\Omega)$ the space of Radon measures $\nu$ in $\Omega$ such that $\rho|\nu|$ is a bounded measure.

**Lemma 3.1** *Let $\Omega$ be a Lipschitz bounded domain. Let $\nu \in \mathfrak{M}_\rho(\Omega)$ and $u \in L^1_{loc}(\Omega)$ be a nonnegative solution of*

$$-\Delta u = \nu \quad \text{in } \Omega.$$

*Then $u \in L^1_\rho(\Omega)$ and there exists a unique positive Radon measure $\mu$ on $\partial\Omega$ such that*

$$u = \mathbb{K}[\mu] + \mathbb{G}[\nu]. \tag{3.1}$$

*Proof.* Let $D$ be a smooth subdomain of $\Omega$ such that $\bar{D} \subset \Omega$. Since $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 1$ it follows that $u$ possesses a trace, say $h_D$, in $W^{1-\frac{1}{p},p}(\partial D)$. Put $v := u - \mathbb{G}^D[\nu]$. Then $-\Delta v = 0$ in $D$ and $v \geq 0$ on $\partial D$ and therefore in $D$. If $\{D_n\}$ is an increasing sequence of such domains, converging to $\Omega$, then $\mathbb{G}^{D_n}[\nu] \uparrow \mathbb{G}^\Omega[\nu]$. Thus $v = u - \mathbb{G}^\Omega[\nu]$ is a nonnegative harmonic function in $\Omega$ and consequently possesses a boundary trace $\mu \in \mathfrak{M}(\partial\Omega)$ such that $v = \mathbb{K}[\mu]$.

□

**Lemma 3.2** *Let $\Omega$ be a Lipschitz bounded domain. If $\nu \in \mathfrak{M}_\rho(\Omega)$ and $\mu \in \mathfrak{M}(\partial\Omega)$, there exists a unique $u \in L^1_\rho(\Omega)$ satisfying*

$$\int_\Omega -u\Delta\eta\,dx = \int_\Omega \eta\,d\nu - \int_\Omega \mathbb{K}[\mu]\Delta\eta dx \quad \forall \eta \in X(\Omega). \tag{3.2}$$

*This is equivalent to*

$$u = \mathbb{G}[\nu] + \mathbb{K}[\mu]. \tag{3.3}$$

*The following estimate holds*

$$\begin{aligned} \|u\|_{L^1_\rho(\Omega)} &\leq c\left(\|\nu\|_{\mathfrak{M}_\rho(\Omega)} + \|\mathbb{K}[\mu]\|_{L^1_\rho(\Omega)}\right) \\ &\leq c\left(\|\nu\|_{\mathfrak{M}_\rho(\Omega)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}\right). \end{aligned} \tag{3.4}$$

*In addition, if $d\nu = f dx$ for some $f \in L^1_\rho(\Omega)$ then, for any nonnegative element $\eta \in X(\Omega)$, we have*

$$-\int_\Omega |u|\,\Delta\eta\,dx \leq -\int_\Omega \mathbb{K}[|\mu|]\Delta\eta dx + \int_\Omega \eta f \operatorname{sgn}(u)\,dx, \tag{3.5}$$

*and*

$$-\int_\Omega u_+\Delta\eta\,dx \leq -\int_\Omega \mathbb{K}[\mu_+]\Delta\eta dx + \int_\Omega \eta f \operatorname{sgn}_+(u)\,dx. \tag{3.6}$$

*Proof.* We approximate $\mu$ by a sequence $\{h_nP(x_0,\cdot)\}$ and $\nu$ by a sequence $\{f_n\}$ such that

$$h_nP(x_0,\cdot)\in L^1(\partial\Omega),\quad h_nP(x_0,\cdot)\mathcal{H}_{N-1}\to\mu\quad\text{weakly in measure}$$

and

$$f_n\in L^1_\rho(\Omega),\quad f_n\to\nu\quad\text{weakly relative to } C_\rho(\Omega),$$

where $C_\rho$ denotes the space of functions $\zeta\in C(\Omega)$ such that $\rho\zeta\in L^\infty(\Omega)$. Applying Lemma 2.8 to problem (2.49) ($f,h$ replaced by $f_n,h_n$) and taking the limit we obtain a solution $u\in L^1_\rho(\Omega)$ of (3.2) satisfying (3.4).

Lemma 2.7 implies that any solution $u$ of (3.2) satisfies (3.3). Therefore the solution is unique and hence (3.4) holds for all solutions.

Inequalities (3.5) and (3.6) are proved in the same way as the corresponding inequalities in Lemma 2.8

□

**Definition 3.3** *Let $\Omega$ be a bounded Lipschitz domain and let $g\in\mathcal{G}(\mathbb{R})$. If $\mu\in\mathfrak{M}(\partial\Omega)$, a function $u\in L^1_\rho(\Omega)$ is a weak solution of*

$$\begin{cases} -\Delta u+g(u)=0 & \text{in } \Omega\\ u=\mu & \text{in } \partial\Omega\end{cases}\tag{3.7}$$

*if $g(u)\in L^1_\rho(\Omega)$ and*

$$u+\mathbb{G}[g(u)]=\mathbb{K}[\mu]\tag{3.8}$$

*a.e. in $\Omega$. Equivalently*

$$\int_\Omega(-u\Delta\eta+g(u)\eta)\,dx=-\int_\Omega(\mathbb{K}[\mu]\Delta\eta)\,dx\quad\forall\eta\in X(\Omega).\tag{3.9}$$

*The measure $\mu$ is called the* boundary trace *of $u$ on $\partial\Omega$.*

*Similarly a function $u\in L^1_\rho(\Omega)$ is a weak supersolution, respectively subsolution, of (3.7) if $g(u)\in L^1_\rho(\Omega)$ and*

$$u+\mathbb{G}[g(u)]\geq\mathbb{K}[\mu]\ \text{ respectively }\ u+\mathbb{G}[g(u)]\leq\mathbb{K}[\mu].\tag{3.10}$$

*This is equivalent to (3.9), with $=$ replaced by $\geq$ or $\leq$, holding for every positive $\eta\in X(\Omega)$.*

*Remark.* It follows from this definition and Lemma 2.10 that, if

$$\mu_n\rightharpoonup\mu\ \text{ weakly in } \mathfrak{M}(\partial\Omega),\ u_n\to u,\quad g(u_n)\to g(u)\ \text{ in } L^1_\rho(\Omega),$$

and if

$$u_n=\mathbb{K}[\mu_n]-\mathbb{G}[g(u_n)],$$

then $u=\mathbb{K}[\mu]-\mathbb{G}[g(u)]$.

**Lemma 3.4** *Let $\Omega$ be a Lipschitz bounded domain and let $g \in \mathcal{G}$. Suppose that $\mu \in \mathfrak{M}(\partial\Omega)$ and that there exists a solution of problem* (3.7)*. Then the solution is unique.*

*If $\mu, \mu'$ are two measures in $\mathfrak{M}(\partial\Omega)$, for which problem* (3.7) *possesses solutions $u, u'$ respectively, then the following estimate holds:*

$$\begin{aligned} \left\|u - u'\right\|_{L^1_\rho(\Omega)} + \left\|g(u) - g(u')\right\|_{L^1_\rho(\Omega)} &\leq \left\|\mathbb{K}[\mu - \mu']\right\|_{L^1_\rho(\Omega)}) \\ &\leq \left\|\mu - \mu'\right\|_{\mathfrak{M}(\partial\Omega)}. \end{aligned} \tag{3.11}$$

*If $\mu \leq \mu'$ then $u \leq u'$.*

*In addition, for any nonnegative element $\eta \in X(\Omega)$, we have*

$$-\int_\Omega (|u|\,\Delta\eta - |g(u)|\eta)\,dx \leq -\int_\Omega \mathbb{K}[|\mu|]\Delta\eta dx \tag{3.12}$$

*and*

$$-\int_\Omega (u_+\Delta\eta - g(u)_+\eta)\,dx \leq -\int_\Omega \mathbb{K}[\mu_+]\Delta\eta dx. \tag{3.13}$$

*Proof.* This follows from Lemma 3.2 in the same way that Lemma 2.10 follows from Lemma 2.8. □

**Definition 3.5** *Assume that $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 1$. We say that $u$ possesses a boundary trace $\mu \in \mathfrak{M}(\partial\Omega)$ if, for every Lipschitz exhaustion $\{\Omega_n\}$ of $\Omega$,*

$$\lim_{n\to\infty}\int_{\partial\Omega_n} Zu\,d\omega_n = \int_{\partial\Omega} Z\,d\mu, \tag{3.14}$$

*holds for every $Z \in C(\bar{\Omega})$.*

*Similarly we say that $u$ possesses a trace $\mu$ on a relatively open set $A \subset \partial\Omega$ if* (3.14) *holds for every $Z \in C(\bar{\Omega})$ such that* $\operatorname{supp} Z \subset \Omega \cup A$.

*Remark.* If $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 1$ then, by Sobolev's trace theorem, for every relatively open $(N-1)$- dimensional Lipschitz surface $\Sigma$, $u$ possesses a trace in $W^{1-\frac{1}{p},p}(\Sigma)$. In particular the trace is in $L^1(\Sigma)$. In fact there exists an element of the Lebesgue equivalence class of $u$ such that the trace on $\Sigma$ is precisely the restriction of $u$ to $\Sigma$. When it is relevant, as in (3.14), we assume that $u$ is represented by such an element.

If $u \in W^{1,p}(\Omega)$ then, by the same token, $u$ possesses a trace in $W^{1-\frac{1}{p},p}(\partial\Omega)$. If $\{\Omega_n\}$ is a uniform Lipschitz exhaustion and $h_n$ (resp. $h$) denotes the trace of $u$ on $\partial\Omega_n$ (resp. $\partial\Omega$) then

$$\|h_n\|_{W^{1-\frac{1}{p},p}(\partial\Omega_n)} \to \|h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}.$$

This follows from the continuity of the imbedding

$$W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$$

and the fact that $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Similarly, if $\{\Omega_n\}$ is a Lipschitz exhaustion (not necessarily uniform, but satisfies (2.26)) then

$$\|h_n\|_{L^1(\partial\Omega_n)} \to \|h\|_{L^1(\partial\Omega)}.$$

In particular, if $u \in W_0^{1,p}(\Omega)$ then its boundary trace is zero, in the sense of the above definition.

**Proposition 3.6** *Let $u$ be a weak solution of* (3.7)*. If $\{\Omega_n\}$ is a Lipschitz exhaustion of $\Omega$ then, for every $Z \in C(\bar{\Omega})$,*

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Zu\, d\omega_n = \int_{\partial\Omega} Z\, d\mu, \tag{3.15}$$

*where $\omega_n$ is the harmonic measure of $\Omega_n$ (relative to a point $x_0 \in \Omega_1$).*

*Proof.* If $v := \mathbb{G}[g \circ u]$ then $v \in L^1_\rho(\Omega)$ and $u+v$ is a harmonic function. By (3.8), $u + v = \mathbb{K}^\Omega[\mu]$. Therefore, by Lemma 2.2,

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Z(u+v)\, d\omega_n = \int_{\partial\Omega} Z\, d\mu \tag{3.16}$$

for every $Z \in C(\bar{\Omega})$. As $v \in W_0^{1,p}(\Omega)$ for some $p > 1$ its boundary trace is zero. Therefore (3.16) implies (3.15). $\square$

**Definition 3.7** *A measure $\mu \in \mathfrak{M}(\partial\Omega)$ is called $g$-admissible if $g(\mathbb{K}[|\mu|]) \in L^1_\rho(\Omega)$.*

**Theorem 3.8** *If $\mu$ is $g$-admissible then problem* (3.7) *possesses a unique solution.*

*Proof.* First assume that $\mu > 0$. Under the admissibility assumption, $U = \mathbb{K}[\mu]$ is a supersolution of (3.7). Let $\{D_n\}$ be an increasing sequence of smooth domains such that $\bar{D}_n \subset D_{n+1} \subset \Omega$ and $D_n \uparrow \Omega$. Let $u_n$ be the solution of problem (3.7) in $D_n$ with boundary data $h_n = U\big|_{\partial D_n}$. Then $\{u_n\}$ decreases and the limit $u = \lim u_n$ satisfies (3.7).

In the general case we define $\bar{U} = \mathbb{K}[|\mu|]$ and $U$, $u_n$ as before. By assumption $g(\bar{U}) \in L^1_\rho(\Omega)$ and $\bar{U}$ dominates $|u_n|$ for all $n$. Let $\eta$ be a non-negative function in $X(\Omega)$ and let $\zeta_n$ be the solution of the problem

$$\Delta\zeta = (\Delta\eta)\rho_n/\rho \ \text{ in } D_n, \ \ \zeta = 0 \ \text{ on } \partial D_n.$$

Then $\zeta_n \in X(D_n)$ and, since $\rho_n \to \rho$,

$$(\Delta\zeta_n) \to (\Delta\eta), \quad \zeta_n \to \eta.$$

In addition, $(\Delta\zeta_n)/\rho_n = (\Delta\eta)/\rho$ is bounded and, by (2.41), the sequence $\{\zeta_n/\rho_n\}$ is uniformly bounded.

The solutions $u_n$ satisfy,

$$\int_{D_n} (-u_n\Delta\zeta_n + g(u_n)\zeta_n)\, dx = -\int_{D_n} \mathbb{P}^{D_n}[h_n]\Delta\zeta_n dx. \tag{3.17}$$

The sequence $\{u_k : k > n\}$ is bounded in $W^{1,p}(D_n)$ for every $n$. Consequently there exists a subsequence (still denoted by $\{u_n\}$) which converges pointwise a.e. in $\Omega$. We denote its limit by $u$. Since $\{u_n\}$ is dominated by $\bar{U}$ it follows that

$$\lim_{n\to\infty} \int_{D_n} (-u_n\Delta\zeta_n + g(u_n)\zeta_n)\, dx = \int_\Omega (-u\Delta\eta + g(u)\eta)\, dx.$$

Furthermore,

$$\int_{D_n} \mathbb{P}^{D_n}[h_n]\Delta\zeta_n dx = \int_{D_n} U\Delta\eta(\rho_n/\rho)\, dx \to \int_\Omega U\Delta\eta dx = \int_\Omega \mathbb{K}[\mu]\Delta\eta\, dx.$$

Thus $u$ is the solution of (3.7).

□

*Remark.* If we do not assume that $g(0) = 0$ the admissibility condition becomes,

$$g(\mathbb{K}[\mu_+] + \rho(g(0))_+) \in L^1_\rho(\Omega) \quad \text{and} \quad g(-\mathbb{K}[\mu_-] - \rho(g(0))_-) \in L^1_\rho(\Omega). \tag{3.18}$$

# 4 The boundary trace of positive solutions

As before we assume that $\Omega$ is a bounded Lipschitz domain and $g \in \mathcal{G}$. We denote by $\rho$ the first eigenfunction of $-\Delta$ in $\Omega$ normalized by $\rho(x_0) = 1$ at some (fixed) point $x_0 \in \Omega$.

A function $u \in L^1_{loc}(\Omega)$ is a solution of the equation

$$-\Delta u + g(u) = 0 \quad \text{in } \Omega, \tag{4.1}$$

if $g \circ u \in L^1_{loc}(\Omega)$ and $u$ satisfies the equation in the distribution sense.

A function $u \in L^1_{loc}(\Omega)$ is a supersolution (resp. subsolution) of the equation (4.1) if $g \circ u \in L^1_{loc}(\Omega)$ and

$$-\Delta u + g \circ u \geq 0 \quad (\text{resp. } \leq 0)$$

in the distribution sense.

**Proposition 4.1** *Let $u$ be a positive solution of (4.1). If $g \circ u \in L^1_\rho(\Omega)$ then $u \in L^1_\rho(\Omega)$ and it possesses a boundary trace $\mu \in \mathfrak{M}(\partial\Omega)$, i.e., $u$ is the solution of the boundary value problem (3.7) with this measure $\mu$.*

*Proof.* If $v := \mathbb{G}[g \circ u]$ then $v \in L^1_\rho(\Omega)$ and $u + v$ is a positive harmonic function. Hence $u + v \in L^1_\rho(\Omega)$ and there exists a non-negative measure $\mu \in \mathfrak{M}(\partial\Omega)$ such that $u + v = \mathbb{K}[\mu]$. In view of (3.8), this implies our assertion. □

**Lemma 4.2** *If $u$ is a non-negative solution of (4.1) then $u \in C^1(\Omega)$.*

*Let $\{u_n\}$ be a sequence of non-negative solutions of (4.1) which is uniformly bounded in every compact subset of $\Omega$. Then there exists a subsequence $\{u_{n_j}\}$ which converges in $C^1(\bar{\Omega}')$ for every $\Omega' \Subset \Omega$ to a solution $u$ of (4.1).*

*Proof.* Since $g \circ u \in L^1_{\text{loc}}(\Omega)$ it follows that $u \in W^{1,p}_{\text{loc}}(\Omega)$ for some $p \in [1, N/(N-1))$. Let $\Omega'$ be a smooth domain such that $\Omega' \Subset \Omega$. By the trace imbedding theorem, $u$ possesses a trace $h \in L^1(\partial\Omega')$. If $U$ is the harmonic function in $\Omega'$ with boundary trace $h$ then $u < U$. Thus $u$ (and hence $g \circ u$) is bounded in every compact subset of $\Omega$. By elliptic p.d.e. estimates, $u \in C^1(\Omega)$.

The second assertion of the lemma follows from the first by a standard argument. □

**Theorem 4.3** *(i) Let $u$ be a non-negative supersolution (resp. subsolution) of (4.1). Then $u \in W^{1,p}_{\text{loc}}(\Omega)$ for some $p \in [1, N/(N-1))$. In particular, if $\Omega'$ is a $C^1$ domain such that $\Omega' \Subset \Omega$ then $u$ possesses a trace $h \in L^1(\partial\Omega')$.*

*(ii) If $u$ is a positive supersolution, there exists a non-negative solution $\underline{u} \leq u$ which is the largest among all solutions dominated by $u$.*

*If $u$ is a positive subsolution and $u$ is dominated by a solution $w$ of* (4.1) *then there exists a minimal solution $\bar{u}$ such that $u \leq \bar{u}$. In particular, if $g \in \mathcal{G}$ satisfies the Keller-Osserman condition then such a solution exists.*

(*iii*) *Under the assumptions of (ii), if $g \circ \underline{u} \in L^1_\rho(\Omega)$ (resp. $g \circ \bar{u} \in L^1_\rho(\Omega)$) then the boundary trace of $\underline{u}$ (resp. $\bar{u}$) is also the boundary trace of $u$ in the sense of Definition 3.5.*

*Proof.* First consider the case of a supersolution. Since $-\Delta u + g(u) \geq 0$ there exists a positive Radon measure $\tau$ in $\Omega$ such that

$$-\Delta u + g(u) = \tau \ \text{ in } \Omega.$$

Therefore $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and consequently $u$ possesses an $L^1$ trace on $\partial\Omega'$ for every $\Omega'$ as above.

Next, let $\{\Omega_n\}$ be a $C^1$ exhaustion of $\Omega$ which is also uniformly Lipschitz. Let $v_n$ be the solution of the boundary value problem

$$-\Delta v + g(v) = 0 \ \text{ in } \Omega_n, \ \ v = u \ \text{ on } \partial\Omega_n. \tag{4.2}$$

Since $u$ possesses a trace in $L^1(\partial\Omega_n)$ this boundary value problem possesses a (unique) solution. By the comparison principle $0 \leq v_n \leq u$ in $\Omega_n$. Therefore the sequence $\{v_n\}$ decreases and consequently it converges to a solution $\underline{u}$ of (4.1). Evidently this is the largest solution dominated by $u$.

Now suppose that $g \circ \underline{u} \in L^1_\rho(\Omega)$ (but not necessarily $g \circ u \in L^1_\rho(\Omega)$). By Proposition 4.1, $\underline{u} \in L^1_\rho(\Omega)$ and $\underline{u}$ possesses a boundary trace $\mu$. By the definition of $v_n$,

$$\int_{\partial\Omega_n} u d\omega_n = \int_{\partial\Omega_n} P^{\Omega_n}(x_0, y)u(y)dS = v_n(x_0) + \int_{\Omega_n} G^{\Omega_n}(x, x_0)g(v_n(x))dx$$
$$\to \underline{u}(x_0) + \int_\Omega G^\Omega(x, x_0)g(\underline{u}(x))dx.$$

Hence, taking a subsequence if necessary, we may assume that

$$u\chi_{\partial\Omega_n}\omega_n \rightharpoonup \mu'$$

where $\mu'$ is a measure on $\partial\Omega$ such that

$$\mu'(\partial\Omega) = \underline{u}(x_0) + \int_\Omega G^\Omega(x, x_0)g(\underline{u}(x))dx.$$

On the other hand, as $\mu$ is the boundary trace of $\underline{u}$,

$$\underline{u}(x_0) + \int_\Omega G^\Omega(x, x_0)g(\underline{u}(x))dx = \mu(\partial\Omega).$$

Thus $\mu(\partial\Omega) = \mu'(\partial\Omega)$. However, as $\underline{u} \leq u$, we have $\mu \leq \mu'$. This implies that $\mu = \mu'$.

Next we treat the case of a subsolution. The proof of (i) is the same as before. We turn to (ii). In the present case, the corresponding sequence $\{v_n\}$ is increasing and, in general, may not converge. But, as we assume that $u$ is dominated by a solution $w$, the sequence converges to a solution $\bar{u}$ which is clearly the smallest solution above $u$. In particular, if $g$ satisfies the Keller-Osserman condition then $\{v_n\}$ is uniformly bounded in every compact subset of $\Omega$ and consequently converges to a solution.

The proof of (iii) for subsolutions is again the same as in the case of supersolutions.

□

**Corollary 4.4** I. *Let $u$ be a non-negative supersolution of* (4.1)*. Let $A$ be a relatively open subset of $\partial\Omega$. Suppose that, for every Lipschitz domain $\Omega'$ such that*

$$\Omega' \subset \Omega, \quad \partial\Omega' \cap \partial\Omega \subset A, \tag{4.3}$$

*we have*

$$g \circ u \in L^1_\rho(\Omega'). \tag{4.4}$$

*Then both $u$ and $\underline{u}$ possess traces on $A$ and the two traces are equal.*

II. *Let $u$ be a non-negative subsolution of* (4.1)*. Let $A$ be a relatively open subset of $\partial\Omega$. Suppose that for every Lipschitz domain $\Omega'$ satisfying* (4.3) *we have*

$$g \circ \bar{u} \in L^1_\rho(\Omega'). \tag{4.5}$$

*Then both $u$ and $\bar{u}$ posses traces on $A$ and the two traces are equal.*

*Proof.* Let $u$ be a supersolution and let $\Omega'$ be a domain as above. Denote by $\rho'$ the first eigenfunction of $-\Delta$ in $\Omega'$ normalized by $\rho'(x_0) = 1$ for some $x_0 \in \Omega'$. Since $\rho' \leq c\rho$, (4.3) implies that $g \circ u \in L^1_{\rho'}(\Omega')$. Let $\underline{u}'$ denote the largest solution of (4.1) in $\Omega'$ dominated by $u$. Then $g \circ \underline{u}' \in L^1_{\rho'}(\Omega')$ and, by Theorem 4.3, $\underline{u}' \in L^1_\rho(\Omega')$ and $\underline{u}'$ has a trace $\nu'$ on $\partial\Omega'$ which is also the boundary trace of $u$ on $\partial\Omega$.

Let $\{\Omega_n\}$ be an increasing uniformly Lipschitz sequence of domains such that $\partial\Omega_n \cap \Omega$ is a $C^1$ surface, $D_n := \Omega \setminus \Omega_n$ is Lipschitz and

$$F_n := \partial\Omega_n \setminus \Omega \subset F^0_{n+1} \subset A, \quad \cup\Omega_n = \Omega, \quad \cup F^0_n = A,$$

where $F^0_n$ is the relative interior of $F_n$. Denote by $\underline{u}_n$ the largest solution dominated by $u$ in $\Omega_n$ and observe that $\{\underline{u}_n\}$ is decreasing and converges to a solution. Obviously this is the largest solution dominated by $u$, namely, $\underline{u}$.

Let $\tau_n$ be the trace of $\underline{u}_n$ on $\partial\Omega_n$. Put $\nu_n = \tau_n \chi_{F_n}$. Recall that $\tau_n$ is also the trace of $u$ so that

$$\nu'_n = \tau_n - \nu_n = u\chi_{\partial\Omega_n \setminus F_n} dS.$$

*Assertion A. There exists a Radon measure $\nu$ on $A$ such that $\nu_n \rightharpoonup \nu$ and $\nu$ is the trace of $u$, as well as of $\underline{u}$, on $A$.*

Let $E$ be a compact subset of $A$ and denote,

$$n(E) := \inf\{m \in \mathbb{N} : \ E \subset F^0_m\}.$$

In view of the fact that, for $n \geq n(E)$, $\nu_n$ is the trace of $u$, relative to $\Omega_n$, on a set $F^0_{n(E)}$ in which $E$ is strongly contained and the fact that $\{\Omega_n\}$ is Lipschitz, Lemma 2.6 implies that the set $\{\nu_n(E) : n \geq n(E)\}$ is bounded. By taking a sequence if necessary we may assume that

$$\nu_n \lfloor_E \rightharpoonup \nu_E.$$

Applying this procedure to $E = F_m$ for each $m \in \mathbb{N}$ and then using the diagonalization method we obtain a subsequence, again denoted by $\{\nu_n\}$, such that

$$\nu_n \rightharpoonup \nu$$

where $\nu$ is a Radon measure on $A$ (not necessarily bounded).

Next we wish to show that $\nu$ is the trace of $u$ on $A$ relative to $\Omega$. To this purpose we construct a $C^1$ exhaustion of $\Omega$, say $\{D_n\}$, such that $D_n \Subset \Omega_n$ and $\partial D_n = \Gamma_n \cup \Gamma'_n$ where

$$\Gamma'_n = \partial\Omega_n \cap \{y \in \Omega : \mathrm{dist}\,(y, F_n) \geq \epsilon_n\}$$
$$\Gamma_n \subset \{y \in \Omega_n : \mathrm{dist}\,(y, F_n) < \epsilon_n\},$$

where $0 < \epsilon_n < \frac{1}{2}\mathrm{dist}\,(F_n, \partial\Omega \setminus A)$ is chosen so that

$$\mathbb{H}_{N-1}\chi_{\Gamma_n} \rightharpoonup \mathbb{H}_{N-1}\chi_A \ \text{ and } \ u\chi_{\Gamma_n} d\omega^n \rightharpoonup \nu.$$

Here $d\omega^n$ is the harmonic measure in $D_n$. This is possible because, if $\Gamma_n$ is sufficiently close to $\partial\Omega_n$, then

$$u\chi_{\Gamma_n} d\omega^n - \nu_n \chi_{F_n} \rightharpoonup 0.$$

(As usual in this paper, $\nu_n\chi_{F_n}$ denotes the Borel measure in $\mathbb{R}^N$ that is equal to $\nu_n$ on $F_n$ and zero elsewhere.) This implies that $\nu$ is the trace of $u$ on $A$.

Since $\nu_n$ is also the trace of $\underline{u}_n$ on $F_n$ it follows that, if $\Gamma_n$ is sufficiently close to $\partial\Omega_n$,

$$\underline{u}_n\chi_{\Gamma_n}d\omega^n - \nu_n\chi_{F_n} \rightharpoonup 0.$$

As $\underline{u}_n \downarrow \underline{u}$ we deduce that $\nu$ is also the trace of $\underline{u}$ on $A$.

If $u$ is a subsolution the argument is essentially the same. Let $\bar{u}_n$ be the smallest solution that dominates $u$ in $\Omega_n$. Then the sequence $\{\bar{u}_n\}$ is increasing, but it is dominated by a solution $w$. Therefore it converges to a solution and this is the smallest solution dominating $u$, namely, $\bar{u}$. By Theorem 4.3, $\underline{u}_n$ and $u\lfloor$ possess the same trace on $\partial\Omega_n$. Let $\tau_n$ be the trace of $\underline{u}_n$ on $\partial\Omega_n$ and put $\nu_n = \tau_n\chi_{F_n}$. The rest of the proof is as before.

□

**Definition 4.5** *Let $u$ be a positive supersolution, respectively subsolution, of* (4.1)*. A point $y \in \partial\Omega$ is a* regular boundary point *relative to $u$ if there exists an open neighborhood $D$ of $y$ such that $g \circ u \in L^1_\rho(\Omega \cap D)$. If no such neighborhood exists we say that $y$ is a* singular boundary point *relative to $u$.*

*The set of regular boundary points of $u$ is denoted by $\mathcal{R}(u)$; its complement on the boundary is denoted by $\mathcal{S}(u)$. Evidently $\mathcal{R}(u)$ is relatively open.*

**Theorem 4.6** *Let $u$ be a positive solution of* (4.1) *in $\Omega$. Then $u$ possesses a trace on $\mathcal{R}(u)$, given by a Radon measure $\nu$.*

*Furthermore, for every compact set $F \subset \mathcal{R}(u)$,*

$$\int_\Omega (-u\Delta\eta + g(u)\eta)\,dx = -\int_\Omega (\mathbb{K}[\nu\chi_F]\Delta\eta)\,dx \qquad (4.6)$$

*for every $\eta \in X(\Omega)$ such that* $\operatorname{supp}\eta \cap \partial\Omega \subset F$*.*

*Proof.* The first assertion is an immediate consequence of Corollary 4.4.

We turn to the proof of the second assertion. Let $F$ be a compact subset of $\mathcal{R}(u)$ and let $\eta \in X(\Omega)$ be a function such that the following conditions hold for some open set $E_\eta$:

$$\operatorname{supp}\eta \subset \bar{\Omega} \cap E_\eta, \quad F \subset E_\eta \cap \partial\Omega, \quad \bar{E}_\eta \cap \mathcal{S}(u) = \emptyset, \quad x_0 \in D_\eta := \Omega \cap E_\eta.$$

By Definition 4.5, if $D$ is a subdomain of $\Omega$ such that $\bar{D} \cap \mathcal{S}(u) = \emptyset$ then $g \circ u \in L^1_\rho(D)$, where $\rho$ is the first normalized eigenfunction of $\Omega$. Let $E$ be a $C^2$ domain such that

$$\bar{E}_\eta \subset E, \quad \mathbb{H}_{N-1}(\partial\Omega \cap \partial E) = 0, \quad \bar{E} \cap \mathcal{S}(u) = \emptyset.$$

Put $D := E \cap \Omega$ and note that $g \circ u \in L^1_\rho(D)$.

If $\phi$ denotes the first normalized eigenfunction in $D$ then $\phi \leq c\rho$ for some positive constant $c$. Therefore the fact that $g \circ u \in L^1_\rho(D)$ implies that $g \circ u \in L^1_\phi(D)$ and the properties of $\eta$ imply that $\eta \in X(D)$. Hence $u$ possesses a boundary trace $\tau^D$ on $\partial D$ and

$$\int_D (-u\Delta\eta + g(u)\eta)\, dx = -\int_D \left(\mathbb{K}^D[\tau^D]\Delta\eta\right) dx. \tag{4.7}$$

Let $\Gamma = \bar{E} \cap \partial\Omega$ and $\Gamma' = \partial D \setminus \Gamma$; note that $\Gamma \cap \mathcal{S}(u) = \emptyset$ and $\eta$ vanishes in a neighborhood of $\partial E \cap \bar{\Omega}$. Put $\tau^D_\Gamma = \tau^D \chi_\Gamma$ and $\tau^D_{\Gamma'} = \tau^D - \tau^D_\Gamma$. Then $d\tau^D_{\Gamma'} = udS$ on $\Gamma'$ and, as $u \in C(\bar{D} \setminus \Gamma)$,

$$\mathbb{K}^D[\tau^D_{\Gamma'}] \in C(\bar{D} \setminus \Gamma).$$

Furthermore $\eta$ vanishes in a neighborhood of $\Gamma'$ and consequently

$$\begin{aligned}\int_D \left(\mathbb{K}^D[\tau^D_{\Gamma'}]\Delta\eta\right) dx &= \int_D \left(\int_{\partial D\setminus\Gamma} P^D(x,y)u(y)dS_y\right) \Delta\eta(x)dx \\ &= \int_{\partial D\setminus\Gamma} \left(\int_D P^D(x,y)\Delta\eta(x)dx\right) u(y)dS_y = 0.\end{aligned}$$

Thus

$$\int_\Omega (-u\Delta\eta + g(u)\eta)\, dx = -\int_\Omega \mathbb{K}^D[\tau^D_\Gamma]\Delta\eta\, dx. \tag{4.8}$$

(Changing the domain of integration from $D$ to $\Omega$ makes no difference since $\eta$ vanishes in $\Omega \setminus D$.)

Now, $\tau^D_\Gamma$ is the trace of $u$ on $\Gamma$ relative to $D$ while $\nu\chi_\Gamma$ is the trace of $u$ on $\Gamma$ relative to $\Omega$. Since $D \subset \Omega$ it follows that

$$\tau^D_\Gamma \leq \nu\chi_\Gamma. \tag{4.9}$$

Let $\{E^j\}$ be an increasing sequence of $C^2$ domains such that each domain possesses the same properties as $E$ and,

$$\bar{E}^j \cap \partial\Omega = \bar{E} \cap \partial\Omega = \Gamma, \quad \text{and} \quad D^j := E^j \cap \Omega \uparrow \Omega. \tag{4.10}$$

For each $j \in \mathbb{N}$ and $y \in \Gamma$, the function $K^{D^j}(\cdot, y)$ is harmonic in $D^j$, vanishes on $\partial D^j \setminus \{y\}$ and $K^{D^j}(x_0, y) = 1$. Furthermore the sequence $\{K^{D^j}(\cdot, y)\}$ is non-decreasing. Therefore it converges uniformly in compact subsets of $(\Omega \cup \Gamma) \setminus \{y\}$. The limit is the corresponding kernel function in $\Omega$, namely $K^\Omega(\cdot, y)$. (Recall that the kernel function is unique.)

In view of (4.9), the sequence $\{\tau_\Gamma^{D^j}\}$ is bounded. Therefore there exists a subsequence, which we still denote by $\{\tau_\Gamma^{D^j}\}$, such that

$$\tau_\Gamma^{D^j} \rightharpoonup \tau_\Gamma$$

weakly relative to $C(\Gamma)$. Combining these facts we obtain,

$$\mathbb{K}^{D_j}[\tau_\Gamma^{D^j}] \to \mathbb{K}^\Omega[\tau_\Gamma].$$

Hence, by (4.7),

$$\int_\Omega (-u\Delta\eta + g(u)\eta)\,dx = -\int_\Omega \left(\mathbb{K}^\Omega[\tau_\Gamma]\Delta\eta\right)dx. \tag{4.11}$$

Finally, as $\tau_\Gamma^{D_j}$ is the trace of $u$ on $\Gamma$ relative to $D_j$ then, in view of (4.10), the limit $\tau_\Gamma$ is the trace of $u$ on $\Gamma$ relative to $\Omega$, i.e.,

$$\tau_\Gamma = \nu\chi_\Gamma.$$

This relation and (4.11) imply (4.6).

□

**Theorem 4.7** I. *Let $u$ be a positive supersolution of* (4.1) *in* $\Omega$ *and let* $\underline{u}$ *be the largest solution dominated by* $u$*. Then,*

$$\mathcal{S}(u) = \mathcal{S}(\underline{u}), \quad \mathcal{R}(u) = \mathcal{R}(\underline{u}). \tag{4.12}$$

*Both $u$ and $\underline{u}$ possess a trace on $\mathcal{R}(u)$ and the two traces are equal.*
II. *Let $u$ be a positive subsolution of* (4.1) *in* $\Omega$ *and let $\bar{u}$ be the smallest solution which dominates $u$. If $u$ is dominated by a solution $w$ of* (4.1) *then both $u$ and $\bar{u}$ possess a trace on $\mathcal{R}(w)$* (*which is contained in* $\mathcal{R}(u)$) *and the two traces are equal on this set.*

*In particular, if $\mathcal{R}(w) = \mathcal{R}(u)$ then* (4.12)*, with $\underline{u}$ replaced by $\bar{u}$, holds and both $u$ and $\bar{u}$ possess a trace on $\mathcal{R}(u)$, the two traces being equal.*
III. *Let $\nu$ denote the trace of $u$ on $\mathcal{R}(u)$. Then, for every compact set $F \subset \mathcal{R}(u)$,*

$$\int_\Omega (-u\Delta\eta + g(u)\eta)\,dx \begin{cases} \geq -\int_\Omega (\mathbb{K}[\nu\chi_F]\Delta\eta)\,dx, & u \text{ supersolution,} \\ \leq -\int_\Omega (\mathbb{K}[\nu\chi_F]\Delta\eta)\,dx, & u \text{ subsolution} \end{cases} \tag{4.13}$$

*for every $\eta \in X(\Omega)$, $\eta \geq 0$, such that* $\operatorname{supp}\eta \cap \partial\Omega \subset F$.

*Proof.* Part I. is a consequence of Corollary 4.4 I.

The first assertion in II. follows from Corollary 4.4 II. with $A = \mathcal{R}(w)$. The second assertion in II. is an immediate consequence of the first.

By Theorem 4.6, $\underline{u}$ (resp. $\bar{u}$) satisfy (4.6), where $\nu$ is the trace of $\underline{u}$ (resp. $\bar{u}$) on $\mathcal{R}(u)$. Since $\nu$ is also the trace of $u$ on $\mathcal{R}(u)$ we obtain statement III.

□

**Theorem 4.8** *Assume that $g \in \mathcal{G}$ satisfies the Keller-Osserman condition.* ($i$) *Let $u$ be a positive solution of* (4.1) *and let* $\{\Omega_n\}$ *be a Lipschitz exhaustion of* $\Omega$. *If $y \in \mathcal{S}(u)$ then, for every nonnegative $Z \in C(\bar{\Omega})$ such that $Z(y) \neq 0$*

$$\lim \int_{\partial\Omega_n} Zud\omega_n = \infty. \tag{4.14}$$

($ii$) *Let $u$ be a positive supersolution of* (4.1) *and let* $\{\Omega_n\}$ *be a $C^1$ exhaustion of* $\Omega$. *If $y \in \mathcal{S}(u)$ then* (4.14) *holds for every nonnegative $Z \in C(\bar{\Omega})$ such that $Z(y) \neq 0$.*

The proof of satement (i) is essentially the same as for the corresponding result in smooth domains [25, Lemma 2.8] and therefore will be omitted. In fact the assumption that $g$ satisfies the Keller-Osserman condition implies that the set of conditions II in [25, Lemma 2.8] is satisfied. Here too, the Keller-Osserman condition can be replaced by the weaker set of conditions II in the same way as in [25].

Part (ii) is a consequence of Theorem 4.7 and statement (i).

□

**Definition 4.9** *Let $g \in \mathcal{G}$. Let $u$ be a positive solution of* (4.1) *with regular boundary set $\mathcal{R}(u)$ and singular boundary set $\mathcal{S}(u)$. The Radon measure $\nu$ in $\mathcal{R}(u)$ associated with $u$ as in Theorem 4.6 is called the* regular part of the trace *of $u$. The couple $(\nu, \mathcal{S}(u))$ is called the* boundary trace *of $u$ on $\partial\Omega$. This trace is also represented by the (possibly unbounded) Borel measure $\bar{\nu}$ given by*

$$\bar{\nu}(E) = \begin{cases} \nu(E), & \text{if } E \subset \mathcal{R}(u) \\ \infty, & \text{otherwise.} \end{cases} \tag{4.15}$$

*The boundary trace of $u$ in the sense of this definition will be denoted by* $\mathrm{tr}_{\partial\Omega} u$.

*Let*

$$V_\nu := \sup\{u_{\nu\chi_F} : \ F \subset \mathcal{R}(u), \ F \text{ compact}\} \tag{4.16}$$

*where $u_{\nu\chi_F}$ denotes the solution of* (3.7) *with $\mu = \nu\chi_F$. Then $V_\nu$ is called the* semi-regular component *of $u$.*

*Remark.* Let $\tau$ be a Radon measure on a relatively open set $A \subset \partial\Omega$. Suppose that for every compact set $F \subset A$, $u_{\tau\chi_F}$ is defined. If $V_\tau$ is defined as above, it need not be a solution of (4.1) or even be finite. However, if $g$ satisfies the Keller–Osserman condition or if $u_{\tau\chi_F}$ is dominated by a solution $w$, independent of $F$, then $V_\tau$ is a solution.

**Definition 4.10** *A compact set $F \subset \partial\Omega$ is removable relative to* (4.1) *if the only non-negative solution $u \in C(\bar{\Omega} \setminus F)$ which vanishes on $\bar{\Omega} \setminus F$ is the trivial solution $u = 0$.*

*Remark.* In the case of power nonlinearities in smooth domains there exists a complete characterization of removable sets (see [24] and the references therein). In a later section we shall derive such a characterization for a family of Lipschitz domains.

**Lemma 4.11** *Let $g \in \mathcal{G}$ and assume that $g$ satisfies the Keller-Osserman condition. Let $F \subset \partial\Omega$ be a compact set and denote by $\mathcal{U}_F$ the class of solutions $u$ of* (4.1) *which satisfy the condition,*

$$u \in C(\bar{\Omega} \setminus F), \quad u = 0 \ \text{ on } \partial\Omega \setminus F. \tag{4.17}$$

*Then there exists a function $U_F \in \mathcal{U}_F$ such that*

$$u \leq U_F \quad \forall u \in \mathcal{U}_F.$$

*Furthermore, $\mathcal{S}(U_F) =: F' \subset F$; $F'$ need not be equal to $F$.*

The proof is standard and will be omitted.

**Definition 4.12** *$U_F$ is called the* maximal solution *associated with $F$. The set $F' = \mathcal{S}(U_F)$ is called the $g$-kernel of $F$ and denoted by $k_g(F)$.*

*Note.* The situation $\mathcal{S}(U_F) \subsetneq F$ occurs if and only if there exists a closed set $F' \subset F$ such that $F \setminus F'$ is a non-empty removable set. In this case $U_F = U_{F'}$.

**Lemma 4.13** *Let $F_1, F_2$ be two compact subsets of $\partial\Omega$. Then,*

$$F_1 \subset F_2 \Longrightarrow U_{F_1} \leq U_{F_2} \tag{4.18}$$

*and*

$$U_{F_1 \cup F_2} \leq U_{F_1} + U_{F_2}. \tag{4.19}$$

*If $F$ is a compact subset of $\partial\Omega$ and $\{N_k\}$ is a decreasing sequence of relatively open neighborhoods of $F$ such that $\bar{N}_{k+1} \subset N_k$ and $\cap N_k = F$ then*

$$U_{\bar{N}_k} \to U_F \tag{4.20}$$

*uniformly in compact subsets of $\Omega$.*

*Proof.* The first statement is an immediate consequence of the definition of maximal solution.

Next we verify (4.20). By (4.18) the sequence $\{U_{\bar{N}_k}\}$ decreases and therefore it converges to a solution $U$. Clearly $U$ has trace zero outside $F$ so that $U \leq U_F$ On the other hand, for every $k$, $U_{\bar{N}_k} \geq U_F$. Hence $U = U_F$

We turn to the verification of (4.19). Let $u$ be a positive solution of (5.1) which vanishes on $\partial\Omega \setminus (F_1 \cup F_2)$. We shall show that there exists solutions $u_1, u_2$ of (5.1) such that

$$u_i = 0 \ \text{ on } \partial\Omega \setminus F_i, \ \ u \leq u_1 + u_2. \tag{4.21}$$

First we prove this statement in the case where $F_1 \cap F_2 = \emptyset$. Let $E_1, E_2$ be $C^1$ domains such that $\bar{E}_1 \cap \bar{E}_2 = \emptyset$ and $F_i \subset E_i \cap \partial\Omega$, (i=1,2). Let $\{\Omega_n\}$ be a Lipschitz exhaustion of $\Omega$ and put $A_{n,i} = \partial\Omega_n \cap E_i$, (i=1,2). Let $v_{n,i}$ be the solution of (5.1) in $\Omega_n$ with boundary data $u\chi_{A_{n,i}}$ and $v_n$ be the solution in $\Omega_n$ with boundary data $u(1 - \chi_{A_{n,1} \cup A_{n,2}})$. Then

$$u \leq v_n + v_{n,1} + v_{n,2}.$$

By taking a subsequence if necessary we may assume that the sequences $\{v_n\}$, $\{v_{n,1}\}$, $\{v_{n,2}\}$ converge. Then $\lim v_{n,i} = U_i$ where $U_i$ vanishes on $\partial\Omega \setminus E_i$, (i=1,2). In addition, as the trace of $u$ on $\partial\Omega \setminus (F_1 \cup F_2)$ is zero, we have $\lim v_n = 0$. Thus

$$u \leq U_1 + U_2.$$

Now take decreasing sequences of $C^1$ domains $\{E_{k,1}\}$, $\{E_{k,2}\}$ such that

$$\bar{E}_{k,1} \cap \bar{E}_{k,2} = \emptyset, \quad F_i \subset E_{k,i} \cap \partial\Omega, \quad \bar{E}_{k,i} \cap \partial\Omega \downarrow F_i \quad i = 1, 2.$$

Construct $U_{k,i}$ corresponding to $E_{k,i}$ in the same way that $U_i$ corresponds to $E_i$. Then,

$$u \leq U_{k,1} + U_{k,2}$$

and, by (4.20), taking a subsequence if necessary,

$$u_i := \lim_{k\to\infty} U_{k,i} = 0 \ \text{ on } \partial\Omega \setminus F_i \ \ , \quad i = 1, 2.$$

This proves (4.21) in the case where $F_1, F_2$ are disjoint.

In the general case, let $\{N_j\}$ be a decreasing sequence of relatively open neighborhoods of $F_1 \cap F_2$ such that

$$\bar{N}_{j+1} \subset N_j, \quad \cap N_j = F_1 \cap F_2.$$

Put $F'_{j,2} = F_2 \setminus N_j$. Let $\{M_j\}$ be a decreasing sequence of relatively open neighborhoods of $F_1$ such that

$$\bar{M}_{j+1} \subset M_j, \quad \cap M_j = F_1, \quad \bar{M}_j \cap F'_{j,2} = \emptyset.$$

Put $F'_{j,1} := \bar{M}_j$.

Let $v_j$ be the largest solution dominated by $u$ and vanishing on the complement of $F'_{j,1} \cup F'_{j,2}$:

$$\begin{aligned}\partial\Omega \setminus (F'_{j,1} \cup F'_{j,2}) &= \partial\Omega \setminus \big((F_1 \cup F_2) \setminus (N_j \setminus \bar{M}_j)\big) \\ &= (\partial\Omega \setminus (F_1 \cup F_2)) \cup (N_j \setminus \bar{M}_j).\end{aligned}$$

Furthermore, $(u - U_{\bar{N}_j \setminus M_j})_+$ is a subsolution which is dominated by $u$ and vanishes on the complement of $F'_{j,1} \cup F'_{j,2}$. Therefore $v_j$ satisfies

$$u \geq v_j \geq (u - U_{\bar{N}_j \setminus M_j})_+,$$

which implies,

$$0 \leq u - v_j \leq U_{\bar{N}_j \setminus M_j} \leq U_{\bar{N}_j}.$$

By (4.20), $U_{\bar{N}_j} \downarrow U_{F_1 \cap F_2}$. Taking a converging subsequence $v_{j_i} \to v$ we obtain

$$0 \leq u - v \leq U_{F_1 \cap F_2}.$$

By the previous part of the proof there exist solutions $v_{j,1}$, $v_{j,2}$, whose boundary trace is supported in $F'_{j,1}$ and $F'_{j,2}$ respectively, such that

$$v_j \leq v_{j,1} + v_{j,2}.$$

Taking a subsequence we may assume convergence of $\{v_{j,1}\}$ and $\{v_{j,2}\}$. Then $u_i = \lim v_{j,i}$ has boundary trace supported in $F_i$. Finally,

$$u \leq v + U_{F_1 \cap F_2} \leq u_1 + u_2 + U_{F_1 \cap F_2}$$

and $\mathrm{tr}_{\partial\Omega} u_1$ is supported in $F_1$ while $\mathrm{tr}_{\partial\Omega}(u_2 + U_{F_1 \cap F_2})$ is supported in $F_2$. Since $u - u_1$ is a subsolution dominated by the supersolution $u_2 + U_{F_1 \cap F_2}$ there exists a solution $w_2$ between them and we obtain

$$u \leq u_1 + w_2$$

where $\mathrm{tr}_{\partial\Omega} w_2$ is supported in $F_2$.

□

The next theorem deals with some aspects of the generalized boundary value problem:

$$\begin{aligned} -\Delta u + g \circ u = 0, \quad u \geq 0 \ \text{ in } \Omega, \\ \mathrm{tr}_{\partial\Omega} = (\nu, F), \end{aligned} \tag{4.22}$$

where $F \subset \partial\Omega$ is a compact set and $\nu$ is a (non-negative) Radon measure on $\partial\Omega \setminus F$.

**Theorem 4.14** *Let $g \in \mathcal{G}$ and assume that $g$ is convex and satisfies the Keller-Osserman condition.*
EXISTENCE. *The following set of conditions is necessary and sufficient for existence of a solution $u$ of (4.22):*
*(i) For every compact set $E \subset \partial\Omega \setminus F$, the problem*

$$-\Delta u + g(u) = 0 \ \text{ in } \Omega, \ \ u = \nu\chi_E \ \text{ on } \partial\Omega, \tag{4.23}$$

*possesses a solution.*
*(ii) If $k_g(F) = F'$, then $F \setminus F' \subset \mathcal{S}(V_\nu)$.*
*When this holds,*

$$V_\nu \leq u \leq V_\nu + U_F. \tag{4.24}$$

*Furthermore if $F$ is a removable set then (4.22) possesses exactly one solution.*

UNIQUENESS. *Given a compact set $F \subset \partial\Omega$, assume that*

$$U_E \text{ is the unique solution with trace } (0, k_g(E)) \tag{4.25}$$

*for every compact $E \subset F$. Under this assumption:*
(a) *If $u$ is a solution of (4.22) then*

$$\max(V_\nu, U_F) \leq u \leq V_\nu + U_F. \tag{4.26}$$

(b) *Equation (5.1) possesses at most one solution satisfying (4.26).*
(c) *Condition (4.25) is necessary and sufficient in order that (4.22) posses at most one solution.*

MONOTONICITY.
(d) *Let $u_1, u_2$ be two positive solutions of (4.1) with boundary traces $(\nu_1, F_1)$ and $(\nu_2, F_2)$ respectively. Suppose that $F_1 \subset F_2$ and that $\nu_1 \leq \nu_2\chi_{F_1} =: \nu_2'$. If (4.25) holds for $F = F_2$ then $u_1 \leq u_2$.*

*Proof.* First assume that there exists a solution $u$ of (4.22). By Theorem 4.6 condition (i) holds. Consequently $V_\nu$ is well defined by (4.16).

Since $V_\nu \le u$ the function $w := u - V_\nu$ is a subsolution of (4.1). Indeed, as $g$ is convex and $g(0) = 0$ we have

$$g(a) + g(b) \le g(a + b) \quad \forall a, b \in \mathbb{R}_+. \tag{4.27}$$

Therefore

$$0 = -\Delta w + (g(u) - g(V_\nu) \ge -\Delta w + g(w).$$

By Theorem 4.3, as $g$ satisfies the Keller-Osserman condition, there exists a solution $\bar{w}$ of (4.1) which is the smallest solution dominating $w$.

By Theorem 4.7, the traces of $w$ and $\bar{w}$ are equal on $A = \mathcal{R}(u) \subset \mathcal{R}(\bar{w})$. Clearly the trace of $w$ on $\mathcal{R}(u)$ is zero. The definitions of $V_\nu$ and $\bar{w}$ imply,

$$\max(V_\nu, \bar{w}) \le u \le V_\nu + \bar{w}. \tag{4.28}$$

Therefore

$$\mathcal{S}(\bar{w}) \cup \mathcal{S}(V_\nu) = \mathcal{S}(u).$$

In addition, as $\bar{w}$ has trace zero in $\partial\Omega \setminus F$, it follows, by the definition of the maximal function, that

$$\bar{w} \le U_F \text{ and consequently } \mathcal{S}(\bar{w}) \subset k_g(F).$$

These observations imply that condition (ii) must hold. Inequality (4.24) follows from (4.28) and this inequality implies that if $F$ is a removable set then (4.22) possesses exactly one solution.

Now we assume that conditions (i) and (ii) hold and prove existence of a solution. The function $V_\nu$ is well defined and $V_\nu + U_F$ is a supersolution of (4.1) whose boundary trace is $(\nu, F)$. Therefore, by Theorem 4.7, the largest solution dominated by it has the same boundary trace, i.e. solves (4.22).

Next assume that condition (4.25) is satisfied. It is obvious that (4.25) is necessary for uniqueness. In addition, (4.25) implies that $U_F \le u$ and consequently (4.24) implies (4.26). It is also clear that (b) implies the sufficiency part of (c).

Therefore it remains to prove statements (b) and (d). Let $u$ be the smallest solution dominating the subsolution $\max(V_\nu, U_F)$ and let $v$ be the largest solution dominated by $V_\nu + U_F$.

To establish (b) we must show that $u = v$. By (4.26) $v - u \le V_\nu$. In addition the subsolution $v - u$ has trace zero on $\partial\Omega \setminus F$. Therefore

$$v - u \le \min(V_\nu, U_F). \tag{4.29}$$

Let $\{N_k\}$ be a decreasing sequence of open sets converging to $F$ such that $N_{k+1} \Subset N_k$. Assuming for a moment that *$\nu$ is a finite measure,* the trace of $V_\nu$ on $N_k$ is $\nu_k := \nu\chi_{N_k}$ and it tends to zero as $k \to \infty$. Therefore, in this case,

$$\min(V_\nu, U_F) \le V_{\nu_k} \to 0$$

and hence $u = v$. Of course this also implies uniqueness (statement (c)) in the case where $\nu$ is a finite measure.

In the general case we argue as follows. Let $v_k$ be the unique solution with boundary trace $(\nu'_k, \bar{N}_k)$ where $\nu'_k = \nu(1 - \chi_{\bar{N}_k})$. By taking a subsequence if necessary, we may assume that $\{v_k\}$ converges to a solution $v'$. By (4.26),

$$\max(V_{\nu'_k}, U_{\bar{N}_k}) \le v_k \le V_{\nu'_k} + U_{\bar{N}_k}$$

and, by the previous part of the proof, $v_k$ is the largest solution dominated by $V_{\nu'_k} + U_{\bar{N}_k}$. We claim that if $w$ is a solution of (5.1) then

$$V_\nu \le w \le V_\nu + U_F \Longrightarrow w \le V_{\nu'_k} + U_{\bar{N}_k}. \tag{4.30}$$

Indeed,

$$w \le V_\nu + U_F \Longrightarrow w \le V_{\nu'_k} + V_{\nu_k} + U_F \Longrightarrow w \le V_{\nu'_k} + U_{\bar{N}_k} + U_F \Longrightarrow w \le V_{\nu'_k} + 2U_{\bar{N}_k}.$$

Thus

$$0 \le w - V_{\nu'_k} \le 2U_{\bar{N}_k}$$

which implies

$$w - V_{\nu'_k} \le U_{\bar{N}_k},$$

because any solution (or subsolution) dominated by $2U_{\bar{N}_k}$ is also dominated by $U_{\bar{N}_k}$.

Hence $v_k \ge v$ and consequently $v' \ge v$.

By (4.20) $U_{\bar{N}_k} \downarrow U_F$ and by definition $V_{\nu'_k} \uparrow V_\nu$. Therefore

$$\max(V_\nu, U_F) \le v' \le V_\nu + U_F.$$

Since $v$ is the largest solution dominated by $V_\nu + U_F$ and $v \le v'$ it follows that $v = v'$.

Let $u_k$ be the unique solution with boundary trace $(\nu'_k, k_g(F))$. By (4.26),

$$\max(V_{\nu'_k}, U_{k_g(F)}) \le u_k \le V_{\nu'_k} + U_{k_g(F)}.$$

Since $u_k \le u$ and $\{u_k\}$ increases (because $\{V_{\nu'_k}\}$ increases) it follows that $u' = \lim u_k \le u$. Furthermore,

$$\max(V_\nu, U_{k_g(F)}) \le u' \le V_\nu + U_{k_g(F)}.$$

If (4.22) possesses a solution then condition (ii) holds. Therefore for any solution $w$ of (5.1)

$$\max(V_\nu, U_{k_g(F)}) \leq w \Longrightarrow \max(V_\nu, U_F) \leq w.$$

Hence $\max(V_\nu, U_F) \leq u'$ and, as $u' \leq u$ we conclude that $u' = u$.

Finally, for every $\epsilon > 0$,

$$(1-\epsilon)V_{\nu'_k} + \epsilon U_{k_g(F)} \leq u_k$$

and consequently

$$v_k - u_k \leq V_{\nu'_k} + U_{\bar{N}_k} - \big((1-\epsilon)V_{\nu'_k} + \epsilon U_{k_g(F)})\big) =$$
$$U_{\bar{N}_k} - (1-\epsilon)U_{k_g(F)} + \epsilon V_{\nu'_k} \leq U_{\overline{N_k \setminus F}} + U_F - (1-\epsilon)U_{k_g(F)} + \epsilon V_{\nu'_k} \leq$$
$$\epsilon(U_F + V_{\nu'_k}) \to \epsilon(U_F + V_\nu).$$

This implies $u_k = v_k$ and hence $u = v$. This establishes statement (b) and hence the sufficiency in (c).

Finally we establish monotonicity. Let $v_i$ be the unique solution of (5.1) with boundary trace $(\nu_i, F_i)$, (i=1,2). Then $v_i$ is the largest solution dominated by $V_{\nu_i} + U_{F_i}$ (i=1,2). The argument used in proving (4.30) yields

$$V_{\nu_1} \leq w \leq V_{\nu_1} + U_{F_1} \Longrightarrow w \leq V_{\nu_2} + U_{F_2}. \tag{4.31}$$

This implies $v_1 \leq v_2$.

□

# 5 Equation with power nonlinearity in a Lipschitz domain

In this section we study the trace problem and the associated boundary value problem for equation

$$-\Delta u + |u|^{q-1} u = 0 \tag{5.1}$$

in a Lipschitz bounded domain $\Omega$ and $q > 1$. The main difference between the smooth cases and the Lipschitz case is the fact that the notion of critical exponent is pointwise. If $G$ is any domain in $\mathbb{R}^N$ we denote

$$\mathcal{U}(G) := \{ \text{ the set of solutions (5.1) in } G\}. \tag{5.2}$$

and $\mathcal{U}_+(G) = \{u \in \mathcal{U}(G) : u \geq 0 \text{ in } G\}$. Notice that any solution is at least $C^3$ in $G$ and any positive solution is $C^\infty$. The next result is proved separately by Keller [19] and Osserman [29].

**Proposition 5.1** *Let $q > 1$, $\Omega \subset \mathbb{R}^N$ be any domain and $u \geq \in C(\Omega)$ be a weak solution of*

$$-\Delta u + Au^q \leq B \quad \text{in } \Omega. \tag{5.3}$$

*for some $A > 0$ and $B \geq 0$. Then there exist $C_i(N, q) > 0$ ($i = 1, 2$) such that*

$$u(x) \leq C_1 \left( \frac{1}{\sqrt{A}\text{dist}\,(x, \partial\Omega)} \right)^{2/(q-1)} + C_2 \left( \frac{B}{A} \right)^{1/q} \quad \forall x \in \Omega. \tag{5.4}$$

For a solution of (5.1) in $\Omega$ which vanishes on the boundary except at one point, we have a more precise estimate.

**Proposition 5.2** *Let $q > 1$, $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $y \in \partial\Omega$ and $u \in \mathcal{U}_+(\Omega)$ is continuous in $\overline{\Omega} \setminus \{y\})$ and vanishes on $\partial\Omega \setminus \{y\}$. Then there exists $C_3(N, q, \Omega) > 0$ and $\alpha \in (0, 1]$ such that*

$$u(x) \leq C_3 \,(\text{dist}\,(x, \partial\Omega))^{\alpha}\, |x - y|^{-2/(q-1)-\alpha} \quad \forall x \in \Omega. \tag{5.5}$$

*Furthermore $\alpha = 1$ if $\Omega$ is a $W^{2,s}$ domain with $s > N$.*

*Proof.* By translation we can assume that $y = 0$. Let $\tilde{u}$ be the extension of $u_+$ by zero outside $\overline{\Omega} \setminus \{0\}$. Then it is a subsolution of (5.1) in $\mathbb{R}^N \setminus \{0\}$ (see [14] e.g.). Thus

$$\tilde{u}(x) \leq C_1 |x|^{-2/(q-1)} \quad \forall x \neq 0,$$

and, with the same estimate for $u_-$, we derive

$$|u(x)| \leq C_1 |x|^{-2/(q-1)} \quad \forall x \in \Omega. \tag{5.6}$$

Next we define the transformation $T_k$ ($k > 0$) by $T_k[u](x) = k^{-2/(q-1)} u(k^{-1}x)$, valid for any $x \in \Omega_k = k\Omega$. Then $u_k := T_k[u]$ satisfies the same equation as $u$ in $\Omega_k$, is continuous in $\overline{\Omega}_k \setminus \{0\}$ and vanishes on $\partial\Omega_k \setminus \{0\}$. Then

$$u_k(x) \leq C_1 |x|^{-2/(q-1)} \quad \forall x \in \Omega_k,$$

thus, by elliptic equation theory in uniformly Lipschitz domains, (which is the case if $k \geq 1$)

$$\|u_k\|_{C^{\alpha}(\Omega_k \cap (B_{7/4} \setminus B_{5/4}))} \leq C \,\|u_k\|_{L^{\infty}(\Omega_k \cap (B_2 \setminus B_1))} = C_2.$$

This implies

$$|u(k^{-1}x') - u(k^{-1}z')| \leq C_2 k^{-2/(q-1)-\alpha} |x' - z'|^{\alpha} \quad \forall (x, z) \in \Omega_k \times \Omega_k : 5/4 \leq |x'|, |z'| \leq 7/4.$$

Let $(x,z)$ in $\Omega\times\Omega$ close enough to 0. First, if $5/7\leq |x|/|z|\leq 7/5$ there exists $k\geq 1$ such that $5/4\leq |kx|,|kz|\leq 7/4$. Then

$$|u(x)-u(z)|\leq C_3|x|^{-2/(q-1)-\alpha}|x-z|^{\alpha}.$$

If we take in particular $x$ such that $z=\mathrm{Proj}_{\partial\Omega}(\mathrm{x})$ satisfies the above restriction, we derive

$$u(x)\leq C_3|x|^{-2/(q-1)-\alpha}\left(\mathrm{dist}\,(x,\partial\Omega)\right)^{\alpha}.$$

Because $\Omega$ is Lipschitz, it is easy to see that there exists $\beta\in(0,1/2)$ such that whenever $\mathrm{dist}\,(x,\partial\Omega)=|x-\mathrm{Proj}_{\partial\Omega}(\mathrm{x})|\leq\beta|x|$, there holds

$$5/7\leq |x|/\left|\mathrm{Proj}_{\partial\Omega}(\mathrm{x})\right|\leq 7/5.$$

Next we suppose $|x-\mathrm{Proj}_{\partial\Omega}(\mathrm{x})|>\beta|x|$. Then, by the Keller-Osserman estimate,

$$u(x)\leq C|x|^{-2/q-1)-\alpha}|x|^{\alpha}\leq C\beta^{-\alpha}|x|^{-2/q-1)-\alpha}\left|x-\mathrm{Proj}_{\partial\Omega}(\mathrm{x})\right|^{\alpha},$$

which is (5.5). If we assume that $\partial\Omega$ is $W^{2,s}$, with $s>N$, then we can perform a change $W^{2,s}$ of coordinates near 0 with transforms $\partial\Omega\cap B_R(0)$ into $\mathbb{R}^N_+\cap B_R(0)$ and the equation into

$$-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial\tilde u}{\partial x_j}\right)+|\tilde u|^{q-1}\tilde u=0,\quad\text{in }\mathbb{R}^N_+\cap B_R(0)\setminus\{0\},\tag{5.7}$$

where the $a_{ij}$ are the partial derivatives of the coordinates and thus belong to $W^{1,s}(B_{R)}$. By developping, $\tilde u$ satisfies

$$-\sum_{i,j}a_{ij}\frac{\partial^2\tilde u}{\partial x_i\partial x_j}-\sum_j b_j\frac{\partial\tilde u}{\partial x_j}+|\tilde u|^{q-1}\tilde u=0.$$

Notice that, since $s>N$, the $a_{ij}$ are continuous while the $b_i$ are in $L^s$. The same regularity holds uniformly for the rescaled form of $\tilde u_k:=T_k[\tilde u]$. By the Agmon-Douglis-Nirenberg estimates $\tilde u_k$ belongs to $W^{2,s}$. Since $s>N$, $\tilde u$ satisfies an uniform $C^1$ estimates, which implies that we can take $\alpha=1$. $\square$

### 5.1 Analysis in a cone

The removability question for solutions of (5.1) near the vertex of a cone has been studied in [11], and we recall this result below.

If we look for separable solutions of (5.1) under the form $u(x) = u(r,\sigma) = r^{\beta}\omega(\sigma)$, where $(r,\sigma) \in \mathbb{R}^+ \times S^{N-1}$ are the spherical coordinates, one finds immediately $\beta = -2/(q-1)$ and $\omega$ is a solution of

$$-\Delta'\omega - \lambda_{N,q}\omega + |\omega|^{q-1}\omega = 0 \tag{5.8}$$

on $S^{N-1}$ with

$$\lambda_{N,q} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right). \tag{5.9}$$

Thus, a solution of (5.1) in the cone $C_S = \{(r,\sigma) : r > 0, \sigma \in S \subset S^{N-1}\}$, vanishing on $\partial C_S \setminus \{0\}$, has the form $u(r,\sigma) = r^{-2/(q-1)}\omega(\sigma)$ if and only if $\omega$ is a solution of (5.8) in $S$ which vanishes on $\partial S$. The next result [11, Prop 2.1] gives the the structure of the set of positive solutions of (5.8).

**Proposition 5.3** *Let $\lambda_S$ be the first eigenvalue of the Laplace-Beltrami operator $-\Delta'$ in $W_0^{1,2}(S)$. Then*
*(i) If $\lambda_S \geq \lambda_{N,q}$ there exists no solution to (5.8) vanishing on $\partial S$.*
*(ii) If $\lambda_S < \lambda_{N,q}$ there exists a unique positive solution $\omega = \omega_S$ to (5.8) vanishing on $\partial S$. Furthermore $S \subset S' \Longrightarrow \omega_S \leq \omega_{S'}$.*

The following is a consequence of Proposition 5.3.

**Proposition 5.4** [11] *Assume $\Omega$ a bounded domain with a purely conical part with vertex 0, that is*

$$\Omega \cap B_{r_0}(0) = C_S \cap B_{r_0}(0) = \{x \in \cap B_{r_0}(0) \setminus \{0\} : x/|x| \in S\} \cup \{0\}$$

*and that $\partial\Omega \setminus \{0\}$ is smooth. Then, if $\lambda_S \geq \lambda_{N,q}$, any solution $u \in \mathcal{U}(\Omega)$ which is continuous in $\overline{\Omega} \setminus \{0\}$ and vanishes on $\partial\Omega \setminus \{0\}$ is identically 0.*

*Remark.* If $S \subset S^{N-1}$ is a domain and $\lambda_S$ the first eigenvalue of the Laplace-Beltrami operator $-\Delta'$ in $W_0^{1,2}(S)$ we denote by $\tilde{\alpha}_S$ and $\alpha_S$ the positive root and the absolute value of the negative root respectively, of the equation

$$X^2 + (N-2)X - \lambda_S = 0.$$

Thus

$$\begin{aligned} \tilde{\alpha}_S &= \frac{1}{2}\left(2 - N + \sqrt{(N-2)^2 + 4\lambda_S}\right), \\ \alpha_S &= \frac{1}{2}\left(N - 2 + \sqrt{(N-2)^2 + 4\lambda_S}\right). \end{aligned} \tag{5.10}$$

It is straightforward that

$$\lambda_S \geq \lambda_{N,q} \Longleftrightarrow \alpha_S \geq \frac{2}{q-1},$$

and, in case of equality, the exponent $q = q_S$ satisfies $q_S = 1 + 2/\alpha_S$.

In subsection 6.2 we compute the Martin kernel $K$ and the first eigenfunction $\rho$ of $-\Delta$ for cones with $k$-dimensional edge. In particular, if $k = 0$ and $C_S$ is the cone with vertex at the origin and 'opening' $S \subset S^{N-1}$, we have

$$K^{C_S}(x,0) = |x|^{-\alpha_S}\omega_S(\sigma), \quad \rho(x) = |x|^{\tilde{\alpha}_S}\omega_S(\sigma). \tag{5.11}$$

Combining the removability result with the admissibility condition Theorem 3.8, we obtain the following.

**Theorem 5.5** *The problem*

$$\begin{aligned} &-\Delta u + |u|^{q-1}u = 0 \quad \text{in } C_S, \\ &u \in C(\bar{C}_S \setminus \{0\}), \quad u = 0 \quad \text{on } \partial C_S \setminus \{0\} \end{aligned} \tag{5.12}$$

*possesses a non-trivial solution if and only if*

$$1 < q < q_S = 1 + 2/\alpha_S.$$

*Under this condition the following statements hold.*
(a) *For every $k \neq 0$ there exists a unique solution $v_k$ of* (5.1) *with boundary trace $k\delta_0$. In addition we have*

$$v_k/v_1(x) \to k \quad \text{uniformly as } x \to 0. \tag{5.13}$$

(b) *Equation* (5.1) *possesses a unique solution $U$ in $C_S$ such that $\mathcal{S}(U) = \{0\}$ and its trace on $\partial C_S \setminus \{0\}$ is zero. This solution satisfies*

$$|x|^{\frac{2}{q-1}}U(x) = U(x/|x|) = \omega_S(x/|x|) \tag{5.14}$$

*and*

$$U = v_\infty := \lim_{k\to\infty} v_k. \tag{5.15}$$

*Proof.* (a) By (5.11),

$$\int_{C_S \cap B_1} K^q(x,0)\rho(x)\,dx \leq C\int_0^1 r^{\tilde{\alpha}_S - q\alpha_S + N - 1}dr < \infty,$$

since

$$\tilde{\alpha}_S - q\alpha_S + N - 1 = 1 - (q-1)\alpha_S > -1.$$

Thus $q$ is admissible for $C_S \cap B_1$ at 0. By Theorem 3.8, for every $k \in \mathbb{R}$, there exists a unique solution of (5.1) with boundary trace $k\delta_0$.

Observe that, for every $a, j > 0$, $\tilde{v}_j(x) := a^{2/(q-1)} v_j(ax)$ is a solution of (5.1) in $C_S$. This solution has boundary trace $k\delta_0$ where $k = a^{2/(q-1)} j$. Because of uniqueness, $\tilde{v}_j = v_k$. Thus

$$v_k(x) = a^{2/(q-1)} v_j(ax), \quad k = a^{2/(q-1)} j. \tag{5.16}$$

This implies (5.13).

(b) Let $w$ be a solution in $C_S$ such that $\mathcal{S}(w) = \{0\}$ and its trace on $\partial C_S \setminus \{0\}$ is zero. We claim that

$$w \geq v_\infty := \lim k \to \infty v_k. \tag{5.17}$$

Indeed, for every $S' \Subset S$, $k > 0$,

$$\int_{aS'} w \, d\omega_a \to \infty, \quad \limsup \int_{aS'} v_k d\omega_a < \infty \text{ as } a \to 0$$

where $d\omega_a$ denotes the harmonic measure for a bounded Lipschitz domain $\Omega_a$ such that $aS' \subset \partial\Omega_a$ and $\Omega_a \uparrow C_S$. Therefore, using the classical Harnack inequality up to the boundary, $w/v_k \to \infty$ as $|x| \to 0$ in $C_{S'}$. In addition, either by Hopf's maximum principle (if $S$ is smooth) or by the boundary Harnack principle (if $S$ is merely Lipschitz),

$$c^{-1} v_1 \leq w \leq c v_1 \text{ in } C_{S \setminus S'}.$$

This inequality together with (5.16) yields,

$$c^{-1} v_k \leq w \leq c v_k \text{ in } C_{S \setminus S'}$$

with $c$ independent of $k$. Therefore $c^{-1} v_k \leq w$ in $C_S$. If $1/c > k/cj > 1$ then $\frac{k}{j} v_j \leq v_k \leq cw$ and consequently $v_j < w$. Here we used the fact that $\frac{k}{j} v_j$ is a subsolution with boundary trace $k\delta_0$.

Let $U_0$ be the maximal solution with trace 0 on $\partial C_S \setminus \{0\}$ and singular boundary point at 0. Then

$$U_0(x) = a^{2/(q-1)} U_0(ax) \quad \forall a > 0, \ x \in C_S,$$

because $a^{2/(q-1)} U_0(ax)$ is again a solution which dominates every solution with trace 0 on $\partial C_S \setminus \{0\}$ and singular boundary point at 0. Hence,

$$U_0(x) = |x|^{-2/(q-1)} U_0(x/|x|) = |x|^{-2/(q-1)} \omega_S(x/|x|). \tag{5.18}$$

The second equality follows from the uniqueness part in Proposition 5.3 since the function $x \to U_0(x/|x|)$ is continuous in $\bar{S}$ and vanishes on $\partial S$.

Inequality (5.17) implies that $v_\infty$ is the minimal positive solution such that $\mathcal{S}(w) = \{0\}$ and its trace on $\partial C_S \setminus \{0\}$ is zero. Using this fact we prove in the same way that $v_\infty$ satisfies

$$v_\infty(x) = |x|^{-2/(q-1)} v_\infty(x/|x|) = |x|^{-2/(q-1)} \omega_S(x/|x|).$$

This implies (5.15) and the uniqueness in statement (b).

□

In the next theorem we describe the precise asymptotic behavior of solutions in a conical domain with mass concentrated at the vertex.

**Theorem 5.6** *Let $C_S$ be a cone with vertex 0 and opening $S \subset S^{N-1}$ and assume that $1 < q < q_S = 1 + 2/\alpha_S$. Denote by $\phi_S$ the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S)$ normalized by $\max \phi_S = 1$. Then the function*

$$\Phi_S = x^{-\alpha_S} \phi_S(x/|x|),$$

*with $\alpha_S$ as in (5.10), is harmonic in $C_S$ and vanishes on $\partial C_S \setminus \{0\}$. Thus there exists $\gamma > 0$ such that the boundary trace of $\Phi_S$ is the measure $\gamma\delta_0$. Put $\Phi_1 := \frac{1}{\gamma}\Phi_S$.*

*Let $r_0 > 0$ and denote $\Omega_S = C_S \cap B_{r_0}(0)$. For every $k \in \mathbb{R}$, let $u_k$ be the unique solution of (5.1) in $\Omega$ with boundary trace $k\delta_0$. Then*

$$u_k(x) = k\Phi_1(x)(1 + o(1)) \quad \text{as } x \to 0. \tag{5.19}$$

*If $v_k$ is the unique solution of (5.1) in $C_S$ with boundary trace $k\delta_0$ then*

$$u_k/v_k \to 1 \ \text{ and } \ v_k/(k\Phi_1) \to 1 \quad \text{as } x \to 0. \tag{5.20}$$

*The function $u_\infty = \lim_{k\to\infty} u_k$ is the unique positive solution of (5.1) in $\Omega_S$ which vanishes on $\partial\Omega_S \setminus \{0\}$ and is strongly singular at 0 (i.e., 0 belongs to its singular set). Its asymptotic behavior at 0 is given by,*

$$u_\infty(x) = |x|^{-\frac{2}{q-1}} \omega_S(x/|x|)(1 + o(1)) \qquad \text{as } x \to 0. \tag{5.21}$$

*Proof. Step 1: Construction of a fundamental solution.* Put

$$\Phi(x) = |x|^{-\alpha_S} \phi_S(x/|x|), \quad \tilde{\Phi}(x) = |x|^{\tilde{\alpha}_S} \phi_S(x/|x|) \tag{5.22}$$

with $\alpha_S$, $\tilde{\alpha}_S$ as in (5.10). Then $\Phi$ and $\tilde{\Phi}$ are harmonic in $C_S$, $\Phi$ vanishes on $\partial C_S \setminus \{0\}$ and $\tilde{\Phi}$ vanishes on $\partial C_S$. Furthermore, since $q < 1 + 2/\alpha_S$,

$$\int_{C_S \cap B_1(0)} \Phi^q \rho dx < \infty.$$

Therefore the boundary trace of $\Phi$ is a bounded measure concentrated at the vertex of $C_S$, which means that the trace is $\gamma\delta_0$ for some $\gamma > 0$. (Here $\delta_0$ denotes the Dirac measure on $\partial C_S$ concentrated at the origin.)

The function

$$\Psi(x) = \frac{1}{\gamma}(\Phi(x) - r_0^{\tilde{\alpha}_S - \alpha_S}\tilde{\Phi}(x))$$

is harmonic and positive in $\Omega_S$ and vanishes on $\partial\Omega_S \setminus \{0\}$. Its boundary trace is $\delta_0$.

*Step 2: Weakly singular behaviour.* By Theorem 3.8 , for any $k \geq 0$, there exists a unique function $u_k \in L^q_\rho(\Omega_S)$ with trace $k\delta_0$ and by (3.8)

$$u_k(x) = k\Psi(x) - \mathbb{G}[|u_k|^q]. \tag{5.23}$$

Since $|x|^{\alpha_S} u_k$ is bounded, we set

$$v(t,\sigma) = r^{\alpha_S} u_k(r,\sigma), \quad t = -\ln r.$$

Then $v$ satisfies

$$v_{tt} + (2\alpha_S + 2 - N)v_t + \lambda_S v + \Delta' v - e^{(\alpha_S(q-1)-2)t}\,|v|^{q-1}\,v = 0 \tag{5.24}$$

in $D_{S,t_0} := [t_0,\infty)\times S$ (with $t_0 := -\ln r_0$) and vanishes on $[t_0,\infty)\times\partial S$. Since $0 \leq u_k(x) \leq k\Psi(x)$, $v$ is uniformly bounded, and, since $\alpha_S(q-1) - 2 < 0$, $v(t,.)$ is uniformly bounded in $C^\alpha(\overline{S})$ for some $\alpha \in (0,1)$. Furthermore, $\nabla' v(t,.)$ (by definition $\nabla'$ is the covariant gradient on $S^{N-1}$) is bounded in $L^2(S)$, independently of $t$. Set

$$y(t) = \int_S v(t,\sigma)\phi_S dV(\sigma), \quad F(t) = \int_S (|v|^{q-1}\,v)(t,\sigma)\phi_S dV(\sigma).$$

From (5.24), it follows

$$\frac{d}{dt}\left(e^{(2\alpha_S+2-N)t}y'\right) = e^{((q+1)\alpha_S - N)t}F,$$

where $dV$ is the volume measure on $S^{N-1}$. By (5.10), $\gamma := 2\alpha_S + 2 - N > 0$, then

$$y'(t) = e^{-\gamma(t-t_0)}y'(t_0) + e^{-\gamma t}\int_{t_0}^t e^{((q+1)\alpha_S - N)s}F(s)ds,$$

and

$$\left|y'(t)\right| \leq c_1 e^{-\gamma(t-t_0)} + c_2 e^{(\alpha_S(q-1)-2)t}.$$

This implies that there exists $k^* \in \mathbb{R}^+$ such that

$$\lim_{t\to\infty} y(t) = k^*. \tag{5.25}$$

Next we use the fact that the following Hilbertian decomposition holds

$$L^2(S) = \oplus_{k=1}^{\infty} ker(-\Delta' - \lambda_k I)$$

where $\lambda_k$ is the $k$-th eigenvalue of $-\Delta'$ in $W_0^{1,2}(S)$ (and $\lambda_S = \lambda_1$). Let $\tilde{v}$ and $\tilde{F}$ be the projections of $v$ and $|v|^{q-1} v$ onto $ker(-\Delta' - \lambda_S I)^\perp$. Since

$$\tilde{v}_{tt} + (2\alpha_S + 2 - N)\tilde{v}_t + \lambda_S \tilde{v} + \Delta'\tilde{v} - e^{(\alpha_S(q-1)-2)t}\tilde{F} = 0 \tag{5.26}$$

we obtain, by multiplying by $\tilde{w}$ and integrating on $S$,

$$V'' + (2\alpha_S + 2 - N)V' - (\lambda_2 - \lambda_S)V + e^{(\alpha_S(q-1)-2)t}\Phi \geq 0,$$

where $V(t) = \|\tilde{v}(t,.)\|_{L^2(S)}$ and $\Phi(t) = \left\|\tilde{F}(t,.)\right\|_{L^2(S)}$. The associated o.d.e.

$$z'' + (2\alpha_S + 2 - N)z' - (\lambda_2 - \lambda_S)z + e^{(\alpha_S(q-1)-2)t}\Phi = 0,$$

admits solutions under the form

$$z(t) = a_1 e^{-\mu_1 t} + a_2 e^{\mu_2 t} + d(t)e^{(\alpha_S(q-1)-2)t}$$

where $-\mu_1$ and $\mu_2$ are respectively the negative and the positive roots of

$$X^2 + (2\alpha_S + 2 - N)X - (\lambda_2 - \lambda_S) = 0,$$

and $|d(t)| \leq c\Phi$ if $\alpha_S(q-1)-2 \neq -\mu_1$, or $|d(t)| \leq ct^1\Phi$ if $\alpha_S(q-1)-2 = -\mu_1$. Applying the maximum principle to (5.26), we derive

$$\|\tilde{v}(t,.)\|_{L^2(S)} \leq \|\tilde{v}(t_0,.)\|_{L^2(S)} e^{-\mu_1(t-t_0)} + d(t)e^{(\alpha_S(q-1)-2)t} \quad \forall t \geq t_0. \tag{5.27}$$

By the standard elliptic regularity results in Lipschitz domains [13], we obtain from (5.27), for any $t > t_0 + 1$,

$$\|\tilde{v}(t,.)\|_{C^\alpha(S)} \leq c_1 \|\tilde{v}\|_{L^2((t-1,t+1)\times S)} + c_2 \left\|e^{(\alpha_S(q-1)-2)s}\tilde{F}\right\|_{L^\infty((t-1,t+1)\times S)}, \tag{5.28}$$

for some $\alpha \in (0,1]$ depending of the regularity of $\partial S$. Thus

$$\|\tilde{v}(t,.)\|_{C^\alpha(S)} \leq ce^{-\mu_1 t} + c' te^{(\alpha_S(q-1)-2)t}. \tag{5.29}$$

Combining (5.25) and (5.29) we obtain that

$$|x|^{\alpha_S}\, u_k(x) - k^*\phi_S(x/|x|) \to 0 \quad \text{as } x \to 0 \tag{5.30}$$

in $C^\alpha(S)$. Furthermore $0 \le k^* \le k$.

*Step 3: Identification of $k^*$.*

Let $\{\Omega_n\}$ be a Lipschitz exhaustion of $\Omega_S$ and denote by $\omega_n$ (resp. $\omega$) the harmonic measure on $\partial\Omega_n$ (resp. $\partial\Omega_S$). By Proposition 3.6

$$\lim_{n\to\infty}\int_{\partial\Omega_n} u_k\, d\omega_n = k.$$

On the other hand, by (5.30),

$$u_k/(k^*|x|^{-\alpha_S}\phi_S) \to 1 \ \text{ as } \ x \to 0.$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\int_{\partial\Omega_n} u_k\, d\omega_n &= k^* \lim_{n\to\infty}\int_{\partial\Omega_n} |x|^{-\alpha_S}\phi_S\, d\omega_n \\ &= k^*\gamma \lim_{n\to\infty}\int_{\partial\Omega_n} \Phi_1\, d\omega_n = k^*\gamma.\end{aligned}$$

Thus

$$k = k^*\gamma. \tag{5.31}$$

This and (5.30) imply (5.19).

Further,

$$u_k \le v_k \le k\Phi_1$$

since $\Phi_1$ is harmonic in $C_S$. Therefore (5.19) implies (5.20).

*Step 4: Study when $k \to \infty$.* By Theorem 5.5, equation (5.1) possesses a unique solution $U$ in $C_S$ such that $U = 0$ on $\partial C_S \setminus \{0\}$ and $U$ has strong singularity at the vertex, i.e., $0 \in \mathcal{S}(U)$. By (5.14) and (5.15) this solution satisfies

$$U = v_\infty := \lim_{k\to\infty} v_k = |x|^{-\frac{2}{q-1}}\omega_S. \tag{5.32}$$

Let $V$ be the maximal solution in $\Omega_S$ vanishing on $\partial\Omega_S\backslash\{0\}$. Its extension by zero to $C_S$ is a subsolution and consequently, $V \le U$.

Let $w$ be the unique solution of (5.1) in $\Omega_S$ such that $w = U$ on $\partial\Omega_S \cap B_{r_0}(0)$ and $w = 0$ on the remaining part of the boundary. Then $w < U$

so that $U - w$ is a subsolution of (5.1) in $\Omega_S$ which vanishes on $\partial\Omega_S \setminus \{0\}$. Therefore $U - w \leq V$. Thus

$$U - w \leq V \leq U \text{ and } U/V \to 1 \text{ as } x \to 0. \tag{5.33}$$

*Assertion 1.* If $u$ is a solution of (5.1) in $\Omega_S$ such that

$$u = 0 \text{ on } \partial\Omega_S \setminus \{0\} \text{ and } u/U \to 1 \text{ as } x \to 0$$

then $u = V$.

By (5.33) $u/V \to 1$ as $x \to 0$. Therefore, by a standard application of the maximum principle, $u = V$.

Let $u$ be an arbitrary positive solution in $\Omega_S$ vanishing on $\partial\Omega_S \setminus \{0\}$. Denote by $u^*$ its extension by zero to $C_S$. Then $u^*$ is a subsolution and, by Theorem 4.3, there exists a solution $\bar{u}$ of (5.1) in $C_S$ which is the smallest solution dominating $u^*$. The solution $\bar{u}$ can be obtained from $u^*$ as follows. Let $\{r_n\}$ be a sequence decreasing to zero, $r_1 < r_0$, and denote

$$D_n = C_S \setminus B_{r_n}(0), \quad h_n = u^* \lfloor_{\partial D_n}.$$

Let $w_n$ be the solution of (5.1) in $D_n$ such that $w_n = h_n$ on the boundary. Then $\{w_n\}$ increases and

$$\bar{u} = \lim w_n. \tag{5.34}$$

If $u$ has strong singularity at the origin then, of course, the same is true with respect to $\bar{u}$ and consequently, by Theorem 5.5,

$$\bar{u} = U. \tag{5.35}$$

In the the remaining part of the proof we assume only (5.35) and show that this implies $u = V$.

Let $z$ be the solution of (5.1) in $\Omega_S$ such that $z = U$ on $\partial\Omega_S \cap \partial B_{r_0}$ and 0 on $\partial\Omega_S \cap \partial C_S$. Then $u + z$ is a supersolution in $\Omega_S$. Let

$$\Omega_n = \Omega_S \setminus B_{r_n}(0) = D_n \cap B_{r_0}(0).$$

The trace of $u + z$ on $\partial\Omega_n$ is given by

$$f_n = \begin{cases} U & \text{on } \partial\Omega_n \cap \partial B_{r_0} \\ h_n + z & \text{on } \partial\Omega_n \setminus \partial B_{r_0}. \end{cases}$$

Since $U = \bar{u} \geq u^*$ we have $f_n \geq h_n$. Therefore, if $\tilde{w}_n$ is the solution of (5.1) in $\Omega_n$ such that $\tilde{w}_n = f_n$ on the boundary then

$$w_n \leq \tilde{w}_n \leq u + z \text{ in } \Omega_n.$$

Hence, by (5.34),

$$U \leq u + z.$$

Since $z \to 0$ as $x \to 0$, it follows that

$$\limsup U/u \leq 1 \quad \text{as} \quad x \to 0.$$

Since $u < V$, (5.33) implies that

$$\liminf U/u \geq 1 \quad \text{as} \quad x \to 0.$$

Therefore $U/u \to 1$ as $x \to 0$ and consequently, by Assertion 1, $u = V$. This proves the uniqueness stated in the last part of the theorem and (5.33) implies (5.21).

□

**Corollary 5.7** *Suppose that u is a positive solution of* (5.1) *in* $\Omega_S$ *which vanishes on* $\partial\Omega_S \setminus \{0\}$ *and*

$$\sup_{\Omega_S} |x|^{\alpha_S} u = \infty. \tag{5.36}$$

*Then* $u = u_\infty$.

*Proof.* Let $\bar{u}$ be as in (5.34). Since $\bar{u} \geq u$ it follows that

$$\sup_{\Omega_S} |x|^{\alpha_S} \bar{u} = \infty.$$

By Theorem 5.5 $\bar{u} = U$. The last part of the proof shows that $u = u_\infty$.

□

As a consequence of Theorem 5.6 we obtain the classification of positive solutions of (5.1) in conical domains with isolated singularity located at the vertex. In the case of a half space such a classification was obtained in [14].

**Theorem 5.8** *Let* $C_S$ *be as in Theorem 5.6,* $\Omega_s = C_S \cap B_{r_0}(0)$ *for some* $r_0 > 0$ *and* $1 < q < q_S = 1 + 2/\alpha_S$*. If* $u \in C(\bar{\Omega}_s \setminus \{0\})$ *is a positive solution of (5.1) vanishing on* $\partial C_S \cap B_{r_0}(0) \setminus \{0\}$*, the following alternative holds:*

*Either*

*(i)* $\limsup_{x\to 0} |x|^{-\tilde{\alpha}_S} u(x) < \infty$ *and thus* $u \in C(\bar{\Omega}_s)$.

*or*

*(ii) there exist* $k > 0$ *such that (5.19) holds*

*or*

*(iii) (5.21) holds.*

*Proof.* Let $u_\epsilon$ be the solution of (5.1) in $\Omega_{S,\epsilon} = \Omega_S \setminus B_\epsilon(0)$ with boundary data $u$ on $\Omega_{S,\epsilon} \cap \partial B_\epsilon(0)$ and zero on $\partial\Omega_{S,\epsilon} \setminus \partial B_\epsilon(0)$. Then

$$0 \le u_\epsilon \le u \le u_\epsilon + Z(x) \quad \forall x \in \Omega_{S,\epsilon},$$

where $Z$ is harmonic in $\Omega_S$, vanishes on $\partial\Omega_S \setminus \partial B_{r_0}(0)$ and coincides with $u$ on $C_S \cap \partial B_{r_0}(0)$. Furthermore $0 < \epsilon < \epsilon' \Longrightarrow u_\epsilon \le u_{\epsilon'}$ in $\Omega_{S,\epsilon'}$. Thus $u_\epsilon$ converges, as $\epsilon \to 0$, to a solution $\tilde{u}$ of (5.1) which vanishes on $\partial\Omega_S \setminus \{0\}$ and satisfies

$$0 \le \tilde{u}(x) \le u(x) \le \tilde{u}(x) + Z(x) \quad \forall x \in \Omega_S. \tag{5.37}$$

If

$$\limsup_{x\to 0} |x|^{\alpha_S}\, \tilde{u}(x) < \infty, \tag{5.38}$$

it follows from Theorem 5.6-Step 2, that there exists $k^* \ge 0$ such that

$$\tilde{u}(x) = k^*\, |x|^{-\alpha_S}\, \phi_S(x/|x|)(1 + o(1)) \quad \text{as } x \to 0. \tag{5.39}$$

If $k^* > 0$ then $u$ satisfies (ii). If $k^* = 0$, it is straightforward to see that, for any $\epsilon > 0$, $\tilde{u}(x) \le \epsilon\, |x|^{-\alpha_S}$. Thus

$$u(x) \le Z(x) = c\, |x|^{\tilde{\alpha}_S}\, \phi_S(x/|x|)(1 + o(1)) \quad \text{as } x \to 0, \tag{5.40}$$

by standard expansion of harmonic functions at 0.

Finally, if

$$\limsup_{x\to 0} |x|^{\alpha_S}\, \tilde{u}(x) = \infty, \tag{5.41}$$

then, by Corollary 5.7, $\tilde{u} = u_\infty$ and consequently, by Theorem 5.6, $\tilde{u}$ – and therefore $u$ – satisfies (5.21). □

## 5.2 Analysis in a Lipschitz domain

In a general Lipschitz bounded domain tangent planes have to be replaced by asymptotic cones, and these asymptotic cones can be inner or outer.

**Definition 5.9** *Let $\Omega$ be a bounded Lipschitz domain and $y \in \partial\Omega$. For $r > 0$, we denote by $\mathcal{C}^I_{y,r}$ (resp. $\mathcal{C}^O_{y,r}$) the set of all open cones $C_{s,y}$ with vertex at $y$ and smooth opening $S \subset \partial B_1(y)$ such that $C_{s,y} \cap B_r(y) \subset \Omega$ (resp. $\Omega \cap B_r(y) \subset C_{s,y}$). Further we denote*

$$C^I_{y,r} := \bigcup \left\{C_{S,y} : C_{S,y} \in \mathcal{C}^I_{y,r}\right\}, \quad C^O_{y,r} := \bigcap \left\{C_{S,y} : C_{S,y} \in \mathcal{C}^O_{y,r}\right\} \tag{5.42}$$

*and*

$$C_y^I := \bigcup_{r>0} C_{y,r}^I, \quad C_y^O := \bigcap_{r>0} C_{y,r}^O. \tag{5.43}$$

*The cone $C_y^I$ (resp. $C_y^O$) is called the limiting inner cone (resp. outer cone) at $y$. Finally we denote*

$$\begin{aligned} S_{y,r}^I &:= C_{y,r}^I \cap \partial B_1(y), \quad S_{y,r}^O := C_{y,r}^O \cap \partial B_1(y), \\ S_y^I &:= C_y^I \cap \partial B_1(y), \qquad S_y^O := C_y^O \cap \partial B_1(y). \end{aligned} \tag{5.44}$$

*Remark.* In this definition, we identify $\partial B_1(y)$ with the manifold $S^{N-1}$. Notice that the following monotonicity holds

$$0 < s < r \Longrightarrow \begin{cases} C_{y,r}^I \subset C_{y,s}^I \\ C_{y,s}^O \subset C_{y,r}^I. \end{cases} \tag{5.45}$$

**Definition 5.10** *If $C_S$ is a cone with vertex $y$ and opening $S$ and if $\lambda_S$ is the first eigenvalue of $-\Delta'$ in $W_0^{1,2}(S)$, we denote*

$$\alpha_S = \frac{1}{2}\left(N - 2 + \sqrt{(N-2)^2 + 4\lambda_S}\right), \quad \textit{and} \quad q_S = 1 + 2/\alpha_S. \tag{5.46}$$

*Thus $q_S$ is the critical value for the cone $C_S$ at its vertex.*

*Remark.* As $r \mapsto S_{y,r}^I$ is nondecreasing, it follows that $r \mapsto \lambda_{S_{y,r}^I}$ is nonincreasing and consequently $r \mapsto q_{S_{y,r}^I}$ is nondecreasing. It is classical that

$$\lim_{r\to 0} \lambda_{S_{y,r}^I} = \lambda_{S_y^I}. \tag{5.47}$$

A similar observation holds with respect to $S_{y,r}^O$ if we interchange the terms ‘nondecreasing’ and ‘nonincreasing’. In particular

$$\lim_{r\to 0} \lambda_{S_{y,r}^O} = \lambda_{S_y^O}. \tag{5.48}$$

In view of (5.46) we conclude that,

$$\lim_{r\to 0} q_{S_{y,r}^I} = q_{S_y^I}, \quad \lim_{r\to 0} q_{S_{y,r}^O} = q_{S_y^O}. \tag{5.49}$$

We also need the following notation:

**Definition 5.11** *Let $\Omega$ be a bounded Lipschitz domain. For every compact set $E \subset \partial\Omega$ denote,*

$$q^*_E = \lim_{r\to 0} \inf \left\{ q_{S^I_{z,r}} : z \in \partial\Omega,\ \operatorname{dist}(z, E) < r \right\}, \tag{5.50}$$

*If $E$ is a singleton, say $\{y\}$, we replace $q^*_E$ by $q^*_y$.*

*Remark.* For a cone $C_S$ with vertex $y$, $q^*_y \le q_S$. However if $C_S$ is contained in a half space then $q^*_y = q_S$. On the other hand, if $C_S$ strictly contains a half space then $q^*_y < q_S$.

If $\Omega$ is the complement of a bounded convex domain then, for every $y \in \partial\Omega$,

$$q^*_y = (N+1)/(N-1) \tag{5.51}$$

Indeed $q_{c,y} \ge (N+1)/(N-1)$. But for $\mathbb{H}_{N-1}$-a.e. point $y \in \partial\Omega$ there exists a tangent plane and consequently $q_{c,y} = (N+1)/(N-1)$. This readily implies (5.51).

Since $\Omega$ is Lipschitz, there exists $r_\Omega > 0$ such that, for every $r \in (0, r_\Omega)$ and every $z \in \partial\Omega$, there exists a cone $C$ with vertex at $z$ such that $C \cap B_r(z) \subset \bar\Omega$. Denote

$$a(r, y) := \inf \left\{ q_{S^I_{z,r}} : z \in \partial\Omega \cap B_r(y) \right\} \quad \forall r \in (0, r_\Omega),\ y \in \partial\Omega.$$

Then,

$$\begin{aligned} q^*_E &:= \lim_{r\to 0} \inf\{a(r, y) : y \in E\} \\ &\le \inf\{\lim_{r\to 0} a(r, y) : y \in E\} = \inf\{q^*_y : y \in E\}. \end{aligned} \tag{5.52}$$

Indeed, the monotonicity of the function $r \mapsto q_{S^I_{y,r}}$ (for each fixed $y \in \partial\Omega$) implies

$$q^*_y = \lim_{r\to 0} a(r, y) = \sup_{0<r<r_\Omega} a(r, y). \tag{5.53}$$

As

$$q^*_E = \lim_{r\to 0} \inf\{a(r, y) : y \in E\}$$

inequality (5.52) follows immediately from (5.53).

Finally we observe that, if $E$ is a compact subset of $\partial\Omega$ then

$$(E)_r := \{z \in \partial\Omega : \operatorname{dist}(z, E) \le r\} \Longrightarrow q^*_{(E)_r} \uparrow q^*_E \ \text{ as } \ r \downarrow 0. \tag{5.54}$$

In order to deal with boundary value problems in a general Lipschitz domain $\Omega$ we must study the question of q-admissibility of $\delta_y$, $y \in \partial\Omega$. This question is addressed in the following:

**Theorem 5.12** *If $y \in \partial\Omega$ and $1 < q < q_{S^I_y} := 1 + 2/\alpha_{S^I_y}$ then*

$$\int_\Omega K^q(x,y)\rho(x)dx < \infty. \tag{5.55}$$

*Furthermore, if $E$ is a compact subset of $\partial\Omega$ and $1 < q < q^*_E$ then, there exists $M > 0$ such that,*

$$\int_\Omega K^q(x,y)\rho(x)dx \leq M \quad \forall y \in E. \tag{5.56}$$

*Proof.* We recall some sharp estimates of the Poisson kernel due to Bogdan [3]. Set $\kappa = 1/2(\sqrt{1+K^2})$, where $K$ is the Lipschitz constant of the domain, seen locally as the graph of a function from $\mathbb{R}^{N-1}$ into $\mathbb{R}$. Let $x_0 \in \Omega$ and set $\phi(x) := G(x,x_0)$. Then there exists $c_1 > 0$ such that for any $y \in \partial\Omega$ and $x \in \Omega$ satisfying $|x-y| \leq r_0$, there holds

$$c_1^{-1}\frac{\phi(x)}{\phi^2(\xi)}|x-y|^{2-N} \leq K(x,y) \leq c_1\frac{\phi(x)}{\phi^2(\xi)}|x-y|^{2-N}, \tag{5.57}$$

for any $\xi$ such that $B_{\kappa|x-y|}(\xi) \subset \Omega \cap B_{|x-y|}(y)$. This implies

$$c_2^{-1}\frac{\phi^{q+1}(x)}{\phi^{2q}(\xi)}|x-y|^{(2-N)q} \leq K^q(x,y)\rho(x) \leq c_2\frac{\phi^{q+1}(x)}{\phi^{2q}(\xi)}|x-y|^{(2-N)q} \tag{5.58}$$

for some $c_2$ since $\phi$ and $\rho$ are comparable in $B_{r_0}(y)$, uniformly with respect to $y$ (provided we have chosen $r_0 \leq \operatorname{dist}(x_0,\partial\Omega)/2$. Let $C_{s,y}$ be a smooth cone with vertex at $y$ and opening $S := C_{s,y} \cap \partial B_1(y)$, such that $\overline{C}_{s,y} \cap \partial B_{r_0}(y) \subset \Omega$. We can impose to the point $\xi$ in inequality (5.57) to be such that $\xi/|\xi| := \Xi_0 \in S$, or, equivalently, such that $|\xi - y| \leq \gamma \operatorname{dist}(\xi,\partial\Omega)$ for some $\gamma > 1$ independent of $\xi$, $|x-y|$ and $y$. Then, by Carleson estimate [2, Lemma 2.4] and Harnack inequality, there exists $c_5$ independent of $y$ such that there holds

$$\frac{\phi(\xi)}{\phi(x)} \geq c_3 \tag{5.59}$$

for all $x \in \Omega \cap B_{r_0}(y)$ and all $\xi$ as above. Consequently, (5.58) yields to

$$K^q(x,y)\rho(x) \leq c_4\phi^{1-q}(\xi)|x-y|^{(2-N)q}. \tag{5.60}$$

There exists a separable harmonic function $v$ in $C_{s,y}$ under the form

$$v(z) = |z-y|^{\alpha_S+2-N}\phi_S((z-y)/|z-y|)$$

where $\phi_S$ is the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S)$ normalized by $\max \phi_S = 1$, $\lambda_S$ the corresponding eigenvalue and $\alpha_S$ is given by (5.10). By the maximum principle,

$$v(z) \leq c_5 \phi(z) \quad \forall z \in C_{S,y} \cap B_{r_0}(y). \tag{5.61}$$

Therefore there exists $c_6 > 0$ such that

$$\phi(\xi) \geq c_6 \left|\xi - y\right|^{\alpha_S + 2 - N}. \tag{5.62}$$

Because $|x-y| \geq |\xi - y| \geq \kappa \left|x - y\right|/2$, from the choice of $\xi$, it follows

$$K^q(x,y)\rho(x) \leq \frac{c_7}{|x-y|^{(q-1)\alpha_S + N - 2}} \quad \forall x \in \Omega \cap B_{r_0}(y). \tag{5.63}$$

Clearly, if we choose $q$ such that $1 < q < q_{S^I_y} := 1 + 2/\alpha_{S^I_y}$, then $q < 1 + 2/\alpha_{S^I_{r,y}}$ for some $r$ small enough and we can take $C_{S,y} = C^I_{y,r}$. Thus (5.55) follows.

We turn to the proof of (5.56). To simplify the notation we assume that $q < q^*_{\partial\Omega}$. The argument is the same in the case $q < q^*_E$.

If we assume $q < \lim_{r\to 0} \inf\{q_{S^I_{z,r}} : z \in \partial\Omega\}$, then for $\epsilon > 0$ small enough, there exists $r_\epsilon > 0$ such that

$$0 < r \leq r_\epsilon \Longrightarrow 1 < q < \inf\{q_{S^I_{z,r}} : z \in \partial\Omega\} - \epsilon \quad \forall 0 < r \leq r_\epsilon.$$

Notice that the shape of the cone may vary, but, since $\partial\Omega$ is Lipschitz there exists a fixed relatively open subdomain $S^* \subset \partial B_1$ such that for any $y \in \partial\Omega$, there exists an isometry $\mathcal{R}_y$ of $\mathbb{R}^N$ with the property that $\mathcal{R}_y(\overline{S^*}) \subset S^I_{y,r}$ for all $0 < r \leq r_\epsilon$. Here we use the fact that $r \mapsto S^I_{y,r}$ is increasing when $r$ decreases. If we take $\xi$ such that $\xi/|\xi| = \Xi_0 \in \mathcal{R}_y(S^*)$, then the constants in Bogdan estimate (5.57) and Carleson inequality (5.59) are independent of $y \in \partial\Omega$ if we replace $r_0$ by $\inf\{r_\epsilon, r_0\}$. Hereafter we shall assume that $r_\epsilon \leq r_0$. Set

$$v_S(t) = |t-y|^{\alpha_S + 2 - N} \phi_S((t-y)/|t-y|)$$

with $S = S^I_{y,r_\epsilon}$. Then $v_S$ is well defined in the cone $C_{S,y}$ with vertex $y$ and opening $S$. Let

$$\Sigma_{cr_\epsilon} := \{t \in \Omega : \text{dist}\,(t, \partial\Omega) = cr_\epsilon\}.$$

Because $\partial\Omega$ is Lipschitz, we can choose $0 < c < 1$ such that $C_{S,y} \cap \Sigma_{cr_\epsilon} \subset B_{r_\epsilon}(z)$. Then we can compare $v_S$ and $\phi$ on the set $\Sigma_{cr_\epsilon}$. It follows by maximum principle that estimate (5.61) is still valid with a constant may depend on $r_\epsilon$, but not on $y$. Because

$$\min_{\mathcal{R}_y(S^*)} \phi_{S^I_{y,r_\epsilon}} \geq c_8$$

where $c_8$ is independent of $y$, (5.62) holds under the form

$$\phi(\xi) \geq c_6 |\xi - y|^{\alpha_{S^I_{y,r_\epsilon}}+2-N}, \tag{5.64}$$

where, we recall it, $\xi$ satisfies $\xi/|\xi| \in \mathcal{R}_y(S^*)$, and is associated to any $x \in B_{r_\epsilon}(y) \cap \Omega$ by the property that $B_{\kappa|x-y|}(\xi) \subset B_{|x-y|}(y) \cap \Omega$, and thus $|x-y| \geq |\xi - y| \geq \kappa |x-y|/2$. Then (5.63) holds uniformly with respect to $y$, with $r_0$ replaced by $r_\epsilon$. This implies (5.56). $\square$

The next proposition partially complements Theorem 5.12.

**Proposition 5.13** *Let $y \in \partial\Omega$ and $q > q_{S^O_y}$. Then any solution of (5.1) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ is identically* 0.

*Remark.* This proposition implies that, if $q > q_{S^O_y}$,

$$\int_\Omega K^q(x,y)\rho(x)dx = \infty. \tag{5.65}$$

Otherwise $\delta_y$ would be admissible.

*Proof.* We consider a local outer smooth cone with vertex at $y$, $C_2$, such that $\overline{\Omega} \cap B_{r_0}(y) \setminus \{0\} \subset C_2 \cap B_{r_0}(y) := C_{2,r_0}$. We denote by $S^* = C_2 \cap \partial B_1(y)$ its opening. For $\epsilon > 0$ small enough, we consider the doubly truncated cone $C^\epsilon_{2,r_0} = \cap C_{2,r_0} \setminus B_\epsilon(y)\}$ and the solution $v := v_\epsilon$ to

$$\begin{cases} -\Delta v + v^q = 0 & \text{in } C^\epsilon_{2,r_0} \\ \qquad v = \infty & \text{on } \partial B_\epsilon(y) \cap C_2 \\ \qquad v = \infty & \text{on } \partial B_{r_0}(y) \cap C_2 \\ \qquad v = 0 & \text{on } \partial C_2 \cap B_{r_0}(y) \setminus \overline{B}_\epsilon(y), \end{cases} \tag{5.66}$$

where $q \geq q_{S^*} := 1 + 2/\alpha_{S^*}$, and $\alpha_{S^*}$ is expressed by (5.10) with $S$ replaced by $S^*$. Then $v_\epsilon$ dominates in $C^\epsilon_{2,r_0} \cap \Omega$ any positive solution $u$ of (5.1) in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$. Letting $\epsilon \to 0$, $v_\epsilon$ converges to $v_0$ which satisfies

$$\begin{cases} -\Delta v + v^q = 0 & \text{in } C_{2,r_0} \\ \qquad v = \infty & \text{on } \partial B_{r_0} \cap C_2 \\ \qquad v = 0 & \text{on } \partial C_2 \cap B_{r_0}(y). \end{cases} \tag{5.67}$$

Furthermore $u \leq v_0$ in $B_{r_0} \cap \Omega$. Because $q_{S^*}$ is the critical exponent in $C_2$ , the singularity at 0 is removable, which implies that $v(x) \to 0$ when $x \to 0$ in $C_2$. Thus $u_+(x) \to 0$ when $x \to 0$ in $\Omega$. Thus $u_+ = 0$. But we can take any cone with vertex $y$ containing $\Omega$ locally in $B_r(y)$ for $r > 0$. This implies that for any $q > q_{S^O_y}$, any solution of (5.1) which vanishes on $\partial\Omega \setminus \{0\}$ is non-positive. In the same way it is non-negative. $\square$

**Definition 5.14** *If $y \in \partial\Omega$ we say that an exponent $q \geq 1$ is:*

*(i) Admissible at $y$ if*

$$\|K(.,y)\|_{L^q_\rho(\Omega)} < \infty,$$

*and we set*

$$q_{1,y} = \sup\{q > 1 : q \text{ admissible at } y\}.$$

*(ii) Acceptable at $y$ if there exists a solution of (5.1) with boundary trace $\delta_y$, and we set*

$$q_{2,y} = \sup\{q > 1 : q \text{ acceptable at } y\}.$$

*(iii) Super-critical at $y$ if any solution of (5.1) which is continuous in $\Omega\setminus\{0\}$ and vanishes on $\partial\Omega \setminus \{0\}$ is identically zero, and we set*

$$q_{3,y} = \inf\{q > 1 : q \text{ super-critical at } y\}.$$

**Proposition 5.15** *Assume $\Omega$ is a bounded Lipschitz domain and $y \in \partial\Omega$. Then*

$$q_{S^I_y} \leq q_{1,y} \leq q_{2,y} \leq q_{3,y} \leq q_{S^O_y}. \tag{5.68}$$

*If $1 < q < q_{2,y}$ then, for any real $a$ there exists exactly one solution of* (5.1) *with boundary trace $\gamma\delta_y$.*

*Proof.* It follows from Theorem 5.12 that $q_{S^I_y} \leq q_{1,y}$ and from Proposition 5.13 that $q_{3,y} \leq q_{S^O_y}$. It is clear from the definition and Theorem 3.8 that $q_{1,y} \leq q_{2,y} \leq q_{3,y}$. Thus (5.68) holds.

Now assume that $q < q_{2,y}$ so that there exists a solution $u$ with boundary trace $\delta_y$. By the maximum principle $u > 0$ in $\Omega$. If $a \in (0,1)$ then $au$ is a subsolution of (5.1) with boundary trace $a\delta_y$ and $au < u$. Therefore by Corollary 4.4 II, the smallest solution dominating $au$ has boundary trace $a\delta_y$. If $a > 1$ then $au$ is a supersolution and the same conclusion follows from Corollary 4.4 I. If $v_a$ is the (unique) solution of (5.1) with boundary trace $a\delta_y$ then $-v$ is the (unique) solution with boundary trace $-a\delta_y$.

□

**Theorem 5.16** *Assume $y \in \partial\Omega$ is such that $S^O_y = S^I_y = S$, let $\lambda_S$ be the first eigenvalue of $-\Delta'$ in $W^{1,2}_0(S)$ and denote*

$$q_{c,y} := 1 + 2/\alpha_S \tag{5.69}$$

*with $\alpha_S$ as in* (5.10)*. Then $q_{1,y} = q_{2,y} = q_{3,y} = q_{c,y}$ and*
(i) *if $1 < q < q_{c,y}$ then $\delta_y$ is admissible;*

(ii) *if* $q > q_{c,y}$ *then the only solution of* (5.1) *in* $\Omega$ *vanishing on* $\partial\Omega \setminus \{y\}$ *is the trivial solution.*
(iii) *if* $q = q_{c,y}$ *and* $u$ *is a solution of* (5.1) *in* $\Omega$ *vanishing on* $\partial\Omega \setminus \{y\}$ *then*

$$u = o(1)|x-y|^{-\frac{2}{q-1}} \quad \text{as } x \to y \text{ in } \Omega. \tag{5.70}$$

*Remark.* We know that, in the conical case, the conclusion of statement (ii) holds for $q = q_{c,y}$ as well. Consequently, in a polyhedral domain $\Omega$, an isolated singularity at a point $y \in \partial\Omega$ is removable if $q \geq q_c(y)$. We do not know if this holds in general Lipschitz domains.

*Proof.* The above assertion, except for statement (iii), is an immediate consequence of Proposition 5.15, Definition 5.10 and the remark following that definition.

It remains to prove (iii). We may assume that $u > 0$. Otherwise we observe that $|u|$ is a subsolution of (5.1) and by Theorem 4.3(ii) there exists a solution $v$ dominating it. It is easy to verify that the smallest solution dominating $|u|$ vanishes on $\partial\Omega \setminus \{y\}$.

For any $r > 0$ let $u_r$ be the extension of $u$ by zero to $D_r := C_{S_r^O} \cap B_r(y)$. Thus $u_r$ is a subsolution in $D_r$, $u_r \in C(\bar{D}_r \setminus \{y\})$ and $u_r = 0$ on $(\partial C_{S_r^O} \cap B_r(y)) \setminus \{y\}$. The smallest solution above it, say $\tilde{u}_r$ is in $C(\bar{D}_r \setminus \{y\})$ and $\tilde{u}_r = 0$ on $(\partial C_{S_r^O} \cap B_r(y)) \setminus \{y\}$. By a standard argument this implies that there exists a positive solution $\tilde{v}_r$ in $D_r$ such that $\tilde{v}_r$ vanishes on $\partial D_r \setminus \{y\}$ and

$$u_r \leq 2\tilde{v}_r \ \text{ in } \ D_r.$$

We extend this solution by zero to the entire cone $C_{S_r^O}$, obtaining a subsolution $\tilde{w}_r$ and finally (again by Theorem 4.3(ii)) a solution $w_r$ in $C_{S_r^O}$ which vanishes on $\partial C_{S_r^O} \setminus \{y\}$ and satisfies

$$u_r \leq 2w_r \ \text{ in } \ D_r.$$

Observe that $q_{S_r^O} \downarrow q_{c,y}$ as $r \downarrow 0$. If $q_{c,y} = q_{S_r^O}$ for some $r > 0$ then the existence of a solution $w_r$ as above is impossible. Therefore we conclude that $q_{c,y} < q_{S_r^O}$ and therefore, by Theorem 5.5, there exists a solution $v_{\infty,r}$ in $C_{S_r^O}$ such that

$$v_{\infty,r}(x) = |x-y|^{-\frac{2}{q-1}} \omega_{S_r^O}((x-y)/|x-y|) \quad \forall x \in C_{S_r^O}.$$

This solution is the maximal solution in $C_{S_r^O}$ so that

$$w_r \leq v_{\infty,r} \ \text{ in } \ D_r.$$

But, since $q = q_{S^O}$, it follows that $\omega_{S^O_r} \to 0$ as $r \to 0$. This implies (5.70).
□

The next result provides an important ingredient in the study of general boundary value problems in Lipschitz domains.

**Theorem 5.17** *Assume that $q > 1$, $\Omega$ is a bounded Lipschitz domain and $u \in \mathcal{U}_+(\Omega)$. If $y \in \mathcal{S}(u)$ and $q < q^*_y$ then, for every $k > 0$, the measure $k\delta_y$ is admissible and*

$$u \geq u_{k\delta_y} \quad \forall k \geq 0. \tag{5.71}$$

*Remark.* If $q > q^*_y$, (5.71) need not hold. For instance, consider the cone $C_S$ with vertex at the origin, such that $S \subset S^{N-1}$ is a smooth domain and $S^{N-1} \setminus S$ is contained in an open half space. Then $q_{c,0} > (N+1)/(N-1)$ while $q_{c,x} = (N+1)/(N-1)$ for any $x \neq 0$ on the boundary of the cone. Thus $q^*(0) < q_{c,0}$. Suppose that $q \in (q^*_0, q_{c,0})$. Let $F$ be a closed subset of $\partial C_S$ such that $0 \in F$ but $0$ is a $C_{2/q,q'}$-thin point of $F$. Let $u$ be the maximal solution in $C_S$ vanishing on $\partial C_S \setminus F$. Then $0 \in \mathcal{S}(u)$ but (5.71) does not hold for any $k > 0$.

*Proof.* Up to an isometry of $\mathbb{R}^N$, we can assume that $y = 0$ and represent $\partial\Omega$ near 0 as the graph of a Lipschitz function. This can be done in the following way: we define the cylinder $C'_R := \{x = (x', x_N) : x' \in B'_R\}$ where $B'_R$ is the $(N-1)$-ball with radius $R$. We denote, for some $R > 0$ and $0 < \sigma < R$,

$$\partial\Omega \cap C'_R = \{x = (x', \eta(x')) : x' \in B'_R\},$$

and

$$\Sigma_{\delta,\sigma} = \{x = (x', \eta(x') + \delta) : x' \in B'_\sigma\},$$

and assume that, if $0 < \delta \leq R$,

$$\Omega^R_\delta = \{x = (x', x_N) : x' \in B'_R,\ \eta(x') < x_N < \eta(x') + R\} \subset \Omega.$$

We can also assume that $\eta(0) = 0$. Although the two harmonic measures in $\Omega$ and $\partial\Omega \cap C'_R$ differ, it follow by Dahlberg's result that there exists a constant $c > 0$ such that, if $\delta < \delta_0 \leq R/2$,

$$c^{-1}\omega^{x_0}_\Omega(E) \leq \omega^{x_0}_{\Omega^R_\delta}(E + \epsilon\mathbf{e}_N) \leq c\,\omega^{x_0}_\Omega(E),$$

for any Borel set $E \subset \partial\Omega \cap C'_\delta$. Therefore, if we set

$$M_{\epsilon,\sigma} = \int_{\Sigma_{\epsilon,\sigma}} u(x) d\omega^{x_0},(x),$$

it follows that $\lim_{\epsilon\to 0} M_{\epsilon,\sigma} = \infty$ since $0 \in \mathcal{S}(u)$. We can suppose that $\sigma$ is small enough so that there exists $\hat{q} \in (q, q_y^*)$ and $M > 0$ such that, for any $p \in [1, \hat{q}]$

$$\int_\Omega K^p(x,z)\rho(x)dx \le M \quad \forall z \in \partial\Omega \cap B_\sigma. \tag{5.72}$$

For fixed $k$ there exists $\epsilon = \epsilon(\delta) > 0$ such that $M_{\epsilon,\sigma} = k$. There exists a uniform Lipschitz exhaustion $\{\Omega_\epsilon\}$ of $\Omega$ with the following properties:

(i) $\Omega_\epsilon \cap C'_R \cap \{x = (x', x_N) : a < x_N < b\} = \Sigma_{\epsilon,R}$, for some fixed $a$ and $b$.

(ii) The $\Omega_\epsilon$ and $\Omega$ have the same Lipschitz character $L$.

It follows that the Poisson kernel $K^{\Omega_\epsilon}$ in $\Omega_\epsilon$ respectively endows the same properties (5.72) as $K$ except $\Omega$ has to be replaced by $\Omega_\epsilon$, $\rho$ by $\rho_\epsilon := \operatorname{dist}(., \partial\Omega_\epsilon$ and $z$ has to belong to $\partial\Omega_\epsilon \cap B_\sigma$. Next, we consider the solution $v = v_{\epsilon(\sigma)}$ of

$$\begin{cases} -\Delta v + v^q = 0 & \text{in } \Omega_\epsilon \\ \qquad\quad v = u\chi_{\Sigma_{\epsilon,\sigma}} & \text{in } \partial\Omega_\epsilon \end{cases} \tag{5.73}$$

By the maximum principle, $u \ge v$ in $\Omega_\epsilon$. Furthermore $v \le \mathbb{K}^{\Omega_\epsilon}[u\chi_{\Sigma_{\epsilon,\sigma}}]$. Let $\hat{q} = (q + \tilde{q}_\sigma)/2$ and $\omega \subset \Omega$ be a Borel subset. By convexity

$$\int_\omega \left(\mathbb{K}^{\Omega_\epsilon}[u\chi_{\Sigma_{\epsilon,\sigma}}]\right)^{\hat{q}} \rho(x)dx \le M\, M_{\epsilon,\sigma}.$$

Thus, by Hölder's inequality

$$\int_\omega \left(\mathbb{K}^{\Omega_\epsilon}[u\chi_{\Sigma_{\epsilon,\sigma}}]\right)^q \rho(x)dx \le \left(\int_\omega \rho(x)dx\right)^{1-q/\hat{q}} (M\, M_{\epsilon,\sigma})^{q/\hat{q}}.$$

By standard a priori estimates, $v_{\epsilon(\sigma)} \to v_0$ (up to a subsequence) a.e. in $\Omega$, thus $v^q_{\epsilon(\sigma)} \to v^q_0$. By Vitali's theorem and the uniform integrability of the $\{v_{\epsilon(\sigma)}\}$, $v_{\epsilon(\sigma)} \to v_0$ in $L^q_\rho(\Omega)$. Because

$$v_{\epsilon(\sigma)} + \mathbb{G}^{\Omega_\epsilon}[v^q_{\epsilon(\sigma)}] = \mathbb{K}^{\Omega_\epsilon}[u\chi_{\Sigma_{\epsilon,\sigma}}]$$

where $\mathbb{G}^{\Omega_\epsilon}$ is the Green operator in $\Omega_\epsilon$, and

$$\mathbb{K}^{\Omega_\epsilon}[u\chi_{\Sigma_{\epsilon,\sigma}}] \to M_{\epsilon,\sigma}K(.,y) = kK(.,y)$$

as $\sigma \to 0$, it follows that $u \ge v_0$, and $v_0$ satisfies

$$v_0 + \mathbb{G}^\Omega[v_0^q] = kK(.,y).$$

Then $v_0 = u_{k\delta_y}$, which ends the proof. □

**Corollary 5.18** *Let $\{y_j\}_{j=1}^n \subset \partial\Omega$ be a set of points such that*

$$q < \inf\{q^*_{y_j} : j = 1, ..., n\}. \tag{5.74}$$

*Then, for any set of positive numbers $k_1, \cdots, k_n$, there exists a unique solution $u_\mu$ of (5.1) in $\Omega$ with boundary trace $\mu = \sum_{j=1}^n k_j \delta_{y_j}$.*

*If $u \in \mathcal{U}_+(\Omega)$ and $\{y_j\}_{j=1}^n \subset \mathcal{S}(u)$ then $u \geq u_\mu$.*

*Proof.* From Theorem 5.17, $u \geq u_{k_j\delta_{y_j}}$ for any $j = 1, ..., n$. Thus $u \geq \tilde{u}_{\{k\}} = \max(u_{k_j\delta_{y_j}})$, which is a subsolution with boundary trace $\sum_j k_j\delta_{y_j}$. But $\tilde{v}_{\{k\}}$, the solution with boundary trace $\sum_j k_j\delta_{y_j}$ is the smallest solution above $\tilde{u}_{\{k\}}$. Therefore the conclusion of the corollary holds. □

As a consequence one obtains

**Theorem 5.19** *Let $E \subset \partial\Omega$ be a closed set and assume that $q < q^*_E$. Then, for every $\mu \in \mathfrak{M}(\Omega)$ such that $\operatorname{supp}\mu \subset E$ there exists a (unique) solution $u_\mu$ of (5.1) in $\Omega$ with boundary trace $\mu$.*

*If $\{\mu_n\}$ is a sequence in $\mathfrak{M}(\Omega)$ such that $\operatorname{supp}\mu_n \subset E$ and $\mu_n \rightharpoonup \mu$ weak* then $u_{\mu_n} \to u_\mu$ locally uniformly in $\Omega$.*

*If $u \in \mathcal{U}_+(\Omega)$ and $q < q^*_{\mathcal{S}(u)}$ then, for every $\mu \in \mathfrak{M}(\Omega)$ such that $\operatorname{supp}\mu \subset \mathcal{S}(u)$,*

$$u_\mu \leq u. \tag{5.75}$$

*Proof.* Without loss of generality we assume that $\mu \geq)$. Let $\{\mu_n\}$ be a sequence of measures on $\partial\Omega$ of the form

$$\mu_n = \sum_{j=1}^{k_n} a_{j,n}\delta_{y_{j,n}}$$

where $y_{j,n} \in E$, $a_{j,n} > 0$ and $\sum_{j=1}^{k_n} a_{j,n} = \|\mu\|$, such that $\mu_n \rightharpoonup \mu$ weakly*. Passing to a subsequence if necessary, $u_{\mu_n} \to v$ locally uniformly in $\Omega$. In order to prove the first assertion it remains to show that $v = u_\mu$.

If $0 < r$ is sufficiently small, there exists $\hat{q}_r \in (q, q^*_E)$ and $M_r > 0$ such that, for any $p \in [1, \hat{q}_r]$ and every $z \in \partial\Omega$ such that $\operatorname{dist}(z, E) < r$, estimate (5.72) holds. It follows that the family of functions

$$\{K(\cdot, z) : z \in \partial\Omega,\ \operatorname{dist}(z, E) < r\}$$

is uniformly integrable in $L^q_\rho(\Omega)$ and consequently the family

$$\{\mathbb{K}[\nu]; \nu \in \mathfrak{M}(\partial\Omega),\ \|\nu\|_{\mathfrak{M}} \leq 1,\ \operatorname{supp}\nu \subset \{z \in \partial\Omega : \operatorname{dist}(z, E) < r\}\}$$

is uniformly integrable in $L^q_\rho(\Omega)$. By a standard argument (using Vitali's convergence theorem) this implies that $v = u_\mu$. This proves the first two assertions of the theorem.

The last assertion is an immediate consequence of the above together with Corollary 5.18. Indeed, if $E = \mathcal{S}(u)$ then, by Corollary 5.18, $u \geq u_{\mu_n}$. Therefore $u \geq u_\mu$.

$\square$

**Proposition 5.20** *Let $y \in \partial\Omega$ and $1 < q < q_{S^I_y}$. Then there exists a maximal solution $u := U_y$ of (5.1) such that $tr(U_y) = (\{y\}, 0)$. It satisfies*

$$\liminf_{\substack{x \to y \\ \frac{x-y}{|x-y|} \to \sigma}} |x-y|^{2/(q-1)} U_y(x) \geq \omega_{S^I_y}(\sigma), \tag{5.76}$$

*uniformly on any compact subset of $S^I_y$, where $\omega_{S^I_y}$ is the unique positive solution of*

$$\begin{cases} -\Delta'\omega - \lambda_{N,q}\omega + |\omega|^{q-1}\omega = 0 & \text{in } S^I_y \\ \omega = 0 & \text{on } \partial S^I_y, \end{cases} \tag{5.77}$$

*normalized by $\omega(\sigma_0) = 1$ for some fixed $\sigma_0 \in S^I_y$.*

*For $r > 0$ small enough, we denote by $\omega_{S^O_{y,r}}$ the unique positive solution of*

$$\begin{cases} -\Delta'\omega - \lambda_{N,q}\omega + |\omega|^{q-1}\omega = 0 & \text{in } S^O_{y,r} \\ \omega = 0 & \text{on } \partial S^O_{y,r}, \end{cases} \tag{5.78}$$

*normalized in the same way. Then*

$$\limsup_{\substack{x \to y \\ \frac{x-y}{|x-y|} \to \sigma}} |x-y|^{2/(q-1)} U_y(x) \leq \omega_{S^O_{y,r}}(\sigma). \tag{5.79}$$

*Finally, if $S^O_y = S^I_y = S$, then*

$$\lim_{\substack{x \to y \\ \frac{x-y}{|x-y|} \to \sigma}} |x-y|^{2/(q-1)} U_y(x) = \omega_S(\sigma). \tag{5.80}$$

*Proof.* We recall that $C^I_{y,r}$ (resp. $C^O_{y,r}$) is a r-inner cone (resp. r-outer cone) at $y$ with opening $S^I_{y,r} \subset \partial B_1(y)$ (resp. $S^O_{y,r} \subset \partial B_1(y)$). This is well defined for a $r > 0$ small enough so that $q < q_{S^I_{y,r}}$. We denote by $\omega_{S^I_{y,r}}$ the unique positive solution of

$$\left\{\begin{aligned} -\Delta'\omega - \lambda_{N,q}\omega + |\omega|^{q-1}\omega = 0 \quad & \text{in } S^I_{y,r} \\ \omega = 0 \quad & \text{on } \partial S^I_{y,r}. \end{aligned}\right. \tag{5.81}$$

We construct $U_y \in \mathcal{U}_+(\Omega)$, vanishing on $\partial\Omega \setminus \{y\}$ in the following way. For $0 < \epsilon < r$, we denote by $v := U_{y,\epsilon}$ the solution of

$$\left\{\begin{aligned} -\Delta v + |v^{q-1}|v = 0 \quad & \text{in } \Omega \setminus \overline{B}_\epsilon(y) \\ v = 0 \quad & \text{in } \partial\Omega \setminus \overline{B}_\epsilon(y) \\ v = \infty \quad & \text{in } \Omega \cap \partial B_\epsilon(y). \end{aligned}\right.$$

Let $v := V^I_\epsilon$ (resp. $v := V^O_\epsilon$) be the solution of

$$\left\{\begin{aligned} -\Delta v + |v^{q-1}|v = 0 \quad & \text{in } C_{S^I_{y,r}} \setminus \overline{B}_\epsilon(y) \qquad (\text{resp. } C_{S^O_{y,r}} \setminus \overline{B}_\epsilon(y)) \\ v = 0 \quad & \text{in } \partial C_{S^I_{y,r}} \setminus \overline{B}_\epsilon(y) \qquad (\text{resp. } \partial C_{S^O_{y,r}} \setminus \overline{B}_\epsilon(y)) \\ v = \infty \quad & \text{in } C_{S^I_{y,r}} \cap \partial B_\epsilon(y) \quad (\text{resp. } C_{S^O_{y,r}} \cap \partial B_\epsilon(y)). \end{aligned}\right.$$

Then there exist $m > 0$ depending on $r$, but not on $\epsilon$, such that

$$V^I_\epsilon(x) - m \leq U_{y,\epsilon}(x) \leq V^O_\epsilon(x) + m \tag{5.82}$$

for all $x \in C^I_{y,r} \setminus \{B_\epsilon(y)\}$ for the left-hand side inequality, and $x \in \partial\Omega \cap B_r(y) \setminus \{B_\epsilon(y)\}$ for the right-hand side one. When $\epsilon \to 0$, $V^I_\epsilon$ converges to the explicit separable solution $x \mapsto |x-y|^{-2/(q-1)}\omega_{S^I_{y,r}}$ in $C_{S^I_{y,r}}$ (the positive cone with vertex generated by $S^I_{y,r}$). Similarly $V^O_\epsilon$ converges to the explicit separable solution $x \mapsto |x-y|^{-2/(q-1)}\omega_{S^O_{y,r}}$ in $C_{S^O_{y,r}}$. Furthermore $\epsilon < \epsilon' \Longrightarrow U_{y,\epsilon} \leq U_{y,\epsilon'}$. If $U_y = \lim_{\epsilon\to 0}\{U_{y,\epsilon}\}$, there holds

$$|x-y|^{-2/(q-1)}\omega_{S^I_{y,r}}\left(\frac{x-y}{|x-y|}\right) - m \leq U_y(x) \leq |x-y|^{-2/(q-1)}\omega_{S^I_{y,r}}\left(\frac{x-y}{|x-y|}\right) + m. \tag{5.83}$$

These inequalities imply

$$\liminf_{\substack{x \to y \\ \frac{x-y}{|x-y|} \to \sigma}} |x-y|^{2/(q-1)}U_y(x) \geq \omega_{S^I_{y,r}}(\sigma), \tag{5.84}$$

Inequality (5.79) is obtained in a similar way. Since $\lim_{r\to 0}\omega_{S^I_{y,r}} = \omega_{S^I_y}$ uniformly in compact subsets of $S^I_y$ we also obtain (5.76). If $S^O_y = S^I_y = S$, then $\omega_{S^I_y} = \omega_{S^O_y} = \omega_S$, thus (5.80) holds. $\square$

*Remark.* Because $U_y$ is the maximal solution which vanishes on $\partial\Omega\setminus\{y\}$, the function $u_{\infty\delta_y} = \lim_{k\to\infty} u_{k\delta_y}$ also satisfies inequality (5.79). *We conjecture that $u_{\infty\delta_y}$ always satisfies estimate (5.76).* This is true if the outer and inner cone at $y$ are the same. In fact in that case we obtain a much stronger result:

**Theorem 5.21** *Assume $y \in \partial\Omega$ is such that $S^O_y = S^I_y = S$ and $q < q_{c,y}$. Then $U_y = u_{\infty\delta_y}$.*

*Proof.* Without loss of generality we can assume that $y = 0$ and will denote $B_r = B_r(0)$ for $r > 0$. Let $C^I_r$ (resp. $C^O_r$) be a cone with vertex 0, such that $\overline{C^I_r \cap B_r} \setminus \{0\} \subset \Omega$ (resp. $\Omega \cap B_r \subset C^O_r$). We recall that the characteristic exponents $\alpha_{S^I_0}$ and $\alpha_{S^O_0}$ are defined according to Definition 5.9 and Definition 5.10. Since

$$\alpha_{S^I_0} = \lim_{r\to 0} \alpha_{S^I_{0,r}} = \lim_{r\to 0} \alpha_{S^O_{0,r}} = \alpha_{S^O_0} < 2/(q-1),$$

we can choose $r$ such that

$$q\alpha_{S^I_{0,r}} - \alpha_{S^O_{0,r}} < 2 - (q-1)(\alpha_{S^I_{0,r}} - \alpha_{S^O_{0,r}}), \tag{5.85}$$

and for simplicity, we set $\alpha_{S^I_{0,r}} = \alpha_I$, $\alpha_{S^O_{0,r}} = \alpha_O$ and

$$\gamma_r = \frac{q-1}{2 + \alpha_O - q\alpha_I}.$$

*Step 1.* We claim that there exists $c > 0$ and $c^* > 0$ such that, for any $m > 0$

$$u_{m\delta_0}(x) \geq c^* m|x|^{-\alpha_O} \quad \forall x \in B_{cm^{-\gamma_r}} \cap C^I_r. \tag{5.86}$$

Since $mK(.,0)$ is a super-solution for (5.1),

$$u_{m\delta}(x) \geq mK(x,0) - m^q \int_\Omega G(z,x)K^q(z,0)dz.$$

If we assume that $x \in \overline{C^I_r \cap B_r}$, then dist $(x, \partial\Omega) \geq \theta|x|$ for some $\theta > 0$ since $\overline{C^I_r \cap B_r} \setminus \{0\} \subset \Omega$. Using Bogdan's estimate and Harnack inequality we derive

$$K(x,0) \geq c_1 \frac{|x|^{2-N}}{G(x,x_0)},$$

for some fixed point $x_0$ in $\Omega$. But the Green function in $\Omega \cap B_r$ is dominated by the Green function in $C^O_r \cap B_r$, thus $G(x,x_0) \leq c_2|x|^{\tilde\alpha_O}$ where $\tilde\alpha_O = 2 - N + \alpha_O$. This implies

$$K(x,0) \geq c_3|x|^{-\alpha_O} \quad \forall x \in C^I_r \cap B_r. \tag{5.87}$$

Similarly (and it is a very rough estimate)

$$K(x,0) \leq c_4|x|^{-\alpha_I} \quad \forall x \in \Omega$$

Because $G(x,z) \leq c_5|x-z|^{2-N}$, we obtain

$$\int_\Omega G(z,x)K^q(z,0)dz \leq c_6 \int_{B_R} |x-z|^{2-N}|z|^{-\alpha_I}\,dz.$$

We write

$$\begin{aligned}\int_{B_R} |x-z|^{2-N}|z|^{-q\alpha_I}\,dz = &\int_{B_{2|x|}} |x-z|^{2-N}|z|^{-q\alpha_I}\,dz \\ &+ \int_{B_R\setminus B_{2|x|}} |x-z|^{2-N}|z|^{-q\alpha_I}\,dz.\end{aligned}$$

But

$$\int_{B_{2|x|}} |x-z|^{2-N}|z|^{-q\alpha_I}\,dz = |x|^{2-q\alpha_I}\int_{B_2(0)} |\xi-t|^{2-N}|t|^{-q\alpha_I}\,dt$$

where $\xi = x/|x|$ is fixed. In the same way

$$\begin{aligned}\int_{B_R\setminus B_{2|x|}} |x-z|^{2-N}|z|^{-q\alpha_I}\,dz &\leq \int_{B_R\setminus B_{2|x|}} |z|^{2-N-q\alpha_I}\,|x|^{2-q\alpha_I}\,dz \\ &\leq |x|^{2-q\alpha_I}\int_{B_{R/|x|}\setminus B_2} |t|^{2-N-q\alpha_I}\,dt \\ &\leq c_7|x|^{2-q\alpha_I}\int_2^{R/|x|} s^{1-q\alpha_I}\,ds.\end{aligned}$$

Thus

$$\int_{B_R\setminus B_{2|x|}} |x-z|^{2-N}|z|^{-q\alpha_I}\,dz \leq \begin{cases} c_8 & \text{if } 1-q\alpha_{_I} > -1 \\ c_8\,|\ln|x|| & \text{if } 1-q\alpha_{_I} = -1 \\ c_8|x|^{2-q\alpha_I} & \text{if } 1-q\alpha_{_I} < -1. \end{cases} \tag{5.88}$$

Combining (5.87) and (5.88) yields to (5.86).

*Step 2.* There holds

$$u_{\infty\delta_0}(x) \geq \left(|x|^{-2/q-1)} - r^{-2/(q-1)}\right)\omega_{_{S_r^I}}(x/|x|) \quad \forall x \in C_r^I \cap B_r, \tag{5.89}$$

where $\omega_{S_r^I}$ is the unique positive solution of (5.81). For $\ell > 0$, let $u^I_{\ell\delta_0}$ be the solution of

$$\left\{\begin{array}{rl} -\Delta u + u^q = 0 & \text{in } C_r^I \\ u = \ell\delta_0 & \text{on } \partial C_r^I. \end{array}\right. \tag{5.90}$$

By comparing $u^I_{\ell\delta_0}$ with the Martin kernel in $C_r^I$,

$$u^I_{\ell\delta_0}(x) \leq c_{10}\ell|x|^{-\alpha_I} \quad \forall x \in C_r^I. \tag{5.91}$$

Because

$$c_{10}\ell|x|^{-\alpha_I} \leq c^* m|x|^{-\alpha_O} \quad \forall x \quad \text{s.t. } |x| \geq c_{11}\left(\frac{\ell}{m}\right)^{(\alpha_I-\alpha_O)^{-1}}, \tag{5.92}$$

it follows

$$u_{m\delta_0}(x) \geq u^I_{\ell\delta}(x) \quad \forall x \quad \text{s.t. } c_{11}\left(\frac{\ell}{m}\right)^{(\alpha_I-\alpha_O)^{-1}} \leq |x| \leq c^* m^{-\gamma_r}. \tag{5.93}$$

Notice that (5.85) implies

$$\left(\frac{\ell}{m}\right)^{(\alpha_I-\alpha_0)^{-1}} = o(m^{-\gamma_r}) \quad \text{as } m \to \infty.$$

Since $u^I_{\ell\delta_0}(x) \leq |x|^{-2/(q-1)}\omega_{S_r^I}(x/|x|)$, it follows, by the maximum principle, that

$$u_{m\delta_0}(x) \geq u^I_{\ell\delta_0}(x) - r^{-2/(q-1)}\omega_{S_r^I}(x/|x|)$$

for every $x \in C_r^I \cap B_r$ such that $|x| \geq c_{11}\left(\frac{\ell}{m}\right)^{(\alpha_I-\alpha_O)^{-1}}$. Letting successively $m \to \infty$ and $\ell \to \infty$ and using

$$\lim_{\ell\to\infty} u^I_{\ell\delta_0}(x) = |x|^{-2/(q-1)}\omega_{S_r^I}(x/|x|) \quad \forall x \in C_r^I,$$

we obtain (5.89).

*Step 3.* Let $u \in \mathcal{U}_+(\Omega)$, $u$ vanishing on $\partial\Omega \setminus \{0\}$. Because

$$u(x) \leq C_{N,q}|x|^{-2/(q-1)}$$

and $\overline{C_r^I \cap B_r} \setminus \{0\} \subset \Omega$, it is a classical consequence of Harnack inequality that, for any $x$ and $x' \in \overline{C_r^I \cap B_{r/2}}$ such that $2^{-1}|x| \leq |x'| \leq 2|x|$, $u$ satisfies

$$c_{12}^{-1}u(x') \leq u(x) \leq c_{12}u(x'),$$

where $c_{12} > 0$ depends on $N$, $q$ and $\min\left\{\operatorname{dist}(z,\partial\Omega)/|z| : z \in \overline{C_r^I \cap B_r}\right\}$.

*Step 4.* There exists $c_{13} = c_{13}(q,\Omega) > 0$ such that

$$U_0(x) \leq c_{13} u_{\infty\delta}(x) \quad \forall x \in \Omega. \tag{5.94}$$

Because of (5.79) and the fact that for $r > 0$ and any compact subset $K \subset S_{0,r}^I$

$$1 \leq \frac{\omega_{S_{0,r}^O}(\sigma)}{\omega_{S_{0,r}^I}(\sigma)} \leq M \quad \forall \sigma \in K,$$

where $M$ depends on $K$, there exists $c_{14} > 0$ such that

$$1 \leq \frac{U_0(x)}{u_{\infty\delta_0}(x)} \leq c_{14} \quad \forall x \in B_r \text{ s.t. } x/|x| \in K.$$

Using Step 3, there also holds

$$c_{15}^{-1} \leq \min\left\{\frac{U_0(x')}{U_0(x)}, \frac{u_{\infty\delta_0}(x')}{u_{\infty\delta_0}(x)}\right\} \leq \max\left\{\frac{U_0(x')}{U_0(x)}, \frac{u_{\infty\delta_0}(x')}{u_{\infty\delta_0}(x)}\right\} \leq c_{15} \quad \forall x, x' \in B_{r/2}, \tag{5.95}$$

provided $x/|x|$ and $x'/|x'| \in K$ and $2^{-1}|x| \leq |x'| \leq 2|x|$. For $0 < s \leq r/2$, set $\Gamma_s = \Omega \cap \partial B_s$. There exists $n_0 \in \mathbb{N}_*$ and $\kappa \in (0, 1/4)$, independent of $s$, such that for any $x \in \Gamma_s$ such that $x/|x| \in K$, there exists at most $n_0$ points $a_j$ $(j = 1, ...j_x)$ such that $a_j \in \Gamma_s$, $a_1 \in \partial\Omega$, $\kappa s \leq \operatorname{dist}(a_j, \partial\Omega) \leq s$, $|a_j - a_{j+1}| \leq s/2$ for $j = 1, ...j_x$ and $a_{j_x} = x$. Using Proposition 9.1 and the remark hereafter,

$$c^{-1}\frac{U_0(z)}{U_0(a_1)} \leq \frac{u_{\infty\delta_0}(z)}{u_{\infty\delta_0}(a_1)} \leq c\frac{U_0(z)}{U_0(a_1)} \quad \forall z \in \Gamma_s \cap B_{a_0}.$$

Combining with (5.95) we derive

$$U_0(x) \leq c c_{15}^{n_0} u_{\infty\delta_0}(x) \quad \forall x \in \Gamma_s.$$

Because $c c_{15}^{n_0} u_{\infty\delta_0}$ is a super-solution of (5.1) (clearly $c c_{15}^{n_0} > 1$),

$$U_0 \leq c c_{15}^{n_0} u_{\infty\delta_0} \text{ in } \Omega \setminus \overline{B}_s \qquad \forall s \in (0, r].$$

Thus (5.94) follows with $c_{13} = c c_{15}^{n_0}$.

*Step 5.* End of the proof. It is based upon an idea introduced in [22]. If we assume $U_0 > u_{\infty\delta_0}$, the convexity of $x \mapsto x^q$ implies that the function

$$v = u_{\infty\delta_0} - \frac{1}{2c_{13}}(U_0 - u_{\infty\delta_0})$$

is a super solution such that

$$au_{\infty\delta_0} \leq v < u_{\infty\delta_0}$$

where $a = \frac{1+c_{13}}{2c_{13}} < 1$. Since $au_{\infty\delta_0}$ is a subsolution, it follows that there exists a solution $w$ such that

$$au_{\infty\delta_0} < w < v < u_{\infty\delta_0}.$$

But this is impossible because, for any $a \in (0,1)$, the smallest solution dominating $au_{\infty\delta_0}$ is $u_{\infty\delta_0}$.

□

The next result extends a theorem of Marcus and Véron [22].

**Theorem 5.22** *Assume that $\Omega$ is a bounded Lipschitz domain such that $S_y^O = S_y^I = S_y$ for every $y \in \partial\Omega$. Further, assume that*

$$1 < q < q^*_{\partial\Omega}.$$

*Then for any outer regular Borel measure $\bar{\nu}$ on $\partial\Omega$ there exists a unique solution $u$ of (5.1) such that $tr_{\partial\Omega}(u) = \bar{\nu}$.*

*Proof.* We assume $\bar{\nu} \sim (\nu, F)$ in the sense of Definition 4.9 where $F$ is a closed subset of $\partial\Omega$ and $\nu$ a Radon measure on $\mathcal{R} = \partial\Omega \setminus F$. We denote by $U_F$ the maximal solution of (5.1) defined in Lemma 4.11. Because $q < q^*_{\partial\Omega}$, for any $y \in F$ there exists $u_{\infty\delta_y}$ (and actually $u_{\infty\delta_y} = U_y$ by Theorem 5.21). Then $U_F \geq u_{\delta y}$ by Lemma 4.13, thus $\mathcal{S}(U_F) = F' = F$ with the notation of Definition 4.12. By Theorem 5.19, any Radon measure is q-admissible thus for any compact subset $E \subset \mathcal{R}$ there exist a unique solution $u_{\nu\chi_E}$ of (5.1) with boundary trace $\nu\chi_E$. Therefore there exists a solution with boundary trace $\bar{\nu}$ and, by Theorem 4.14, its uniqueness is reduced to showing that $U_F$ is the unique solution with boundary trace $(0, F)$. Assume $u_F$ is any solution with trace $(0, F)$. By Theorem 5.17 and Theorem 5.21, there holds

$$u_F(x) \geq u_{\infty\delta_y}(x) = U_y(x) \qquad \forall y \in F,\ \forall x \in \Omega. \tag{5.96}$$

Next we prove:

*Assertion. There exists $C > 0$ depending on $F$, $\Omega$ and $q$ such that*

$$U_F(x) \leq Cu_F(x) \qquad \forall x \in \Omega. \tag{5.97}$$

There exists $r_0 > 0$ and a circular cone $C_0$ with vertex 0 and opening $S_0 \subset \partial B_1$ such that for any $y \in \partial\Omega$ there exists an isometry $\mathcal{R}_y$ of $\mathbb{R}^N$ such that $\mathcal{R}_y(\overline{C}_0) \cap B_{r_0}(y) \subset \Omega \cup \{y\}$. We shall denote by $C_1$ a fixed sub-cone of $C_0$ with vertex 0 and opening $S_1 \Subset S_0$. In order to simplify the geometry, we shall assume that both $C_0$ and $C_0$ are radially symmetric cones. If $x \in \Omega$ is such that $\text{dist}\,(x, \partial\Omega) \leq r_0/2$, either

(i) there exists some $y \in \mathcal{S}$ and an isometry $\mathcal{R}_y$ such that $\mathcal{R}_y(\overline{C}_0) \cap B_{r_0}(y) \subset \Omega \cup \{y\}$ and $(x-y)/|x-y| \in S_1$,

(ii) or such a $y$ and $\mathcal{R}_y$ does not exist.

In the first case, it follows from Proposition 5.20 and Theorem 5.21 that

$$u_F(x) \geq c_1 |x-y|^{-2/(q-1)}. \tag{5.98}$$

Furthermore, the constant $c_1$ depends on $r$, $S$ $q$ and $\Omega$, but not on $u_F$. By (5.5)

$$U_F(x) \leq c_2 \left(\text{dist}\,(x, \partial\Omega)\right)^{-2/(q-1)}. \tag{5.99}$$

Since in case (i), there holds $\text{dist}\,(x, \partial\Omega) \geq c_3|x-y|$ for some $c_3 > 1$ depending on $S_0$ and $S_1$, it follows that (5.97) holds with $c = c_1 c_2^{2/(q-1)}/c_3$.

In case (ii), $x$ does not belong to any cone radially symmetric cones with opening $S_1$ and vertex at some $y \in \mathcal{S}$. Therefore, there exists $c_4 < 1$ depending on $C_1$ such that

$$\text{dist}\,(x, \partial\Omega) \leq c_4 \text{dist}\,(x, \mathcal{S}). \tag{5.100}$$

We denote $r_x := \text{dist}\,(x, \mathcal{S})$. If

$$\text{dist}\,(x, \partial\Omega) \leq \min\{c_4, 10^{-1}\} r_x, \tag{5.101}$$

there exists $\xi_x \in \partial\Omega$ such that $|x - \xi_x| \text{dist}\,(x, \partial\Omega)$. Then $B_{9r_x/10}(\xi_x) \subset B_{r_x}(x)$. We can apply Proposition 9.1 in $\Omega \cap B_{9r_x/10}(\xi_x)$. Since $x \in B_{r_x/5}(\xi_x)$, there holds

$$c_5^{-1} \frac{u_F(z)}{U_F(z)} \leq \frac{u_F(x)}{U_F(x)} \leq c_5 \frac{u_F(z)}{U_F(z)} \qquad \forall z \in B_{r_x/5}(\xi_x) \cap \Omega. \tag{5.102}$$

We can take in particular $z$ such that $|z - \xi_x| = r_x/5$ and $\text{dist}\,(z, \partial\Omega) = \max\{\text{dist}\,(t, \partial\Omega) : t \in B_{r_x/5}(\xi_x) \cap \Omega\}$. Since the distance from $z$ to $\mathcal{S}$ is comparable to $\text{dist}\,(z, \partial\Omega)$, there exist $n_0 \in \mathbb{N}_*$ depending on the geometry of $\Omega$ and $n_0$ points $\{a_j\}$ with the properties that $\text{dist}\,(a_j, \partial\Omega) \geq \text{dist}\,(z, \partial\Omega)$, $B_{r_x/10}(a_j) \cap B_{r_x/10}(a_{j+1}) \neq \emptyset$ for $j = 1, ..., n_0 - 1$, $a_1 = z$ and $a_{n_0}$ have the property (i) above, that is there exists some $y \in \mathcal{S}$ and an isometry $\mathcal{R}_y$ such that $\mathcal{R}_y(\overline{C}_0) \cap B_{r_0}(y) \subset \Omega \cup \{y\}$ and $(a_{n_0} - y)/|a_{n_0} - y| \in S_1$. By classical Harnack inequality (see Theorem 5.21 Step 3), there holds

$$u_F(a_j) \geq c_6 u_F(a_{j+1}) \quad \text{and} \quad U_F(a_j) \geq c_6^{-1} U_F(a_{j+1})$$

for some $c_6 > 1$ depending on $N$, $q$ and $\Omega$ via the cone $C_0$. Therefore

$$U_F(x) \leq c_5 c_6^{2n_0} \frac{u_F(a_{n_0})}{U_F(a_{n_0})} u_F(x) \leq c_7 u_F(x), \tag{5.103}$$

which implies (5.97) from case (i) applied to $a_{n_0}$.

Finally, if (5.100) holds, but also

$$\text{dist}\,(x, \partial\Omega) \geq \min\{c_4, 10^{-1}\} r_x, \tag{5.104}$$

this means that $\text{dist}\,(x, \partial\Omega)$ is comparable to $r_x$. Then we can perform the same construction as in the case (5.101) holds, except that we consider balls $B_{\text{dist}\,(x,\partial\Omega)/4}(a_j)$ in order to connect $x$ to a point $a_{n_0}$ satisfying (i). The number $n_0$ is always independent of $u_F$. Thus we derive again estimate (5.97) provided $\text{dist}\,(x, \partial\Omega) \leq r_0/2$. In order to prove that this holds in whole $\Omega$, we consider some $0 < r_1 \leq r_0/2$ such that $\Omega'_{r_1} := \{x \in \Omega : \text{dist}\,(x, \Omega) > r_1\}$ is connected. The function $v$ solution of

$$\left\{ \begin{aligned} -\Delta v + v^q &= 0 \quad \text{in } \Omega'_{r_1} \\ v &= c_1 u_F \quad \text{in } \partial\Omega'_{r_1} \end{aligned} \right. \tag{5.105}$$

is larger that $U_F$ in $\Omega'_{r_1}$. Since $c_1 u_F$ is a super solution, $v \leq c_1 u_F$ in $\Omega'_{r_1}$. This implies that (5.97) holds in $\Omega$.

Inequality (5.97) implies uniqueness by the same argument as in the proof of Theorem 5.21, Step 5. $\square$

# 6 The Martin kernel and critical values for a cone.

## 6.1 The geometric framework

An N-dim polyhedra $P$ is the bounded domain bordered by a finite number of hyperplanes. Thus characteristic elements of the boundary of $P$ are the faces (subsets of an hyperplane), the vertex (intersection of $N$ hyperplanes) and a wide variety of N-k dimensional edges, where k ranges from 2 to N. An N-k dimensional edge, i.e. an intersection of k hyperplanes, will be described by the characteristic angles of these hyperplanes.

We recall that the spherical coordinates in $\mathbb{R}^N = \{x = (x_1, ...x_N)\}$ are expressed by

$$\begin{cases} x_1 = r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\sin\theta_1 \\ x_2 = r\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2\cos\theta_1 \\ x_3 = r\sin\theta_{N-1}\sin\theta_{N-2}...\cos\theta_2 \\ . \\ . \\ . \\ x_{N-1} = r\sin\theta_{N-1}\cos\theta_{N-2}, \\ x_N = r\cos\theta_{N-1} \end{cases} \tag{6.1}$$

where $\theta_1 \in [0, 2\pi]$ and $\theta_\ell \in [0, \pi]$ for $\ell = 2, 3, ..., N-1$ (the $\theta_j$ are the Euler angles). Thus the "angular" component $\sigma \in S^{N-1}$ of the spherical coordinates $(r, \sigma)$ of $x \in \mathbb{R}^N$ is denoted by $\sigma = (\theta_1, ..., \theta_{N-1})$.

We consider an unbounded *non-degenerate k-dihedron* defined as follows. Let $k \in [2, N] \cap \mathbb{N}$ and let $A$ be given by

$$A = \begin{cases} (0, \alpha_1) \times \prod_{j=2}^{k-1}(\alpha_j, \alpha'_j) & \text{if } k > 2 \\ (0, \alpha_1) & \text{if } k = 2 \end{cases}$$

where

$$0 < \alpha_1 < 2\pi, \quad 0 < \alpha_j < \alpha'_j < \pi \quad j = 2, ..., k-1.$$

We denote by $S_A$ the spherical domain

$$S_A = \{x \in \mathbb{R}^N : |x| = 1,\, \sigma \in A \times \prod_{j=k}^{N-1}[0, \pi]\} \subset S^{N-1}\} \tag{6.2}$$

and by $D_A$ the corresponding k-dihedron,

$$D_A = \{x = (r, \sigma) : r > 0, \sigma \in S_A\}.$$

The *edge* of $D_A$ is the (N-k)-dimensional space

$$d_A = \{x : x_1 = x_2 = ... = x_k = 0\}. \tag{6.3}$$

## 6.2 Separable harmonic functions and the Martin kernel in a k-dihedron.

In the system of spherical coordinates, the Laplacian takes the form

$$\Delta u = \partial_{rr}u + \frac{N-1}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^{N-1}} u$$

where the Laplace-Beltrami operator $\Delta_{_{S^{N-1}}}$ is expressed by induction by

$$\begin{aligned}\Delta_{_{S^{N-1}}} u = & \frac{1}{(\sin\theta_{N-1})^{N-2}} \frac{\partial}{\partial\theta_{N-1}} \left( (\sin\theta_{N-1})^{N-2} \frac{\partial u}{\partial\theta_{N-1}} \right) \\ & + \frac{1}{(\sin\theta_{N-1})^2} \Delta_{_{S^{N-2}}} u.\end{aligned} \tag{6.4}$$

and

$$\Delta_{_{S^1}} u = \partial_{\theta_1\theta_1} u \tag{6.5}$$

If we compute the positive harmonic functions in the k-dihedron $D_A$ of the form

$$v(x) = v(r,\sigma) = r^\kappa \omega(\sigma) \quad \text{in } D_A, \quad v = 0 \quad \text{in } \partial D_A \setminus \{0\}.$$

we find that $\kappa$ satisfies the algebraic equation

$$\kappa^2 + (N-2)\kappa - \lambda_A = 0 \tag{6.6}$$

where $\lambda_A$ is the first eigenvalue of $-\Delta_{_{S^{N-1}}}$ in $W_0^{1,2}(S_A)$ and $\omega$ is the corresponding normalized eigenfunction:

$$\begin{cases} \Delta_{_{S^{N-1}}}\omega + \lambda_A\omega = 0 & \text{in } S_A \\ \omega = 0 & \text{on } \partial S_A. \end{cases} \tag{6.7}$$

Thus

$$\begin{aligned}\kappa_+ &= \frac{1}{2}\left(2 - N + \sqrt{(N-2)^2 + 4\lambda_A}\right) \\ \kappa_- &= \frac{1}{2}\left(2 - N - \sqrt{(N-2)^2 + 4\lambda_A}\right).\end{aligned} \tag{6.8}$$

Since

$$S^{N-1} = \left\{\sigma \in \mathbb{R}^{N-1} \times \mathbb{R} : \sigma = (\sigma_2 \sin\theta_{N-1}, \cos\theta_{N-1}),\ \sigma_2 \in S^{N-2}\right\}, \tag{6.9}$$

we look for $\omega := \omega^{\{1\}}$ of the form

$$\omega^{\{1\}}(\sigma) = (\sin\theta_{N-1})^{\kappa_+} \omega^{\{2\}}(\sigma_2), \quad \theta_{N-1} \in (0,\pi), \quad \sigma_2 \in S^{N-2}.$$

Here $S^{N-2} = S^{N-1} \cap \{x_N = 0\}$ and we denote

$$S_A^{\{N-2\}} = S_A \cap \{x_N = 0\}, \quad D_A^{\{N-2\}} := D_A \cap \{x_N = 0\} \subset \mathbb{R}^{N-1}.$$

Then (6.8) jointly with relation (6.4) implies

$$\begin{cases} \Delta_{_{S^{N-2}}}\omega^{\{2\}} + (\lambda_A - \kappa_+)\omega^{\{2\}} = 0 & \text{on } S_A^{\{N-2\}} \\ \omega^{\{2\}} = 0 & \text{on } \partial S_A^{\{N-2\}}. \end{cases} \tag{6.10}$$

Since we are interested in $\omega^{\{2\}}$ positive, $\lambda_A^{\{2\}} := \lambda_A - \kappa_+$ must be the first eigenvalue of $-\Delta_{_{S^{N-2}}}$ in $W_0^{1,2}(S_A^{\{N-2\}})$.

Next we look for positive harmonic functions $\tilde{u}$ in $D_A^{\{N-2\}}$ such that

$$\tilde{u}(x_1,\dots,x_{N-1}) = r^{\kappa'}\omega(\sigma_2), \quad \tilde{u} = 0 \text{ on } \partial D_A^{\{N-2\}}$$

The algebraic equation which gives the exponents is

$$(\kappa')^2 + (N-3)\kappa' - \lambda_A^{\{2\}} = 0.$$

Denote by $\kappa'_+$ the positive root of this equation. By the definition of $\lambda_A^{\{2\}}$,

$$\kappa_+^2 + (N-3)\kappa_+ - \lambda_A^{\{2\}} = \kappa_+^2 + (N-2)\kappa_+ - \lambda_A = 0.$$

Therefore $\kappa'_+ = \kappa_+$. Accordingly, if $k \geq 3$, we set

$$\omega^{\{2\}}(\sigma_2) = (\sin\theta_{N-2})^{\kappa_+}\omega^{\{3\}}(\sigma_3),$$

an find that $\omega^{\{3\}}$ satisfies

$$\begin{cases} \Delta_{_{S^{N-3}}}\omega^{\{3\}} + (\lambda_A - 2\kappa_+)\omega^{\{3\}} = 0 & \text{in } S_A^{\{N-3\}} \\ \omega^{\{3\}} = 0 & \text{on } \partial S_A^{\{N-3\}}, \end{cases} \tag{6.11}$$

where

$$S_A^{\{N-3\}} = S_A \cap \{x_N = x_{N-1} = 0\}.$$

Performing this reduction process (N-k) times, we obtain the following results.

(i) If $k > 2$ then

$$\omega(\sigma) = (\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_k)^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1}) \tag{6.12}$$

where

$$\sigma_{N-k+1} \in S^{k-1} = S^{N-1} \cap \{x_N =, x_{N-1} = \dots = x_{k+1} = 0\},$$

and $\omega' := \omega^{\{N-k+1\}}$ satisfies

$$\begin{cases} \Delta_{_{S^{k-1}}}\omega' + (\lambda_A - (N-k)\kappa_+)\omega' = 0, & \text{in } S_A^{\{k-1\}} \\ \omega' = 0, & \text{on } \partial S_A^{\{k-1\}}, \end{cases} \tag{6.13}$$

$$S_A^{\{k-1\}} = S_A \cap \{x_N = x_{N-1} = ... = x_{k+1} = 0\} \approx A$$

and $\lambda_A - (N-k)\kappa_+$ is the first eigenvalue of the problem. In such a case, it is usually impossible to determine more explicitly $\omega^{\{N-k+1\}}$ and $\lambda_A - (N-k)\kappa_+$, except for some very specific values of $\alpha_j$ and $\alpha'_j$, associated to consecutive zeros of generalized Legendre functions.

(ii) If $k = 2$ then

$$\omega(\sigma) = (\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_2)^{\kappa_+}\omega^{\{N-1\}}(\theta_1) \tag{6.14}$$

where $\sigma_{N-1} \in S^1 \approx \theta_1 \in (0, 2\pi)$, and $\omega^{\{N-1\}}$ satisfies

$$\left\{\begin{aligned} \Delta_{S^1}\omega^{\{N-1\}} + (\lambda_A - (N-2)\kappa_+)\omega^{\{N-1\}} = 0 \quad &\text{on } S_A^{\{1\}} \\ \omega^{\{N-1\}} = 0 \quad &\text{on } \partial S_A^{\{1\}}, \end{aligned}\right. \tag{6.15}$$

with $\partial S_A^{\{1\}} \approx (0, \alpha)$. In this case

$$\kappa_+ = \frac{\pi}{\alpha}, \quad \omega^{\{N-1\}}(\theta_1) = \sin(\pi\theta_1/\alpha), \tag{6.16}$$

and

$$\lambda_A - (N-2)\kappa_+ = \frac{\pi^2}{\alpha^2} \Longrightarrow \lambda_A = \frac{\pi^2}{\alpha^2} + (N-2)\frac{\pi}{\alpha}. \tag{6.17}$$

Observe that $\frac{1}{2} \leq \kappa_+$ with equality holding only in the degenerate case $\alpha = 2\pi$ (which we exclude).

In either case, we find a positive harmonic function $v_A$ in $D_A$, vanishing on $\partial D_A$, of the form

$$v_A(x) = |x|^{\kappa_+}\omega(x/|x|)$$

with $\omega$ as in (6.12) (for $k > 2$) or (6.16) (for k=2).

Similarly we find a positive harmonic function in $D_A$ vanishing on $\partial D_A \setminus \{0\}$, singular at the origin, of the form

$$K'_A(x) = |x|^{\kappa_-}\omega(x/|x|), \quad \kappa_- = 2 - N - \kappa_+.$$

Because of the uniqueness of the kernel function (see **A.2**) $K'_A(x)$ is, up to a multiplicative constant $c_A$, the Martin kernel of the Laplacian in $D_A$, with singularity at 0. The Martin kernel, with singularity at a point $z \in d_A$, is given by

$$K_A(x,z) = c_A\frac{(\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_k)^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x-z|^{N-2+\kappa_+}} \tag{6.18}$$

for every $x \in D_A$. From (6.1)

$$\sin\theta_{N-1}\sin\theta_{N-2}...\sin\theta_k = |x-z|^{-1}\sqrt{x_1^2+x_2^2+...+x_k^2}.$$

Therefore, if we write $x \in \mathbb{R}^N$ in the form $x=(x',x'')$, $x'=(x_1,...,x_k)$, $x''=(x_{k+1},\cdots,x_N)$, we obtain the formula,

$$\begin{aligned} K_A(x,z) &= c_{_A}\frac{|x'|^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1})}{|x-z|^{(N-2+2\kappa_+)}} \\ &= c_{_A}\frac{|x'|^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1})}{(|x'|^2+|x''-z|^2)^{(N-2+2\kappa_+)/2}}. \end{aligned} \tag{6.19}$$

Therefore, the Poisson potential of a measure $\mu \in \mathfrak{M}(d_A)$ is expressed by

$$\begin{aligned} \mathbb{K}[\mu](x) &= c_{_A}|x'|^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1}) \\ &\quad\times\int_{\mathbb{R}^{N-k}}\frac{d\mu(z)}{(|x'|^2+|x''-z|^2)^{(N-2+2\kappa_+)/2}}. \end{aligned} \tag{6.20}$$

### 6.3 The admissibility condition

Consider the boundary value problem

$$\left\{\begin{aligned} -\Delta u+|u|^{q-1}u &= 0 \quad \text{in } D_A \\ u &= \mu \in \mathfrak{M}(\partial D_A). \end{aligned}\right. \tag{6.21}$$

Let $r'=|x'|$, $r''=|x''|$,

$$\Gamma_R = \{x=(x',x''): r' \leq R, r'' \leq R\} \tag{6.22}$$

and let $\rho_R$ denote the first (positive) eigenfunction in $D_{A,R} := D_A \cap \Gamma_R$.

By Definition 3.7, the admissibility condition for a measure $\mu \in \mathfrak{M}(d_A \cap \Gamma_R)$ relative to (6.21) in $D_{A,R}$ is

$$\int_{D_{A,R}} \mathbb{K}^R[|\mu|](x)^q \rho_{_R}(x)dx < \infty. \tag{6.23}$$

where $\mathbb{K}^R$ is the Martin kernel of $-\Delta$ in $D_{A,R}$. Near $d_A$ this kernel behaves like the Martin kernel of the dihedron $D_A$. Furthermore, the first eigenfunction $\rho_{_R}$ of $-\Delta$ in $W_0^{1,2}(D_{A,R})$ behaves like the regular harmonic function $v_A$. Therefore

$$\rho_{_R}(x) \approx (r')^{\kappa_+}\omega^{\{N-k+1\}}(\sigma_{N-k+1}) \tag{6.24}$$

and the admissibility condition for a measure $\mu \in \mathfrak{M}(d_A)$ is

$$\int_{\Gamma_R \cap D_A} \mathbb{K}[|\mu|](x)|^q \rho(x)dx < \infty \quad \forall R > 0, \tag{6.25}$$

with $\Gamma_R$ as in (6.22). By (6.20),

$$\mathbb{K}[|\mu|](x) \leq \; c_A (r')^{\kappa_+} \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z) \tag{6.26}$$

where

$$j(x) = |x|^{-N+2-2\kappa_+} \quad \forall x \in \mathbb{R}^N. \tag{6.27}$$

Therefore, using (6.24), condition (6.25) becomes

$$\int_0^R \int_{|x''|<R} \Big( \int_{\mathbb{R}^{N-k}} j(x', x'' - z) d|\mu|(z) \Big)^q (r')^{(q+1)\kappa_+ + k - 1} dx'' dr' < \infty \tag{6.28}$$

for every $R > 0$.

## 6.4 The critical values.

Relative to the equation

$$-\Delta u + |u|^{q-1} u = 0 \tag{6.29}$$

there exist two thresholds of criticality associated with the edge $d_A$.

The first is the value $q_c^*$ such that, for $q_c^* \leq q$ the whole edge $d_A$ is removable relative to this equation, but for $1 < q < q_c^*$ there exist non-trivial solutions in $D_A$ which vanish on $\partial D_A \setminus d_A$. The second $q_c < q_c^*$ corresponds to the removability of points on $d_A$. For $q \geq q_c$ points on $d_A$ are removable while for $1 < q < q_c$ there exist solutions with isolated point singularities on $d_A$. In the next two propositions we determine these critical values.

**Proposition 6.1** *Assume $q > 1$, $1 \leq k < N$. Then the condition*

$$q < q_c^* := 1 + \frac{2 - k + \sqrt{(k-2)^2 + 4\lambda_A - 4(N-k)\kappa_+}}{\lambda_A - (N-k)\kappa_+} \tag{6.30}$$

*is necessary and sufficient for the existence of a non-trivial solution $u$ of (6.29) in $D_A$ which vanishes on $\partial D_A \setminus d_A$. Furthermore, when this condition holds, there exist non-trivial positive bounded measures $\mu$ on $d_A$ such that $\mathbb{K}[\mu] \in L^q_\rho(\Gamma_R \cap D_A)$.*

*Remark.* The statement remains true in the case $k = N$, which is the case of the cone. In this case $q_c = q_c^* = 1 + (2/\alpha_S)$ in the notation of Section 5. However, in the present notation, $\alpha_S = -\kappa_-$ and a straightforward computation yields:

$$q_c = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}}. \tag{6.31}$$

*Proof.* Recall that $\lambda_A - (N-k)\kappa_+$ is the first eigenvalue in $S_A^{\{k-1\}}$ (see (6.13) and the remarks following it). Let $\kappa'_+, \kappa'_-$ be the two roots of the equation

$$X^2 + (k-2)X - (\lambda_A - (N-k)\kappa_+) = 0,$$

i.e.

$$\kappa'_\pm = \frac{1}{2}\big(2 - k \pm \sqrt{(k-2)^2 + 4(\lambda_A - (N-k)\kappa_+)}\big).$$

Then, by Theorem 5.5 and Theorem 5.6, if $1 < q < 1 - (2/\kappa'_-)$ (note that because of a change in notation the entity denoted by $\alpha_S$ in subsection 5.1 is the same as $-\kappa'_-$ in the present section) there exists a unique solution of (6.29) in the cone $C_{S_A^{k-1}}$ i.e. the cone with opening $S_A^{k-1} \subset S^{k-1} \subset \mathbb{R}^k$ with trace $a\delta_0$ (where $\delta_0$ denotes the Dirac measure at the vertex of the cone and $a > 0$). By Theorem 5.6 this solution satisfies

$$u_a(x) = a\,|x|^{-\alpha}\,\phi(x/\,|x|)(1 + o(1)) \quad \text{as } x \to 0, \tag{6.32}$$

where $\phi$ is the first positive eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_A^{k-1})$ normalized so that $u_1$ possesses trace $\delta_0$.

The function $u$ given by

$$\tilde{u}_a(x', x'') = u_a(x') \quad \forall (x', x'') \in D_A = C_{S_A^{k-1}} \times \mathbb{R}^{N-k},$$

is a nonzero solution of (6.29) in $D_A$ which vanishes on $\partial D_A \setminus d_A$ and has bounded trace on $d_A$.

A simple calculation shows that $1 - (2/\kappa_-)$ equals $q_c^*$ as given in (6.30).

Next, assume that $q \geq q_c^*$ and let $u$ be a solution of (6.29) in $D_A$ which vanishes on $\partial D_A \setminus d_A$.

Given $\epsilon > 0$ let $v_\epsilon$ be the solution of (6.29) in $D_A^{\{N-k-1\}} \setminus \{x' \in \mathbb{R}^k : |x'| \leq \epsilon\}$ such that

$$v_\epsilon(x') = \begin{cases} 0, & \text{if } x' \in \partial D_A^{\{N-k-1\}} \\ \infty, & \text{if } |x'| = \epsilon. \end{cases}$$

Given $R>0$ let $w_R$ be the maximal solution in $\{x''\in\mathbb{R}^{N-k}:|x''|<R\}$.

Then the function $u^*$ given by

$$u^*(x',x'')=v_\epsilon(x')+w_R(x'')$$

is a supersolution of (6.29) in $D_A\setminus\{(x',x''):|x'|>\epsilon,\ |x''|<R\}$ and it dominates $u$ in this domain. But $w_R(x'')\to 0$ as $R\to\infty$ and, by [11], $v_\epsilon(x')\to 0$ as $\epsilon\to 0$. Therefore $u_+=0$ and, by the same token, $u_-=0$.

□

**Proposition 6.2** *Let $A$ be defined as before. Then*

$$\mathbb{K}[\mu]\in L^q_\rho(\Gamma_R\cap D_A)\quad\forall\mu\in\mathfrak{M}(d_A),\quad\forall R>0 \tag{6.33}$$

*if and only if*

$$1<q<q_c:=\frac{\kappa_+ +N}{\kappa_+ +N-2}. \tag{6.34}$$

*This statement is equivalent to the following:*

*Condition* (6.34) *is necessary and sufficient in order that the Dirac measure $\mu=\delta_P$, supported at a point $P\in d_A$, satisfy* (6.33).

*Proof.* It is sufficient to prove the result relative to the family of measures $\mu$ such that $\mu$ is positive, has compact support and $\mu(d_A)=1$. Let $R>1$ be sufficiently large so that the support of $\mu$ is contained in $\Gamma_{R/2}$. The measure $\mu$ can be approximated (in the sense of weak convergence of measures) by a sequence $\{\mu_n\}$ of convex combinations of Dirac measures supported in $d_A\cap\Gamma_{R/2}$. For such a sequence $\mathbb{K}[\mu_n]\to\mathbb{K}[\mu]$ pointwise and $\{\mathbb{K}[\mu_n]\}$ is uniformly bounded in $D_A\setminus\Gamma_{3R/4}$. Therefore it is sufficient to prove the result when $\mu=\delta_0$. In this case the admissibility condition (6.28)) is

$$\int_0^R\int_{|x''|<R} j(x)^q(r')^{(q+1)\kappa_+ +k-1}dx''dr'<\infty,$$

i.e.,

$$\int_0^R\int_0^R |x|^{q(2-N-2\kappa_+)}(r')^{(q+1)\kappa_+ +k-1}(r'')^{N-k-1}dr''dr'<\infty.$$

Substituting $\tau:=r''/r'$ the condition becomes

$$\int_0^R\int_0^{R/r'}(1+\tau^2)^{\frac{q}{2}(2-N-2\kappa_+)}(r')^{q(2-N-\kappa_+)+\kappa_+ +N-1}\tau^{N-k-1}d\tau\,dr'<\infty.$$

This holds if and only if $q < (\kappa_+ + N)/(\kappa_+ + N - 2)$.

$\square$

*Remark.* It is interesting to notice that $k$ does not appear explicitly in (6.34). Furthermore, we observe that

$$\frac{2}{q_c - 1}\left(\frac{2q_c}{q_c - 1} - N\right) = \lambda_A \Longleftrightarrow \kappa_+(\kappa_+ + N - 2) = \lambda_A, \tag{6.35}$$

which follows from (6.8). This implies that there does not exist a nontrivial solution of the nonlinear eigenvalue problem

$$\begin{aligned} -\Delta_{S^{N-1}}\psi - \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\psi + |\psi|^{q-1}\psi = 0 \quad &\text{in } S_{D_A} \\ \psi = 0 \quad &\text{in } \partial S_{D_A} \end{aligned} \tag{6.36}$$

which, in turn, implies that there does not exists a nontrivial solution of (6.29) of the form $u(x) = u(r,\sigma) = |x|^{-2/(q-1)}\psi(\sigma)$, and also no solution of this equation in $D_A$ which vanishes on $\partial D_A \setminus \{0\}$. This is the classical ansatz for the removability of isolated singularities in $d_A$.

# 7 The harmonic lifting of a Besov space $B^{-s,p}(d_A)$.

Denote by $W^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0$, $1 \le p \le \infty$) the Sobolev spaces over $\mathbb{R}^\ell$. In order to use interpolation, it is useful to introduce the Besov space $B^{\sigma,p}(\mathbb{R}^\ell)$ ($\sigma > 0$). If $\sigma$ is not an integer then

$$B^{\sigma,p}(\mathbb{R}^\ell) = W^{\sigma,p}(\mathbb{R}^\ell). \tag{7.1}$$

If $\sigma$ is an integer the space is defined as follows. Put

$$\Delta_{x,y} f = f(x+y) + f(x-y) - 2f(x).$$

Then

$$B^{1,p}(\mathbb{R}^\ell) = \left\{ f \in L^p(\mathbb{R}^\ell) : \frac{\Delta_{x,y} f}{|y|^{1+\ell/p}} \in L^p(\mathbb{R}^\ell \times \mathbb{R}^\ell) \right\}, \tag{7.2}$$

with norm

$$\|f\|_{B^{1,p}} = \|f\|_{L^p} + \left( \int\!\!\int_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|\Delta_{x,y} f|^p}{|y|^{\ell+p}} dx\, dy \right)^{1/p}, \tag{7.3}$$

(with standard modification if $p = \infty$) and

$$B^{m,p}(\mathbb{R}^\ell) = \Big\{ f \in W^{m-1,p}(\mathbb{R}^\ell) : \\ D_x^\alpha f \in B^{1,p}(\mathbb{R}^\ell) \; \forall \alpha \in \mathbb{N}^\ell, \; |\alpha| = m-1 \Big\} \tag{7.4}$$

with norm

$$\|f\|_{B^{m,p}} = \|f\|_{W^{m-1,p}} + \left( \sum_{|\alpha|=m-1} \iint_{\mathbb{R}^\ell \times \mathbb{R}^\ell} \frac{|D_x^\alpha \Delta_{x,y} f|^p}{|y|^{\ell+p}} dx\, dy \right)^{1/p}. \tag{7.5}$$

We recall that the following inclusions hold ([30, p 155])

$$\begin{aligned} W^{m,p}(\mathbb{R}^\ell) &\subset B^{m,p}(\mathbb{R}^\ell) \quad \text{if } p \geq 2 \\ B^{m,p}(\mathbb{R}^\ell) &\subset W^{m,p}(\mathbb{R}^\ell) \quad \text{if } 1 \leq p \leq 2. \end{aligned} \tag{7.6}$$

When $1 < p < \infty$, the dual spaces of $W^{s,p}$ and $B^{m,p}$ are respectively denoted by $W^{-s,p'}$ and $B^{-m,p'}$.

The following is the main result of this section.

**Theorem 7.1** *Suppose that $q_c < q < q_c^*$ and let $A$ be defined as in subsection* **6.1***. Then there exist positive constants $c_1, c_2$, depending on $q, N, k, \kappa_+$, such that for any $R > 1$ and any $\mu \in \mathfrak{M}_+(d_A)$ with support in $B_{R/2}$:*

$$\begin{aligned} & c_1 \|\mu\|^q_{B^{-s,q}(\mathbb{R}^{N-k})} \\ & \leq \int_{D_{A,R}} \mathbb{K}[|\mu|]^q(x)\rho(x)dx \leq c_2(1+R)^\beta \|\mu\|^q_{B^{-s,q}(\mathbb{R}^{N-k})}, \end{aligned} \tag{7.7}$$

*where $s = 2 - \frac{\kappa_+ + k}{q'}$, $\beta = (q+1)\kappa_+ + k - 1$ and $D_{A,R} = D_A \cap \Gamma_R$. If $q = q_c$ the estimate remains valid for measures $\mu$ such that the diameter of* $\operatorname{supp}\mu$ *is sufficiently small (depending on the parameters mentioned before).*

*Remark.* When $q \geq 2$ the norms in the Besov space may be replaced by the norms in the corresponding Sobolev spaces.

Recall the admissibility condition for a measure $\mu \in \mathfrak{M}_+(d_A)$:

$$\int_{D_{A,R}} \mathbb{K}[\mu]^q(x)\rho(x)dx < \infty \quad \forall R > 0$$

and the equivalence (see (6.25)–(6.28))

$$\int_{D_{A,R}} \mathbb{K}[\mu]^q(x)\rho(x)dx \approx J^{A,R}(\mu) := \tag{7.8}$$
$$\int_0^R \int_{B''_R} \Big( \int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{(\tau^2+|x''-z|^2)|)^{(N-2+2\kappa_+)/2}} \Big)^q \tau^{(q+1)\kappa_+ +k-1} dx'' d\tau,$$

where $x=(x',x'')\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, $\tau=|x'|$ and $B''_R=\{x''\in\mathbb{R}^{N-k}:|x''|<R\}$. We denote,

$$\nu = N-2+2\kappa_+. \tag{7.9}$$

If $2\kappa_+$ is an integer, it is natural to relate (7.8) to the Poison potential of $\mu$ in $\mathbb{R}^n_+=\mathbb{R}_+\times\mathbb{R}_{n-1}$ where $n=N-2+2\kappa_+$. We clarify this statement below.

Assuming that $2\le n+k-N$, denote

$$y=(y_1,\widetilde{y},y'')\in\mathbb{R}^n, \quad \widetilde{y}=(y_2,\cdots,y_{n+k-N}), \quad y''=(y_{n+k-N+1},\cdots,y_n).$$

The Poisson kernel in $\mathbb{R}^n_+=\mathbb{R}_+\times\mathbb{R}_{n-1}$ is given by

$$P_n(y)=\gamma_n y_1|y|^{-n} \quad y_1>0, \tag{7.10}$$

for some $\gamma_n>0$, and the Poisson potential of a bounded Borel measure $\mu$ with support in

$$\mathbf{d} := \{y=(0,y'')\in\mathbb{R}^n:\, y''\in\mathbb{R}_{N-k}\}$$

is

$$\mathbb{K}_n[\mu](y)=\gamma_n y_1\int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{\big(y_1^2+|\widetilde{y}|^2+|y''-z|^2\big)^{n/2}} \quad \forall y\in\mathbb{R}^n_+. \tag{7.11}$$

In particular, for $\widetilde{y}=0$,

$$\mathbb{K}_n[\mu](y_1,0,y'')=\gamma_n y_1\int_{\mathbb{R}^{N-k}} \frac{d\mu(z)}{\big(y_1^2+|y''-z|^2\big)^{n/2}}. \tag{7.12}$$

The integral in (7.12) is precisely the same as the inner integral in (7.8).

In fact, it will be shown that, if we set

$$n := \{\nu\} = \inf\{m\in\mathbb{N}: m\ge\nu\}, \tag{7.13}$$

this approach also works when $2\kappa_+$ is not an integer. We note that, for $n$ given by (7.13),

$$n - N + k \geq 2, \tag{7.14}$$

with equality only if $k = 3$ and $\kappa_+ \leq 1/2$ or $k = 2$ and $\kappa_+ \in (1/2, 1]$. Indeed,

$$n - N + k = k + \{2\kappa_+\} - 2$$

and (as $\kappa_+ > 0$) $\{2\kappa_+\} \geq 1$. If $k = 2$ then $\kappa_+ > 1/2$ and consequently $\{2\kappa_+\} \geq 2$. These facts imply our assertion.

We also note that $\kappa_+$ is strictly increasing relative to $\lambda_A$ and

$$\kappa_+ \begin{cases} = 1, & \text{if } D_A = \mathbb{R}^N_+, \\ < 1, & \text{if } D_A \subsetneqq \mathbb{R}^N_+, \\ > 1, & \text{if } D_A \supsetneqq \mathbb{R}^N_+. \end{cases} \tag{7.15}$$

Finally we observe that $\gamma := \lambda_A - (N-k)\kappa_+ > 0$ (see (6.13)) and, by (6.8) and (6.30):

$$\gamma = \kappa_+^2 + (k-2)\kappa_+, \quad q_c^* = 1 + \frac{-(k-2) + \sqrt{(k-2)^2 + 4\gamma}}{\gamma}. \tag{7.16}$$

Therefore $q_c^*$ is strictly decreasing relative to $\gamma$ and consequently also relative to $\kappa_+$.

The proof of the theorem is based on the following important result proved in [31, 1.14.4.]

**Proposition 7.2** *Let $1 < q < \infty$ and $s > 0$. Then for any bounded Radon measure $\mu$ in $\mathbb{R}^{n-1}$ there holds*

$$I(\mu) = \int_{\mathbb{R}^n_+} |\mathbb{K}_n[\mu](y)|^q \, e^{-y_1} y_1^{sq-1} dy \approx \|\mu\|^q_{B^{-s,q}(\mathbb{R}^{n-1})}. \tag{7.17}$$

In the first part of the proof we derive inequalities comparing $I(\mu)$ and $J^{A,R}(\mu)$. Actually, it is useful to consider a slightly more general expression than $I(\mu)$, namely:

$$I^{m,j}_{\nu,\sigma}(\mu) := \int_{\mathbb{R}^{m+j}_+} \left| \int_{\mathbb{R}^m} \frac{y_1 d\mu(z)}{\left(y_1^2 + |\tilde{y}|^2 + |y'' - z|^2\right)^{\nu/2}} \right|^q e^{-y_1} y_1^{\sigma q - 1} dy, \tag{7.18}$$

where $\nu$ is an arbitrary number such that $\nu > m$, $j \geq 1$ and $\sigma > 0$. A point $y \in \mathbb{R}^{m+j}_+$ is written in the form $y = (y_1, \tilde{y}, y'') \in \mathbb{R}_+ \times \mathbb{R}^{j-1} \times \mathbb{R}^m$. We assume that $\mu$ is supported in $\mathbb{R}^m$. Note that,

$$I(\mu) = \gamma_n^q I^{m,j}_{n,s} \quad \text{where} \quad m = N - k, \quad j = n - m = n - N + k. \tag{7.19}$$

Put

$$F_{\nu,m}[\mu](\tau) := \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2 + |y'' - z|^2)^{\nu/2}} \right|^q dy'' \quad \forall \tau \in [0, \infty). \tag{7.20}$$

With this notation, if $j \geq 2$ then

$$I^{m,j}_{\nu,\sigma}(\mu) := \int_0^\infty \int_{\mathbb{R}^{j-1}} F_{\nu,m}[\mu](\sqrt{y_1^2 + |\widetilde{y}|^2}\,) e^{-y_1} y_1^{(\sigma+1)q-1} d\widetilde{y}\, dy_1 \tag{7.21}$$

and if $j = 1$

$$I^{m,1}_{\nu,\sigma}(\mu) := \int_0^\infty F_{\nu,m}[\mu](y_1) e^{-y_1} y_1^{(\sigma+1)q-1}\, dy_1 \tag{7.22}$$

**Lemma 7.3** *Assume that $m < \nu$, $0 < \sigma$, $2 \leq j$ and $1 < q < \infty$. Then there exists a positive constant $c$, depending on $m, j, \nu, \sigma, q$, such that, for every bounded Borel measure $\mu$ with support in $\mathbb{R}^m$:*

$$\frac{1}{c} \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau \leq I^{m,j}_{\nu,\sigma}(\mu) \leq c \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau, \tag{7.23}$$

*where $F_{\nu,m}$ is given by (7.20) and, for every $\tau > 0$,*

$$h_{\sigma,j}(\tau) = \begin{cases} \dfrac{\tau^{(\sigma+1)q+j-2}}{(1+\tau)^{(\sigma+1)q}}, & \text{if } j \geq 2, \\ e^{-\tau} \tau^{(\sigma+1)q-1}, & \text{if } j = 1. \end{cases} \tag{7.24}$$

*Proof.* There is nothing to prove in the case $j = 1$. Therefore we assume that $j \geq 2$.

We use the notation $y = (y_1, \widetilde{y}, y'') \in \mathbb{R} \times \mathbb{R}^{j-1} \times \mathbb{R}^m$. The integrand in (7.21) depends only on $y_1$ and $\rho := |\widetilde{y}|$. Therefore, $I^{m,j}_{\nu,\sigma}$ can be written in the form

$$I^{m,j}_{\nu,\sigma}(\mu) = c_{m,j} \int_0^\infty \int_0^\infty F_{\nu,m}[\mu](\sqrt{y_1^2 + \rho^2}) e^{-y_1} y_1^{(\sigma+1)q-1}\, dy_1 \rho^{j-2} d\rho.$$

We substitute $y_1 = (\tau^2 - \rho^2)^{1/2}$, then change the order of integration and finally substitute $\rho = r\tau$. This yields,

$$\begin{aligned} & c^{-1}_{m,j} I^{m,j}_{\nu,\sigma}(\mu) \\ &= \int_0^\infty \int_\rho^\infty F_{\nu,m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2-\rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau\, d\tau\, d\rho \\ &= \int_0^\infty \int_0^\tau F_{\nu,m}[\mu](\tau) \rho^{j-2} e^{-\sqrt{\tau^2-\rho^2}} (\tau^2 - \rho^2)^{(\sigma+1)q/2-1} \tau\, d\rho\, d\tau \\ &= \int_0^\infty \int_0^1 F_{\nu,m}[\mu](\tau) \tau^{j-2+(\sigma+1)q} e^{-\tau\sqrt{1-r^2}} f(r) dr\, d\tau, \end{aligned}$$

where

$$f(r) = r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

We denote

$$I^j_\sigma(\tau) = \int_0^1 e^{-\tau\sqrt{1-r^2}} f(r)dr,$$

so that

$$I^{m,j}_{\nu,\sigma}(\mu) = c_{m,j} \int_0^\infty F_{\nu,m}[\mu](\tau)\tau^{j-2+(\sigma+1)q} I^j_\sigma(\tau)d\tau. \tag{7.25}$$

To complete the proof we estimate $I^j_\sigma$. Since $j \geq 2$, $f \in L^1(0,1)$ and $I^j_\sigma$ is continuous in $[0,\infty)$ and positive everywhere. Hence, for every $\alpha > 0$, there exists a positive constant $c_\alpha = c_\alpha(\sigma)$ such that

$$\frac{1}{c_\alpha} \leq I^j_\sigma \leq c_\alpha \quad \text{in} \quad [0,\alpha). \tag{7.26}$$

Next we estimate $I^j_\sigma$ for large $\tau$. Since $j \geq 2$,

$$I^j_\sigma \leq 2^{(\sigma+1)q/2-1} \int_0^1 (1-r)^{(\sigma+1)q/2-1} e^{-\tau\sqrt{1-r}} dr.$$

Substituting $r = 1 - t^2$ we obtain,

$$I^j_\sigma \leq 2^{(\sigma+1)q/2} \int_0^1 t^{(\sigma+1)q-1} e^{-t\tau} dt = c(\sigma,q)\tau^{-(\sigma+1)q}. \tag{7.27}$$

On the other hand, if $\tau \geq 2$,

$$\begin{aligned} I^j_\sigma(\tau) &= \int_0^1 (1-t^2)^{(j-3)/2} t^{(\sigma+1)q-1} e^{-\tau t} dt \\ &= \tau^{-(\sigma+1)q} \int_0^\tau (1-(s/\tau)^2)^{(j-3)/2} s^{(\sigma+1)q-1} e^{-s} ds \\ &\geq \tau^{-(\sigma+1)q} 2^{-(j-3)} \int_0^1 s^{(\sigma+1)q-1} e^{-s} ds. \end{aligned} \tag{7.28}$$

Combining (7.25) with (7.26)–(7.28) we obtain (7.23).

□

Next we derive an estimate in which integration over $\mathbb{R}^n_+ = \mathbb{R}^j_+ \times \mathbb{R}^m$ is replaced by integration over a bounded domain, for measures supported in a fixed bounded subset of $\mathbb{R}^m$.

Let $B_R^j(0)$ and $B_R^m(0)$ denote the balls of radius $R$ centered at the origin, in $\mathbb{R}^j$ and $\mathbb{R}^m$ respectively. Denote

$$F_{\nu,m}^R[\mu](\tau)=\int_{B_R^m}\left|\int_{\mathbb{R}^m}\frac{d\mu(z)}{(\tau^2+|y''-z|^2)^{\nu/2}}\right|^q dy'' \quad \forall\tau\in[0,\infty) \tag{7.29}$$

and, if $j\geq 2$,

$$I_{\nu,\sigma}^{m,j}(\mu;R)=\int_{B_R^j\cap\{0<y_1\}}F_{\nu,m}^R[\mu](\sqrt{y_1^2+|\widetilde{y}|^2})e^{-y_1}y_1^{\sigma q-1}d\widetilde{y}\,dy_1. \tag{7.30}$$

where $(y_1,\widetilde{y})\in\mathbb{R}\times\mathbb{R}^{j-1}$. If $j=1$ we denote,

$$I_{\nu,\sigma}^{m,1}(\mu;R)=\int_0^R F_{\nu,m}^R[\mu](y_1)e^{-y_1}y_1^{\sigma q-1}\,dy_1. \tag{7.31}$$

Similarly to Lemma 7.3 we obtain,

**Lemma 7.4** *If $j\geq 1$, there exists a positive constant $c$ such that, for any bounded Borel measure $\mu$ with support in $\mathbb{R}^m\cap B_R$*

$$c^{-1}\int_0^R F_{\nu,m}^R[\mu](\tau)h_{\sigma,j}(\tau)d\tau\leq\ I_{\nu,\sigma}^{m,j}(\mu;R)\leq c\int_0^R F_{\nu,m}^R[\mu](\tau)h_{\sigma,j}(\tau)d\tau \tag{7.32}$$

*with $h_{\sigma,j}$ as in* (7.24)*.*

*Proof.* In the case $j=1$ there is nothing to prove . Therefore we assume that $j\geq 2$.

From (7.30) we obtain,

$$I_{\nu,\sigma}^{m,j}(\mu;R)=c_{m,j}\int_0^R\int_0^{\sqrt{R^2-\rho^2}}F_{\nu,m}^R[\mu](\sqrt{y_1^2+\rho^2})e^{-y_1}y_1^{(\sigma+1)q-1}dy_1\rho^{j-2}d\rho.$$

Substituting $y_1=(\tau^2-\rho^2)^{1/2}$, then changing the order of integration and finally substituting $\rho=r\tau$ we obtain,

$$c_{m,j}^{-1}I_{\nu,\sigma}^{m,j}(\mu;R)=\int_0^R\int_0^1 F_{\nu,\mu}^R[\mu](\tau)\tau^{j-2+(\sigma+1)q}e^{-\tau\sqrt{1-r^2}}f(r)dr\,d\tau.$$

where

$$f(r)=r^{j-2}(1-r^2)^{(\sigma+1)q/2-1}.$$

The remaining part of the proof is the same as for Lemma 7.3.

$\square$

**Lemma 7.5** *Let* $1 < q$*,* $0 < \sigma$ *and assume that* $m < \nu q$ *and* $0 \leq j - 1 < \nu$*. Then there exists a positive constant* $\bar{c}$*, depending on* $j, m, q, \sigma, \nu$*, such that, for every* $R \geq 1$ *and every bounded Borel measure* $\mu$ *with support in* $B_{R/2}(0) \cap \mathbb{R}^m$*,*

$$\begin{aligned}\left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F^R_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \\ \leq \bar{c} R^{(\sigma+1-\nu)q+m+j-1} \|\mu\|^q_{\mathfrak{M}}\end{aligned} \tag{7.33}$$

*with* $h_{\sigma,j}$ *as in* (7.24)*.*

*Proof.* We estimate,

$$\begin{aligned}&\left| \int_0^\infty F_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau - \int_0^R F^R_{\nu,m}[\mu](\tau) h_{\sigma,j}(\tau) d\tau \right| \leq \\ &\int_R^\infty |F_{\nu,m}[\mu]| (\tau) h_{\sigma,j}(\tau) d\tau + \int_0^R \left| F_{\nu,m}[\mu] - F^R_{\nu,m}[\mu] \right| (\tau) h_{\sigma,j}(\tau) d\tau.\end{aligned} \tag{7.34}$$

For every $\tau > 0$,

$$|F_{\nu,m}[\mu]| (\tau) \leq \tau^{-\nu q} \|\mu\|^q_{\mathfrak{M}}. \tag{7.35}$$

Since $j - 1 < \nu q$, it follows that

$$\begin{aligned}\int_R^\infty |F_{\nu,m}[\mu]| (\tau) h_{\sigma,j}(\tau) d\tau &\leq \|\mu\|^q_{\mathfrak{M}} \int_R^\infty \tau^{-\nu q} h_{\sigma,j}(\tau) d\tau \\ &\leq c(\sigma, q) \|\mu\|^q_{\mathfrak{M}} \int_R^\infty \frac{\tau^{(\sigma+1)q+j-2-\nu q}}{(1+\tau)^{(\sigma+1)q}} d\tau \\ &\leq \frac{c(\sigma, q)}{\nu q - j + 1} \|\mu\|^q_{\mathfrak{M}} R^{j-1-\nu q}.\end{aligned} \tag{7.36}$$

Since, by assumption, $\operatorname{supp}\mu \subset B_{R/2}$, we have

$$\begin{aligned}
&\int_0^R \left|F_{\nu,m}[\mu] - F^R_{\nu,m}[\mu]\right|(\tau)h_{\sigma,j}(\tau)d\tau \\
&\quad \le \int_0^R \int_{|y''|>R} \left|\int_{\mathbb{R}^m} \frac{d\mu(z)}{(\tau^2+|y''-z|^2)^{\nu/2}}\right|^q dy'' h_{\sigma,j}(\tau)d\tau \\
&\quad \le \|\mu\|^q_{\mathfrak{M}} \int_0^R \int_{|\zeta|>R/2} (|\tau^2+|\zeta|^2)^{-\nu q/2}\, d\zeta\, h_{\sigma,j}d\tau \\
&\quad \le c(m,q)\,\|\mu\|^q_{\mathfrak{M}} \int_0^R \int_{R/2}^\infty (\tau^2+\rho^2)^{-\nu q/2}\rho^{m-1}\, d\rho\, h_{\sigma,j}d\tau \qquad (7.37) \\
&\quad \le c(m,q)\,\|\mu\|^q_{\mathfrak{M}} \int_0^R \tau^{m-\nu q} \int_{R/2\tau}^\infty (1+\eta^2)^{-\nu q/2}\eta^{m-1}\, d\eta\, h_{\sigma,j}d\tau \\
&\quad \le \frac{c(m,q)}{\nu q - m}\,\|\mu\|^q_{\mathfrak{M}} R^{m-\nu q} \int_0^R \tau^{(\sigma+1)q+j-2}\, d\tau \\
&\quad \le \frac{c(m,q)}{(\nu q - m)((\sigma+1)q+j-1)}\,\|\mu\|^q_{\mathfrak{M}} R^{(\sigma+1)q+j-1+m-\nu q}.
\end{aligned}$$

Combining (7.34)–(7.37) we obtain (7.33).

□

**Corollary 7.6** *For every $R > 0$ put*

$$J^{m,j}_{\nu,\sigma}(\mu;R) := \int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(\sigma+1)q+j-2}d\tau. \qquad (7.38)$$

*Then*

$$\begin{aligned}
&\frac{1}{c}I^{m,j}_{\nu,\sigma}(\mu) - \bar{c}R^\beta\,\|\mu\|^q_{\mathfrak{M}} \le J^{m,j}_{\nu,\sigma}(\mu;R) \le cR^{(\sigma+1)q}I^{m,j}_{\nu,\sigma}(\mu), \\
&\qquad \beta = (\sigma+1-\nu)q + j + m - 1,
\end{aligned} \qquad (7.39)$$

*for every $R > 1$ and every bounded Borel measure $\mu$ with support in $B^m_{R/2}(0) := B_{R/2}(0) \cap \mathbb{R}^m$.*

*Proof.* This is an immediate consequence of Lemma 7.5 and Lemma 7.3.

□

**Lemma 7.7** *Let $m, j$ be positive integers such that $j \ge 1$ and let $1 < q$, $0 < \sigma$. Put $n := m + j$.*

*Then there exist positive constants $c, \bar{c}$, depending on $j, m, q, \sigma$, such that, for every $R > 1$ and every measure $\mu \in \mathfrak{M}_+(B^m_{R/2}(0))$,*

$$\begin{aligned} &\frac{1}{c} \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} - \bar{c} R^{q\left(\sigma - \frac{n-1}{q'}\right)} \|\mu\|^q_{\mathfrak{M}} \leq J^{m,j}_{n,\sigma}(\mu; R) \\ &\leq c R^{(\sigma+1)q} \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} . \end{aligned} \tag{7.40}$$

*If $\sigma < \frac{n-1}{q'}$, there exists $R_0 > 1$ such that, for all $R > R_0$*

$$\frac{1}{2c} \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} \leq J^{m,j}_{n,\sigma}(\mu; R). \tag{7.41}$$

*If $\sigma = \frac{n-1}{q'}$ then, there exists $a > 0$ such that the inequality remains valid for measures $\mu$ such that* $\operatorname{diam}(\operatorname{supp} \mu) \leq a$.

*If, in addition, $\frac{j-1}{q'} < \sigma$ then*

$$\frac{1}{2c} \|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} \leq J^{m,j}_{n,\sigma}(\mu; R) \leq c R^{(\sigma+1)q} \|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} , \tag{7.42}$$

*where $s := \sigma - \frac{j-1}{q'}$.*

*Remark.* Assume that $\mu \geq 0$. Then:
(i) If $\mu \in B^{-\sigma,q}(\mathbb{R}^{n-1})$ and $\frac{j-1}{q'} \geq \sigma$ then $\mu(\mathbb{R}^m) = 0$.
(ii) If $\mu \in B^{-s,q}(\mathbb{R}^m)$ and $\sigma > (n-1)/q'$ then $s > m/q'$ and therefore $B^{s,q'}(\mathbb{R}^m)$ can be embedded in $C(\mathbb{R}^m)$.

*Proof.* Inequality (7.40) follows from (7.39) and Proposition 7.2 (see also (7.19)).

For positive measures $\mu$,

$$\|\mu\|_{\mathfrak{M}} = \mu(\mathbb{R}^{n-1}) \leq \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} .$$

Therefore, if $\sigma < \frac{n-1}{q'}$, (7.40) implies that there exists $R_0 > 1$ such that (7.41) holds for all $R > R_0$.

If $\sigma = \frac{n-1}{q'}$ (7.40) implies that

$$\frac{1}{c} \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} - \bar{c} \|\mu\|^q_{\mathfrak{M}} \leq J^{m,j}_{n,\sigma}(\mu; R).$$

But if $\mu$ is a positive bounded measure such that $\operatorname{diam}(\operatorname{supp} \mu) \leq a$ then

$$\|\mu\|_{\mathfrak{M}} / \|\mu\|^q_{B^{-\sigma,q}(\mathbb{R}^{n-1})} \to 0 \ \text{ as } a \to 0.$$

The last inequality follows from the imbedding theorem for Besov spaces according to which there exists a continuous trace operator $T : B^{\sigma,q'}(\mathbb{R}^{n-1}) \mapsto B^{s,q'}(\mathbb{R}^m)$ and a continuous lifting $T' : B^{s,q'}(\mathbb{R}^m) \mapsto B^{\sigma,q'}(\mathbb{R}^{n-1})$ where $s = \sigma - \frac{n-m-1}{q'}$.

□

If $\nu \in \mathbb{N}$ and $\sigma = s + \frac{\nu-m-1}{q'}$,

$$J^{m,\nu-m}_{\nu,\sigma}(\mu;R) = \int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(\sigma+1)q+\nu-m-2}\,d\tau$$
$$= \int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(s+\nu-m)q-1}\,d\tau.$$

However, if $\mu$ is positive, the expression

$$M^m_{\nu,s}(\mu;R) := \int_0^R F^R_{\nu,m}[\mu](\tau)\tau^{(s+\nu-m)q-1}\,d\tau, \tag{7.43}$$

is meaningful for any real $\nu > m$ and $s > 0$. Furthermore, as shown below, the results stated in Lemma 7.7 can be extended to this general case.

**Theorem 7.8** *Let $1 < q$, $\nu \in \mathbb{R}$ and $m$ a positive integer. Assume that $1 \le \nu - m$ and $0 < s < m/q'$. Then there exists a positive constant $c$ such that, for every bounded positive measure $\mu$ supported in $\mathbb{R}^m \cap B_{R/2}(0)$, $R > 1$,*

$$\frac{1}{c}\,\|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)} \le M^m_{\nu,s}(\mu;R) \le cR^{(s+\nu-m)q+1}\,\|\mu\|^q_{B^{-s,q}(\mathbb{R}^m)}. \tag{7.44}$$

*This also holds when $s = m/q'$, provided that the diameter of $\operatorname{supp}\mu$ is sufficiently small.*

*Proof.* If $\nu$ is an integer and $j := \nu - m$ then this statement is part of Lemma 7.7. Indeed the condition $s > 0$ means that $\sigma = s + \frac{j-1}{q'} > \frac{j-1}{q'}$ and the condition $s < m/q'$ means that $\sigma < \frac{n-1}{q'}$.

Therefore we assume that $\nu \notin \mathbb{N}$. Let $n := \{\nu\}$ and $\theta := n - \nu$ so that $0 < \theta < 1$. Our assumptions imply that $1 \le n - m - 1$ because (as $\nu$ is not an integer) $\nu - m > 1$ and consequently $n - m \ge 2$.

If $a, b$ are positive numbers, put

$$A_\nu := \frac{a^{(s+\nu-m)q-1}}{(a^2+b^2)^{\nu q/2}}.$$

Obviously $A_\nu$ decreases as $\nu$ increases. Therefore, $A_n \le A_\nu \le A_{n-1}$ which in turn implies,

$$M^m_{n,s} \le M^m_{\nu,s} \le M^m_{n-1,s}.$$

By Lemma 7.7, the assertions of the theorem are valid in the case that $\nu = n$ or $\nu = n-1$. Therefore the previous inequality implies that the assertions hold for any real $\nu$ subject to the conditions imposed.

□

By (7.8),

$$J^{A,R} = \int_0^R F^R_{\nu,m}(\tau)\tau^{(q+1)\kappa_+ + k-1} d\tau,$$

where $m = N-k$ and $\nu = N-2+2\kappa_+$. Consequently, by (7.38),

$$J^{A,R} = M^m_{\nu,s}$$

where $s$ is determined by,

$$(s+\nu-m)q - 1 = (q+1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_+.$$

It follows that

$$sq = -(k-2+2\kappa_+)q + (q+1)\kappa_+ + k = k(1-q) + 2q - \kappa_+(q-1)$$

and therefore

$$s = 2 - \frac{k+\kappa_+}{q'}.$$

*Proof of Theorem 7.1.*

Put

$$\nu := N-2+2\kappa_+, \quad s := 2 - \frac{\kappa_+ + k}{q'}, \quad m := N-k. \tag{7.45}$$

Recall that in the case $k=2$ we have $\kappa_+ > 1/2$. Therefore

$$\nu - m - 1 = k - 3 + 2\kappa_+ > 0. \tag{7.46}$$

Furthermore,

$$(s+\nu-m)q - 1 = (q+1)\kappa_+ + k - 1, \quad k = \nu - m + 2 - 2\kappa_.$$

Thus

$$J^{A,R} = \int_0^R F^R_{\nu,m}(\tau)\tau^{(q+1)\kappa_+ + k-1} d\tau = M^m_{\nu,s}.$$

Next we show that $0 < s \leq m/q'$. More precisely we prove

$$0 < s \leq m/q' \iff q_c \leq q < q_c^*. \tag{7.47}$$

Let $\mu$ be a bounded non-negative Borel measure in $B^{-s,q}(\mathbb{R}^m)$. If $s \leq 0$, $B^{-s,q}(\mathbb{R}^m) \subset L^q(\mathbb{R}^m)$. Therefore, in this case, every bounded Borel measure on $\mathbb{R}^m$ is admissible i.e. satisfies (6.33). Consequently, by Proposition 6.2, $q < q_c$. As we assume $q \geq q_c$ it follows that $s > 0$.

If, $s > 0$ and $sq' - m \geq 0$ then $C_{s,q'}(K) = 0$ for every compact subset of $\mathbb{R}^m$ and consequently $\mu(K) = 0$ for any such set. Conversely, if $sq' - m < 0$ then there exist non-trivial positive bounded measures in $B^{-s,q}(\mathbb{R}^m)$. Therefore, by Proposition 6.1, $sq' < m$ if and only if $q < q_c^*$.

In conclusion, $0 < s \leq m/q'$ and $\nu - m \geq 1$; therefore Theorem 7.1 is a consequence of Theorem 7.8.

□

*Remark.* Note that the critical exponent for the imbedding of $B^{2-\frac{\kappa_+ + k}{q'},q'}(\mathbb{R}^{N-k})$ into $C(\mathbb{R}^{N-k})$ is again

$$q = q_c = \frac{N + \kappa_+}{N + \kappa_+ - 2}.$$

# 8 Supercritical equations in a polyhedral domain

In this section $q$ is a real number larger than 1 and $P$ an N-dim polyhedral domain as described in subsection 6.1. Denote by $\{L_{k,j} : k = 1, \ldots, N,\ j = 1, \ldots, n_k\}$ the family of faces, edges and vertices of $P$. In this notation, $L_{1,j}$ denotes one of the open faces of $P$; for $k = 2, \ldots, N-1$, $L_{k,j}$ denotes a relatively open $N-k$-dimensonal edge and $L_{N,j}$ denotes a vertex. For $1 \leq k < N$, the $(N-k)$ dimensional space which contains $L_{k,j}$ is denoted by $\mathbb{R}_j^{N-k}$. If $1 < k < N$, the cylinder of radius $r$ around the axis $\mathbb{R}_j^{N-k}$ will be denoted by $\Gamma_{k,j,r}^{\infty}$ and the subset $A_{k,j}$ of $S^{k-1}$ is defined by

$$\lim_{r\to 0} \frac{1}{r}(\partial\Gamma_{k,j,r}^{\infty} \cap P) = L_{k,j} \times A_{k,j}.$$

$A_{k,j}$ is the 'opening' of $P$ at the edge $L_{k,j}$. For $k = N$ we replace in this definition the cylinder $\Gamma_{N,j,r}^{\infty}$ by the ball $B_r(L_{N,j})$. For $1 < k \leq N$ and $A = A_{k,j}$ we use $d_A$ as an alternative notation for $\mathbb{R}_j^{N-k}$ and denote by $D_A$ the k-dihedron with edge $d_A$ and opening $A$ as in subsection 6.1 (with $S_A$ defined as in (6.2)). For $k = 1$, $D_A$ stands for the half space $\mathbb{R}_j^{N-1} \times (0, \infty)$.

In what follows we denote by $\mathfrak{M}_q^\Omega$ the set of bounded measures $\mu$ on the boundary of a Lipschitz domain $\Omega$ such that the boundary value problem

$$-\Delta u + u^q = 0 \text{ in } \Omega, \;\; u = \mu \text{ on } \partial\Omega \tag{8.1}$$

possesses a solution. A measure $\mu$ in this space is called a *q-good measure.*

The following statements can be proved in the same way as in the case of smooth domains. For the proof in that case see [23].

**I.** $\mathfrak{M}_q^\Omega$ is a linear space and

$$\mu \in \mathfrak{M}_q^\Omega \iff |\mu| \in \mathfrak{M}_q^\Omega.$$

**II.** If $\{\mu_n\}$ is an increasing sequence of measures in $\mathfrak{M}_q^\Omega$ and $\mu := \lim \mu_n$ is a finite measure then $\mu \in \mathfrak{M}_q^\Omega$.

**Proposition 8.1** *Let $\mu$ be a bounded measure on $\partial P$. ($\mu$ may be a signed measure.) For $i = 1, \ldots, N$, $j = 1, \ldots, n_i$, we define the measure $\mu_{k,j}$ on $d_{A_{k,j}}$ by,*

$$\mu_{k,j} = \mu \;\; \text{on } L_{k,j}, \qquad \mu_{k,j} = 0 \;\; \text{on } d_{A_{k,j}} \setminus L_{k,j} \;\; .$$

*Then $\mu \in \mathfrak{M}_q^P$, i.e., problem*

$$-\Delta u + u^q = 0 \;\; \text{in } P, \;\; u = \mu \;\; \text{on } \partial P \tag{8.2}$$

*possesses a solution, if and only if, $\mu_{k,j}$ is a q-good measure relative to $D_{A_{k,j}}$ for all $(k, j)$ as above.*

*Proof.* In view of statement **I** above, it is sufficient to prove the proposition in the case that $\mu$ is non-negative. This is assumed hereafter. If $\mu \in \mathfrak{M}_q^P$ then any measure $\nu$ on $\partial P$ such that $0 \leq \nu \leq \mu$ is a q-good measure relative to $P$. Therefore

$$\mu \in \mathfrak{M}_q^P \Longrightarrow \mu'_{k,j} := \mu\chi_{L_{k,j}} \in \mathfrak{M}_q^P.$$

Assume that $\mu \in \mathfrak{M}_q^P$ and let $u_{k,j}$ be the solution of (8.2) when $\mu$ is replaced by $\mu'_{k,j}$. Denote by $u'_{k,j}$ the extension of $u_{k,j}$ by zero to the k-dihedron $D_{A_{k,j}}$. Then $u'_{k,j}$ is a subsolution of (5.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. In the present case there always exists a supersolution, e.g. the maximal solution of (5.1) in $D_{A_{k,j}}$ vanishing outside $d_{A_{k,j}} \setminus \bar{L}_{k,j}$. Therefore there exists a solution $v_{k,j}$ of this equation in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$, i.e., $\mu_{k,j}$ is q-good relative to $D_{A_{k,j}}$.

Next assume that $\mu \in \mathfrak{M}(\partial P)$ and that $\mu_{k,j}$ is q-good relative to $D_{A_{k,j}}$ for every $(k, j)$ as above. Let $v_{k,j}$ be the solution of (5.1) in $D_{A_{k,j}}$ with boundary data $\mu_{k,j}$. Then $v_{k,j}$ is a supersolution of problem (8.2) with $\mu$ replaced by $\mu'_{k,j}$ and consequently there exists a solution $u_{k,j}$ of this problem. It follows that

$$w := \max\{u_{k,j} : k = 1, \dots, N,\ j = 1, \dots, n_k\}$$

is a subsolution while

$$\bar{w} := \sum_{k=1,\dots,N,\ j=1,\dots,n_k} u_{k,j}$$

is a supersolution of (8.2). Consequently there exists a solution of this problem, i.e., $\mu \in \mathfrak{M}_q^P$.

□

## 8.1 Removable singular sets and 'good measures', I

**Proposition 8.2** *Let $A$ be a Lipschitz domain on $S^{k-1}$, $2 \le k \le N-1$, and let $D_A$ be the $k$-dihedron with opening $A$. Let $\mu \in \mathfrak{M}(\partial D_A)$ be a positive measure with compact support contained in $d_A$ (= the edge of $D_A$). Assume that $\mu$ is $q$-good relative to $D_A$. Let $R > 1$ be large enough so that* $\operatorname{supp}\mu \subset B_R^{N-k}(0)$ *and let $u$ be the solution of* (5.1) *in $D_A^R$ with trace $\mu$ on $d_A^R$ and trace zero on $\partial D_A^R \setminus d_A^R$. Then:*

(i) *For every non-negative $\eta \in C_0^\infty(B_{3R/4}^{N-k}(0))$,*

$$\begin{aligned}\left(\int_{d_A^R} \eta^{q'} d\mu\right) \le cM^{q'} \int_{D_A^R} u^q \rho dx + \\ + cM^{q'} \left(\int_{D_A^R} u^q \rho dx\right)^{\frac{1}{q}} \left(1 + M^{-1} \|\eta\|_{L^{q'}(d_A^R)}\right).\end{aligned} \tag{8.3}$$

*where $M = \|\eta\|_{L^\infty}$ and $\rho$ is the first eigenfunction of $-\Delta$ in $D_A^R$ normalized by $\rho(x_0) = 1$ at some point $x_0 \in D_A^R$. The constant $c$ depends only on $N, q, k, x_0, \lambda_1, R$ where $\lambda_1$ is the first eigenvalue.*

(ii) *For any compact set $E \subset d_A$,*

$$C_{s,q}^{N-k}(E) = 0 \Longrightarrow \mu(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'}, \tag{8.4}$$

*where $C_{s,q}^{N-k}$ denotes the Bessel capacity with the indicated indices in $\mathbb{R}^{N-k}$.*

*Remark.* If we replace $D_A^R$ by $D_A^{\tilde{R},R} = D_A \cap B_{\tilde{R}}^k(0) \cap B_R^{N-k}(0)$, $\tilde{R} > 1$, then the constant $c$ in (i) depends on $\tilde{R}$ but *not on* $R$.

*Proof.* We identify $d_A$ with $\mathbb{R}^{N-k}$ and use the notation

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{N-k}, \quad y = |x'|.$$

Let $\eta \in C_0^\infty(\mathbb{R}^{N-k})$ and let $R$ be large enough so that $\operatorname{supp} \eta \subset B_{R/2}^{N-k}(0)$. Let $w = w_R(t, x'')$ be the solution of the following problem in $\mathbb{R}_+ \times B_R^{N-k}(0)$:

$$\begin{aligned} \partial_t w - \Delta_{x''} w = 0 \qquad & \text{in } \mathbb{R}^+ \times B_R^{N-k}(0), \\ w(0, x'') = \eta(x'') \quad & \text{in } B_R^{N-k}, \\ w(t, x'') = 0 \qquad & \text{on } \partial B_R^{N-k}(0). \end{aligned} \tag{8.5}$$

Thus $w_R(t, \cdot) = S_R(t)[\eta]$ where $S_R(t)$ is the semi-group operator corresponding to the above problem. Denote,

$$H_R[\eta](x', x'') = w_R(|x'|^2, x'') = S_R(y^2)[\eta](x''), \quad y := |x'|. \tag{8.6}$$

We assume, as we may, that $R > 1$. Let $\rho^R$ be the first eigenfunction of $-\Delta_{x''}$ in the ball $B_R^{N-k}(0)$ normalized by $\rho^R(0) = 1$ and let $\rho_A$ be the first eigenfunction of $-\Delta_{x'}$ in $C_A$ (where $C_A$ denotes the cone with opening $A$ in $\mathbb{R}^k$) normalized so that $\rho_A(x'_0) = 1$ at some point $x'_0 \in S_A$. Then $\rho^R \rho_A$ is the first eigenfunction of $-\Delta$ in $\{x \in D_A : |x''| < R\}$. Note that $\rho^R \le 1$ and $\rho^R \to 1$ as $R \to \infty$ in $C^2(I)$ for any bounded set $I \subset \mathbb{R}^{N-k}$.

Let $h \in C^\infty(\mathbb{R})$ be a monotone decreasing function such that $h(t) = 1$ for $t < 1/2$ and $h(t) = 0$ for $t > 3/4$. Put

$$\psi_R(x') = h(|x'|/R)$$

and

$$\zeta_R := \rho_A \psi_R H_R[\eta]^{q'}. \tag{8.7}$$

If $\rho_A^R$ is the first eigenfunction (normalized at $x_0$) of $D_A^R := D_A \cap \Gamma_R$ ($\Gamma_R$ as in (6.22)) then

$$\rho_A \psi_R \le c \rho_A^R \tag{8.8}$$

and $\rho^R \rho_A^R$ is the first eigenfunction in $D_A^R$.

Hereafter we shall drop the index $R$ in $\zeta_R, H_R, w_R$ but keep it in the other notations in order to avoid confusion.

We shall verify that $\zeta \in D_A^R$. To this purpose we compute,

$$\begin{aligned}\Delta\zeta =& -\lambda_1(\rho_A\psi_R)H[\eta]^{q'} + (\rho_A\psi_R)\Delta H[\eta]^{q'} + 2\nabla(\rho_A\psi_R)\cdot\nabla H[\eta]^{q'}\\ =& -\lambda_1\zeta + q'(\rho_A\psi_R)(H[\eta])^{q'-1}\Delta H[\eta]\\ &+ q(q'-1)(\rho_A\psi_R)(H[\eta])^{q'-2}|\nabla H[\eta]|^2\\ &+ 2q'(H[\eta])^{q'-1}\nabla(\rho_A\psi_R)\cdot\nabla H[\eta].\end{aligned} \tag{8.9}$$

In addition,

$$\begin{aligned}\nabla H[\eta] = \nabla_{x'}H[\eta] + \nabla_{x''}H[\eta] &= \partial_y H[\eta]\frac{x'}{y} + \nabla_{x''}H[\eta]\\ &= 2y\partial_t w(y^2,x'')\frac{x'}{y} + \nabla_{x''}H[\eta](x',x'')\end{aligned}$$

and consequently (recall that $y$ stands for $|x'|$),

$$\begin{aligned}&\nabla H[\eta]\cdot\nabla(\rho_A\psi_R)\\ &= 2\partial_t w(y^2,x'')x'\cdot\Big(\psi_R\big(|x'|^{\kappa_+-1}(\kappa_+\frac{x'}{y}\omega_k(x'/y) + |x'|\nabla\omega_k(x'/y))\big) + \rho_A\nabla\psi_R\Big)\\ &= 2\kappa_+\partial_t w(y^2,x'')|x'|^{\kappa_+}\omega_k(x'/y) = 2\partial_t w(y^2,x'')(\kappa_+\rho_A\psi_R + \rho_A x'\cdot\nabla\psi_R).\end{aligned}$$

Since $w = w_R$ vanishes for $|x''| = R$ and $\eta = 0$ in a neighborhood of this sphere, $|\partial_t w(y^2,x'')| \le c\rho^R$. As $\psi_R$ vanishes for $|x'| > 3R/4$ we have $\rho_A\nabla\psi_R \le c\rho_A^R$. Therefore

$$|\nabla H[\eta]\cdot\nabla\rho_A| \le c\rho^R\rho_A^R$$

and, in view of (8.9),

$$|\Delta\zeta| \le c\rho^R\rho_A^R. \tag{8.10}$$

Thus $\zeta \in X(D_A^R)$ and consequently

$$\int_{D_A^R}(-u\Delta\zeta + u^q\zeta)\,dx = -\int_{D_A^R}\mathbb{K}[\mu]\Delta\zeta dx. \tag{8.11}$$

Since $q(q'-1)\rho_A(H[\eta])^{q'-2}|\nabla H[\eta]|^2 \geq 0$, we have

$$\begin{aligned}
&\left|\int_{D_A^R} u\Delta\zeta dx\right| \\
&\leq \int_{D_A^R} u\left(\lambda_1\zeta + q'(H[\eta])^{q'-1}\left(\rho|\Delta H[\eta]| + 2|\nabla\rho.\nabla H[\eta]|\right)\right)dx \\
&\leq \int_{D_A^R} u\left(\lambda_1\zeta + q'\zeta^{1/q}\left(\rho^{1/q'}|\Delta H[\eta]| + 2\rho^{-1/q}|\nabla\rho.\nabla H[\eta]|\right)\right)dx \\
&\leq \left(\int_{D_A^R} u^q\zeta dx\right)^{\frac{1}{q}}\left(\lambda_1\left(\int_{D_A^R}\zeta dx\right)^{\frac{1}{q'}} + q'\left\|L[\eta]\right\|_{L^{q'}(D_A^R)}\right)
\end{aligned} \tag{8.12}$$

where

$$L[\eta] = \rho^{1/q'}|\Delta H[\eta]| + 2\rho^{-1/q}|\nabla\rho.\nabla H[\eta]|. \tag{8.13}$$

By Proposition 2.4

$$-\int_{D_A^R}\mathbb{K}[\mu]\Delta\zeta dx = \int_{d_A^R}\eta^{q'}d\mu. \tag{8.14}$$

Therefore

$$\begin{aligned}
\left(\int_{d_A^R}\eta^{q'}d\mu\right) &\leq \int_{D_A^R} u^q\zeta dx + \\
&+\left(\int_{D_A^R} u^q\zeta dx\right)^{\frac{1}{q}}\left(\lambda_1\left(\int_{D_A^R}\zeta dx\right)^{\frac{1}{q'}} + q'\left\|L[\eta]\right\|_{L^{q'}(D_A^R)}\right).
\end{aligned} \tag{8.15}$$

Next we prove that

$$\left\|L[\eta]\right\|_{L^{q'}(D_A^R)} \leq C\left\|\eta\right\|_{W^{s,q'}(\mathbb{R}^{N-k})} \tag{8.16}$$

starting with the estimate of the first term on the right hand side of (8.13).

$$\begin{aligned}
\Delta H[\eta] &= \Delta_{x'}H[\eta] + \Delta_{x''}H[\eta] = \partial_y^2 H[\eta] + \frac{k-1}{y}\partial_y H[\eta] + \Delta_{x''}H[\eta] \\
&= 2y^2\partial_{tt}w(y^2,x'') + k\partial_t w(y^2,x'') + \Delta_{x''}H[\eta] \\
&= 2y^2\partial_{tt}w(y^2,x'') + (k+1)\partial_t w(y^2,x'').
\end{aligned}$$

Then

$$\int_{\mathbb{R}^N}\rho\,|\Delta H[\eta]|^{q'}\,dx \le c\int_0^1\int_{\mathbb{R}^{N-k}}\left|\partial_{tt}w(y^2,x'')\right|^{q'}dx''y^{\kappa_++2q'+k-1}dy$$

$$+\,c\int_0^1\int_{\mathbb{R}^{N-k}}\left|\partial_t w(y^2,x'')\right|^{q'}dx''y^{\kappa_++k-1}dy$$

$$\le c\int_0^1\int_{\mathbb{R}^{N-k}}|\partial_{tt}w(t,x'')|^{q'}\,dx''t^{(\kappa_++k)/2+q'}\frac{dt}{t}$$

$$+\,c\int_0^1\int_{\mathbb{R}^{N-k}}|\partial_t w(t,x'')|^{q'}\,dx''t^{(\kappa_++k)/2}\frac{dt}{t}$$

$$\le c\int_0^1\left\|t^{2-(1-\frac{\kappa_++k}{2q'}))}\frac{d^2S(t)[\eta]}{dt^2}\right\|^{q'}_{L^{q'}(\mathbb{R}^{N-k})}\frac{dt}{t}$$

$$+\,c\int_0^1\left\|t^{1-(1-\frac{\kappa_++k}{2q'})}\frac{dS(t)[\eta]}{dt}\right\|^{q'}_{L^{q'}(\mathbb{R}^{N-k})}\frac{dt}{t}.$$

Put $\beta=\frac{\kappa_++k}{2q'}$ and note that $0<\beta=\frac{1}{2}(2-s)<1$. By standard interpolation theory,

$$\int_0^1\left\|t^{1-(1-\beta)}\frac{dS(t)[\eta]}{dt}\right\|^{q'}_{L^{q'}(\mathbb{R}^{N-k})}\frac{dt}{t}$$

$$\approx\|\eta\|^{q'}_{[W^{2,q'},L^{q'}]_{1-\beta,q'}}\approx\|\eta\|^{q'}_{W^{2(1-\beta),q'}(\mathbb{R}^{N-k})},$$

and

$$\int_0^1\left\|t^{2-(1-\beta))}\frac{d^2S(t)[\eta]}{dt^2}\right\|^{q'}_{L^{q'}(\mathbb{R}^{N-k})}\frac{dt}{t}$$

$$\approx\|\eta\|^{q'}_{[W^{4,q'},L^{q'}]_{\frac{1}{2}(1-\beta),q'}}\approx\|\eta\|^{q'}_{W^{2(1-\beta),q'}(\mathbb{R}^{N-k})}.$$

The second term on the right hand side of (8.13) is estimated in a similar way:

$$\int_{\mathbb{R}^N}\rho^{-q'/q}\,|\nabla H[\eta]\cdot\nabla\rho|^{q'}\,dx\le c\int_0^1\int_{\mathbb{R}^{N-k}}\left|\partial_t w(y^2,x'')\right|^{q'}dx'y^{\kappa_++k-1}dy$$

$$\le c\int_0^1\int_{\mathbb{R}^{N-k}}\left|\partial_t w(t,x'')\right|^{q'}dx't^{\frac{\kappa_++k}{2}}\frac{dt}{t}$$

$$\le c\int_0^1\left\|t^{1-(\frac{1}{2}-\beta)}\frac{dS(t)[\eta]}{dt}\right\|^{q'}_{L^{q'}(\mathbb{R}^{N-k})}\frac{dt}{t}$$

$$\approx\|\eta\|^{q'}_{W^{2(1-\beta),q'}(\mathbb{R}^{N-k})}.$$

This proves (8.16). Further, (8.15) and (8.16) imply (8.3).

We turn to the proof of part (ii). Let $E$ be a closed subset of $B^{N-k}_{R/2}(0)$ such that $C^{N-k}_{s,q'}(E) = 0$. Then there exists a sequence $\{\eta_n\}$ in $C^\infty_0(d_A)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of $E$ (which may depend on $n$), $\operatorname{supp}\eta_n \subset B^{N-k}_{3R/4}(0)$ and $\|\eta_n\|_{W^{s,q'}} \to 0$. Then, by (8.16),

$$\|L[\eta_n]\|_{L^{q'}(D^R_A)} \to 0.$$

Furthermore

$$\|w\|_{L^{q'}((0,R)\times B^{N-k}_R(0))} \leq c\,\|\eta_n\|_{L^{q'}(B^{N-k}_R(0))}$$

and consequently

$$H[\eta_n] \to 0 \ \text{ in } \ L^{q'}(D^R_A).$$

(Here we use the fact that $k \geq 2$.) In addition

$$0 \leq H[\eta_n] \leq 1, \quad H[\eta_n] \leq c(R - |x'|)$$

with a constant $c$ independent of $n$. Hence (see (8.8))

$$\zeta_{n,R} := \rho_A \psi_R H[\eta_n]^{q'} \leq \rho^R \rho_A \psi_R H[\eta_n]^{q'-1} \leq \rho^R \rho^R_A H[\eta_n]^{q'-1}.$$

As $u^q \rho^R \rho^R_A \in L^1(D^R_A)$ we obtain,

$$\lim_{n\to\infty} \int_{D_A} u^q \zeta_n dx = 0.$$

This fact and (8.15) imply that

$$\int_{d^R_A} \eta_n^{q'} d\mu \to 0.$$

As $\eta_n = 1$ on a neighborhood of $E$ in $\mathbb{R}^{N-k}$ it follows that $\mu(E) = 0$.

□

**Proposition 8.3** *Let $D_A$ be a $k$-dihedron, $1 \leq k < N$. Let $k_+$ be as in* (6.8) *and let $q^*_c$ and $q_c$ be as in Proposition 6.1 and Proposition 6.2 respectively. Assume that $q_c \leq q < q^*_c$. A measure $\mu \in \mathfrak{M}(\partial D_A)$, with compact support contained in $d_A$, is $q$-good relative to $D_A$ if and only if $\mu$ vanishes on every Borel set $E \subset d_A$ such that $C_{s,q'}(E) = 0$, where $s = 2 - \frac{k+\kappa_+}{q'}$.*

*Remark.* We shall use the notation $\mu \prec C_{s,q'}$ to say that $\mu$ vanishes on any Borel set $E \subset (d_A)$ such that $C_{s,q'}(E) = 0$.

In the case $k = N$: $D_A = C_A$ (= the cone with vertex 0 and opening $A$ in $\mathbb{R}^k$) and $q_c = q_c^*$. By Theorem 5.5, $q_c = 1 - \frac{2}{\kappa_-} = \frac{N+\kappa_+}{N+\kappa_+-2}$. (Note the difference in notation; the entity denoted by $\kappa_-$ in section 6 and in the present section is denoted by $-\alpha_S$ in subsection 5.1. See (5.10) and (6.8).) If $1 < q < q_c$ then, again by Theorem 5.5, there exist solutions for every measure $\mu = k\delta_0$ on $\partial C_A$.

In the case $k = 1$, $q_c^* = \infty$, $\kappa_+ = 1$ and $q_c = \frac{N+1}{N-1}$. Thus $s = 2/q$ and the statement of the theorem is well known (see [24]).

*Proof.* In view of the last remark, it remains to deal only with $2 \leq k \leq N-1$. We shall identify $d_A$ with $\mathbb{R}^{N-k}$.

It is sufficient to prove the result for positive measures because $\mu \prec C_{s,q'}$ if and only if $|\mu| \prec C_{s,q'}$. In addition, if $|\mu|$ is a q-good measure then $\mu$ is a q-good measure.

First we show that if $\mu$ is non-negative and q-good then $\mu \prec C_{s,q'}$. If $E$ is a Borel subset of $\partial\Omega$ then $\mu\chi_E$ is q-good. If $E$ is compact and $C_{s,q'}(E) = 0$ then, by Proposition 8.2, $E$ is a removable set. This means that the only solution of (8.1) such that $\mu(\partial\Omega \setminus E) = 0$ is the zero solution. This implies that $\mu\chi_E = 0$, i.e., $\mu(E) = 0$. If $C_{s,q'}(E) = 0$ but $E$ is not compact then $\mu(E') = 0$ for every compact set $E' \subset E$. Therefore, we conclude again that $\mu(E) = 0$.

Next, assume that $\mu$ is a positive measure in $\mathfrak{M}(\partial D_A)$ supported in a compact subset of $\mathbb{R}^{N-k}$.

If $\mu \in B^{-s,q}(\mathbb{R}^{N-k})$ then, by Theorem 7.1 and Theorem 3.8, $\mu$ is q-good relative to $D_A \cap \Gamma_{k,R}$, for every $R > 0$. (As before $\Gamma_{k,R}$ is the cylinder with radius $R$ around the 'axis' $\mathbb{R}^{N-k}$.) This implies that $\mu$ is q-good relative to $D_A$.

If $\mu \prec C_{s,q'}$ then, by a theorem of Feyel and de la Pradelle [12] (see also [1]), there exists a sequence $\{\mu_n\} \subset (B^{-s,q}(\mathbb{R}^{N-k}))_+$ such that $\mu_n \uparrow \mu$. As $\mu_k$ is q-good, it follows that $\mu$ is q-good.

□

**Theorem 8.4** *Let $P$ be an $N$-dimensional polyhedron as described in Proposition 8.1. Let $\mu$ be a bounded measure on $\partial P$, (may be a signed measure). Let $k = 1, \ldots, N$, $j = 1, \ldots, n_k$, and let $L_{k,j}$ and $A_{k,j}$ be defined as at the beginning of section 8. Further, put*

$$s(k,j) = 2 - \frac{k + (\kappa_+)_{k,j}}{q'}, \tag{8.17}$$

*where* $(\kappa_+)_{k,j}$ *is defined as in* (6.8) *with* $A = A_{k,j}$. *Then* $\mu \in \mathfrak{M}_q^P$, *i.e.,* $\mu$ *is a good measure for* (5.1) *relative to* $P$, *if and only if, for every pair* $(k, j)$ *as above and every Borel set* $E \subset L_{k,j}$:
*If* $1 \leq k < N$ *then*

$$\begin{aligned} (q_c)_{k,j} \leq q < (q_c^*)_{k,j},\ C^{N-k}_{s(k,j),q'}(E) = 0 \Longrightarrow \mu(E) = 0 \\ q \geq (q_c^*)_{k,j} \Longrightarrow \mu(L_{N,j}) = 0 \end{aligned} \tag{8.18}$$

*and if* $k = N$, *i.e.,* $L$ *is a vertex,*

$$q \geq (q_c)_{k,j} = \frac{N + 2 + \sqrt{(N-2)^2 + 4\lambda_A}}{N - 2 + \sqrt{(N-2)^2 + 4\lambda_A}} \Longrightarrow \mu(L) = 0. \tag{8.19}$$

*Here* $(q_c^*)_{k,j}$ *and* $(q_c)_{k,j}$ *are defined as in* (6.30) *and* (6.34)*respectively, with* $A = A_{k,j}$.

*If* $1 < q < (q_c)_{k,j}$ *then there is no restriction on* $\mu\chi_{L_{k,j}}$.

*Proof.* This is an immediate consequence of Proposition 8.1 and Proposition 8.3 (see also the Remark following it). In the case $k = N$, $L_{N,j}$ is a vertex and the condition says merely that for $q \geq q_c(L_{N,j}$, $\mu$ does not charge the vertex.

□

### 8.2 Removable singular sets II.

**Proposition 8.5** *Let* $A$ *be a Lipschitz domain on* $S^{k-1}$, $2 \leq k \leq N-1$, *and let* $D_A$ *be the* $k$*-dihedron with opening* $A$. *Let* $u$ *be a positive solution of* (5.1) *in* $D_A^R$, *for some* $R > 0$. *Suppose that* $F = \mathcal{S}(u) \subset d_A^R$ *and let* $Q$ *be an open neighborhood of* $F$ *such that* $\bar{Q} \subset d_A^R$. (*Recall that* $d_A^R = d_A \cap B_R^{N-k}(0)$ *is an open subset of* $d_A$.) *Let* $\mu$ *be the trace of* $u$ *on* $\mathcal{R}(u)$.

*Let* $\eta \in W_0^{s,q'}(d_A^R)$ *such that*

$$0 \leq \eta \leq 1, \quad \eta = 0 \ \ \text{on } Q. \tag{8.20}$$

*Employing the notation in the proof of Proposition 8.2, put*

$$\zeta := \rho_A \psi_R H_R[\eta]^{q'}. \tag{8.21}$$

*Then*

$$\int_{D_A^R} u^q \zeta \, dx \leq c(1 + \|\eta\|_{W^{s,q'}(d_A)})^{q'} + \mu(d_A^R \setminus Q)^q, \tag{8.22}$$

$c$ *independent of* $u$ *and* $\eta$.

*Proof.* First we prove (8.22) for $\eta \in C_0^\infty(d_A^R)$. Let $\sigma_0$ be a point in $A$ and let $\{A_n\}$ be a Lipschitz exhaustion of $A$. If $0 < \epsilon < \operatorname{dist}(\partial A, \partial A_n) = \bar{\epsilon}_n$ then

$$\epsilon\sigma_0 + C_{A_n} \subset C_A.$$

Denote

$$D_A^{R',R''} = D_A \cap [|x'| < R'] \cap [|x''| < R''].$$

Pick a sequence $\{\epsilon_n\}$ decreasing to zero such that $0 < \epsilon_n < \min(\bar{\epsilon}_n/2^n, R/8)$. Let $u_n$ be the function given by

$$u_n(x'x'') = u(x' + \epsilon_n\sigma_0, x'') \quad \forall x \in D_{A_n}^{R_n,R}, \quad R_n = R - \epsilon_n.$$

Then $u_n$ is a solution of (5.1) in $D_{A_n}^{R_n,R}$ belonging to $C^2(\bar{D}_{A_n}^{R_n,R})$ and we denote its boundary trace by $h_n$. Let

$$\zeta_n := \rho_{A_n}\psi_R H_R[\eta]^{q'},$$

with $\psi_R$ and $H_R[\eta]$ as in the proof of Proposition 8.2. By Proposition 2.4

$$-\int_{D_{A_n}^{R_n,R}} \mathbb{P}[h_n]\Delta\zeta_n dx = \int_{B_R^{N-k}(0)} \eta^{q'} h_n d\omega_n \tag{8.23}$$

where $\omega_n$ is the harmonic measure on $d_{A_n}^R$ relative to $D_{A_n}^{R_n,R}$. (Note that $d_{A_n}^R = d_A^R$ and we may identify it with $B_R^{N-k}(0)$.) Hence

$$\int_{D_{A_n}^{R_n,R}} (-u_n\Delta\zeta_n + u_n^q\zeta_n)\, dx = -\int_{B_R^{N-k}(0)} \eta^{q'} h_n\, d\omega_n. \tag{8.24}$$

Further,

$$\int_{B_R^{N-k}(0)} \eta^{q'} h_n\, d\omega_n \to \int_{B_R^{N-k}(0)} \eta^{q'} d\mu \le \mu(d_A^R \setminus Q),$$

because $\eta = 0$ in $Q$. By (8.12), (8.16) we obtain,

$$\begin{aligned} &\left|\int_{D_{A_n}^{R_n,R}} u_n\Delta\zeta_n\, dx\right| \le \\ &c\Big(\int_{D_{A_n}^{R_n,R}} u_n^q\zeta_n dx\Big)^{\frac{1}{q}}\Big(\Big(\int_{D_{A_n}^{R_n,R}} \zeta_n dx\Big)^{\frac{1}{q'}} + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\Big). \end{aligned} \tag{8.25}$$

From the definition of $\zeta_n$ it follows that

$$\int_{D_{A_n}^{R_n,R}} \zeta_n\, dx \le \int_{D_{A_n}^{R_n,R}} \rho_n\, dx \to \int_{D_A^R} \rho\, dx,$$

where $\rho$ (resp. $\rho_n$) is the first eigenfunction of $-\Delta$ in $D_A^R$ (resp. $D_{A_n}^{R_n,R}$) normalized by 1 at some $x_0 \in D_{A_1}^{R_1,R}$. Therefore, by (8.24),

$$\int_{D_{A_n}^{R_n,R}} u_n^q \zeta_n dx \leq c\Big( \int_{D_{A_n}^{R_n,R}} u_n^q \zeta_n dx\Big)^{\frac{1}{q}} \big(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\big) + \mu(d_A^R \setminus Q).$$

This implies

$$\int_{D_{A_n}^{R_n,R}} u_n^q \zeta_n dx \leq c\big(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\big)^{q'} + \mu(d_A^R \setminus Q)^q. \tag{8.26}$$

To verify this fact, put

$$m = \Big(\int_{D_{A_n}^{R_n,R}} u_n^q \zeta_n dx\Big)^{1/q},\ b = \mu(d_A^R \setminus Q),\ a = c\big(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\big)$$

so that (8.26) becomes

$$m^q - am - b \leq 0.$$

If $b \leq m$ then

$$m^{q-1} - a - 1 \leq 0.$$

Therefore,

$$m \leq (a+1)^{\frac{1}{q-1}} + b$$

which implies (8.26). Finally, by the lemma of Fatou we obtain (8.22) for $\eta \in C_0^\infty$. By continuity we obtain the inequality for any $\eta \in W_0^{s,q'}$ satisfying (8.20).

□

**Theorem 8.6** *Let $A$ be a Lipschitz domain on $S^{k-1}$, $2 \leq k \leq N-1$, and let $D_A$ be the k-dihedron with opening $A$. Let $E$ be a compact subset of $d_A^R$ and let $u$ be a non-negative solution of (5.1) in $D_A^R$ (for some $R > 0$) such that $u$ vanishes on $\partial D_A^R \setminus E$. Then*

$$C_{s,q'}^{N-k}(E) = 0, \quad s = 2 - \frac{\kappa_+ + k}{q'} \Longrightarrow u = 0, \tag{8.27}$$

*where $C_{s,q'}^{N-k}$ denotes the Bessel capacity with the indicated indices in $\mathbb{R}^{N-k}$.*

*Proof.* By Proposition 8.2, (8.27) holds under the additional assumption

$$\int_{D_A^R} u^q \rho_R \rho_A^R dx < \infty. \tag{8.28}$$

Indeed, by Proposition 4.1, (8.28) implies that the solution $u$ possesses a boundary trace $\mu$ on $\partial D_A^R$. By assumption, $\mu(\partial D_A^R \setminus E) = 0$. Therefore, by Proposition 8.3, the fact that $C_{s,q'}^{N-k}(E) = 0$ implies that $\mu(E) = 0$. Thus $\mu = 0$ and hence $u = 0$.

We show that, under the conditions of the theorem, if $C_{s,q'}^{N-k}(E) = 0$ then (8.28) holds.

By Proposition 8.5, for every $\eta \in W_0^{s,q'}(d_A^R)$ such that $0 \le \eta \le 1$ and $\eta = 0$ in a neighborhood of $E$,

$$\int_{D_A^R} u^q \zeta \, dx \le c\big(1 + \|\eta\|_{W^{s,q'}(B_R^{N-k}(0))}\big)^{q'}, \tag{8.29}$$

for $\zeta$ as in (8.21). (Here we use the assumption that $u = 0$ on $\partial D_A^R \setminus E$.)

Let $a > 0$ be sufficiently small so that $E \subset B_{(1-4a)R}^{N-k}(0)$. Pick a sequence $\{\phi_n\}$ in $C_0^\infty(\mathbb{R}^{N-k})$ such that, for each $n$, there exists a neighborhood $Q_n$ of $E$, $\bar{Q}_n \subset B_{(1-3a)R}^{N-k}(0)$ and

$$\begin{aligned}
&0 \le \phi_n \le 1 \ \text{ everywhere}, \ \ \phi_n = 1 \ \text{ in } Q_n,\\
&\tilde{\phi}_n := \phi_n \chi_{[|x''|<(1-2a)R]} \in C_0^\infty(\mathbb{R}^{N-k}),\\
&\big\|\tilde{\phi}_n\big\|_{W^{s,q'}(\mathbb{R}^{N-k})} \to 0 \ \text{ as } \ n \to \infty\\
&\eta_n := (1-\phi_n)\lfloor_{[|x''|<R]} \in C_0^\infty(d_A^R),\\
&\eta_n = 0 \text{ in } [(1-a)R < |x''| < R].
\end{aligned} \tag{8.30}$$

Such a sequence exists because $C_{s,q'}^{N-k}(E) = 0$. Applying (8.29) to $\eta_n$ we obtain,

$$\sup \int_{D_A^R} u^q \zeta_n \, dx \le c < \infty, \tag{8.31}$$

where $\zeta_n = \rho_A \psi_R H_R^{q'}[\eta_n]$ (see (8.21)). By taking a subsequence we may assume that $\{\eta_n\}$ converges (say to $\eta$) in $L^{q'}(B_R^{N-k}(0))$ and consequently $H[\eta_n] \to H[\eta]$ in the sense that

$$H_R[\eta_n](x', \cdot) = w_{n,R}(y^2, \cdot) \to w_R(y^2, \cdot) = H_R[\eta](x', \cdot) \ \text{ in } L^{q'}$$

uniformly with respect to $y = |x'|$. It follows that

$$\int_{D_A^R} u^q \zeta \, dx < \infty, \quad \zeta = \rho_A \psi_R H_R^{q'}[\eta]. \tag{8.32}$$

As $\tilde{\phi}_n \to 0$ in $W^{s,q'}(\mathbb{R}^{N-k})$ it follows that $\phi_n \to 0$ and hence $\eta_n \to 1$ a.e. in $B^{N-k}_{(1-2a)R}(0)$. Thus $\eta = 1$ in this ball, $\eta = 0$ in $[(1-a)R < |x''| < R]$ and $0 \le \eta \le 1$ everywhere.

Consequently, given $\delta > 0$, there exists an $N$-dimensional neighborhood $O$ of $d_A \cap B^{N-k}_{(1-2a)R}(0)$ such that

$$1 - \delta < H_R[\eta] < 1 \ \text{ and } \ 1 - \delta < \psi_R/\rho_A^R < 1 \ \text{ in } \ O.$$

Therefore (8.32) implies that

$$\int_{D_A^{(1-3a)R}} u^q \rho^R \rho_A^R \, dx \le c < \infty. \tag{8.33}$$

Recall that the trace of $u$ on $\partial D_A^R \setminus d_A^{(1-4a)R}$ is zero. Therefore $u$ is bounded in $D_A^R \setminus D_A^{(1-3a)R}$. This fact and (8.33) imply (8.28).

$\square$

**Definition 8.7** *Let $\Omega$ be a bounded Lipschitz domain. Denote by $\rho$ the first eigenfunction of $-\Delta$ in $\Omega$ normalized by $\rho(x_0) = 1$ for a fixed point $x_0 \in \Omega$.*

*For every compact set $K \subset \partial\Omega$ we define*

$$M_{\rho,q}(K) = \{\mu \in \mathfrak{M}(\partial\Omega) : \mu \ge 0, \ \mu(\partial\Omega \setminus K) = 0, \ \mathbb{K}[\mu] \in L^1_\rho(\Omega)\}$$

*and*

$$\tilde{C}_{\rho,q'}(K) = \sup\{\mu(K)^q : \ \mu \in M_{\rho,q}(K), \ \int_\Omega \mathbb{K}[\mu]^q \rho \, dx = 1\}.$$

*Finally we denote by $C_{\rho,q'}$ the outer measure generated by the above functional.*

The following statement is verified by standard arguments:

**Lemma 8.8** *For every compact $K \subset \partial\Omega$, $C_{\rho,q'}(K) = \tilde{C}_{\rho,q'}(K)$. Thus $C_{\rho,q'}$ is a capacity and,*

$$C_{\rho,q'}(K) = 0 \iff M_{\rho,q}(K) = \{0\}. \tag{8.34}$$

**Theorem 8.9** *Let $\Omega$ be a bounded polyhedron in $\mathbb{R}^N$. A compact set $K \subset \partial\Omega$ is removable if and only if*

$$C_{s(k,j),q'}(K \cap L_{k,j}) = 0, \tag{8.35}$$

*for $k = 1, \cdot, N$ $j = 1, \cdots, n_k$, where $s(k,j)$ is defined as in (8.17). This condition is equivalent to*

$$C_{\rho,q'}(K) = 0. \tag{8.36}$$

*A measure $\mu \in \mathfrak{M}(\partial\Omega)$ is $q$-good if and only if it does not charge sets with $C_{\rho,q'}$-capacity zero.*

*Proof.* The first assertion is an immediate consequence of Proposition 8.1 and Theorem 8.6. The second assertion follows from the fact that

$$C_{\rho,q'}(K \cap L_{k,j}) = C_{s(k,j),q'}(K \cap L_{k,j}).$$

The third assertion follows from Theorem 8.4 and the previous statement. □



# 9 Appendix–Boundary Harnack inequality

In this section we prove the following

**Proposition 9.1** *Assume $\Omega$ is a bounded Lipschitz domain, $A \subset \partial\Omega$ is relatively open and $q > 1$. Let $(r_0, \lambda_0)$ be the Lipschitz characteristic of $\Omega$ (see subsection 2.1).*

*Let $u_i \in C(\Omega \cup A)$, $i = 1, 2$, be positive solutions of*

$$-\Delta u + u^q = 0 \quad \textit{in} \;\; \Omega,$$

*such that $u_i = 0$ on $A$. Put $S = \partial\Omega \setminus A$ and $d(x, S) = \operatorname{dist}(x, S)$. Let $y \in A$ and let*

$$r := \min(r_0/8, \frac{1}{4} d(y, S))$$

*so that*

$$\partial(B_{4r}(y) \cap \Omega) = (\overline{B}_{4r}(y) \cap \partial\Omega) \cup (\partial B_{4r}(y) \cap \Omega).$$

*Then*

$$c^{-1} \frac{u_1(z')}{u_1(z)} \leq \frac{u_2(z')}{u_2(z)} \leq c \frac{u_1(z')}{u_1(z)} \quad \forall z, z' \in B_r(y) \cap \Omega, \tag{9.1}$$

*where the constant $c > 0$ depends only on $N, q$ and the Lipschitz characteristic of $\Omega$.*

*Proof.* Without loss of generality we assume that $y = 0$.

Let $b = d(0, S)$ and put

$$\tilde{u}_i(x) = b^{-\frac{2}{q-1}} u_1(x/b), \quad i = 1, 2.$$

Then $\tilde{u}_i$ has the same properties as $u_i$ when $\Omega$ is replaced by $\Omega^b = \frac{1}{b}\Omega$, $S$ by $S^b = \frac{1}{b}S$ and $r$ by $\delta = r/b$. Of course $d(0, S^b) = 1$ so that

$$\delta = \min(r_0/(8b), 1/4).$$

The functions $\tilde{u}_i$ satisfy the equation

$$-\Delta\tilde{u}_i + \tilde{u}_i^q = 0 \ \text{ in } \ B_{4\delta}(0) \cap \Omega^b$$

and $\tilde{u}_i = 0$ on $B_{4\delta}(0) \cap \partial\Omega^b$. Therefore, by the Keller–Osserman estimate,

$$\tilde{u}_i \leq c(N, q)\delta^{-2/(q-1)} \ \text{ in } \ \bar{B}_{3\delta}(0) \cap \Omega^b.$$

If $a(x) = \tilde{u}_1^{q-1}$ then $\tilde{u}_1$ satisfies

$$-\Delta\tilde{u}_1 + a(x)\tilde{u}_1 = 0 \quad \text{in} \quad (\frac{1}{b}\Omega) \cap B_1(0),$$

and $a(\cdot)$ is bounded in $\bar{B}_{3\delta}(0)$.

Let $w$ be the solution of

$$\begin{cases} -\Delta w + a(x)w = 0 & \text{in } \ B_{3\delta}(0) \cap \Omega^b \\ w = 0 & \text{on } \ \overline{B}_{3\delta}(0) \cap \frac{1}{b}\partial\Omega \\ w = \tilde{u}_2 & \text{on } \ \partial B_{3\delta}(0) \cap \partial\Omega^b. \end{cases}$$

By applying the boundary Harnack principle in $B_{3\delta}(0) \cap \Omega^b$ (using the slightly more general form derived in [2, Theorem 2.1]) we obtain

$$c^{-1}\frac{\tilde{u}_1(\zeta')}{\tilde{u}_1(\zeta)} \leq \frac{w(\zeta')}{w(\zeta)} \leq c\frac{\tilde{u}_1(\zeta')}{\tilde{u}_1(\zeta)} \quad \forall \zeta, \zeta' \in B_{2\delta}(0) \cap \Omega^b, \tag{9.2}$$

where the constant $c$ depends only on the Lipschitz characteristic of $\Omega^b$ (which is $(r_0/b, \lambda_0 b)$ and therefore 'better' then that of $\Omega$ when $b \leq 1$). Since $w \leq \tilde{u}_2$ the above inequality implies,

$$\frac{w(\zeta')}{\tilde{u}_2(\zeta)} \leq c\frac{\tilde{u}_1(\zeta')}{\tilde{u}_1(\zeta)} \ \text{ and } \ c^{-1}\frac{\tilde{u}_1(\zeta')}{\tilde{u}_1(\zeta)} \leq \frac{\tilde{u}_2(\zeta')}{w(\zeta)}$$

which in turn implies

$$\frac{w(\zeta')}{w(\zeta)} \leq c\frac{\tilde{u}_2(\zeta')}{\tilde{u}_2(\zeta)} \quad \forall \zeta, \zeta' \in B_{2\delta}(0) \cap \Omega^b$$

and therefore

$$\frac{\tilde{u}_1(\zeta')}{\tilde{u}_1(\zeta)} \leq c^2\frac{\tilde{u}_2(\zeta')}{\tilde{u}_2(\zeta)} \quad \forall \zeta, \zeta' \in B_{2\delta}(0) \cap \Omega^b.$$

Switching the roles of $\tilde{u}_1$ and $\tilde{u}_2$ we obtain,

$$\frac{\tilde{u}_2(\zeta')}{\tilde{u}_2(\zeta)} \leq c^2\frac{\tilde{u}_1(\zeta')}{\tilde{u}_1(\zeta)} \quad \forall \zeta, \zeta' \in B_{2\delta}(0) \cap \Omega^b.$$

This completes the proof.

□

# References


[1] P. Baras and M. Pierre, *Singularits liminables pour des quations semi-lineaires*, **Ann. Inst. Fourier (Grenoble) 34**, 185206 (1984).

[2] S. Bauman, *Positive solutions of elliptic equations in nondivergence form and their adjoints*, **Ark. Mat. 22**, 153-173 (1984).

[3] K. Bogdan, *Sharp estimates for the Green function in Lipschitz domains*, **J. Math. Anal. Appl. 243**, 326-337 (2000).

[4] H. Brezis and F. Browder, *Sur une proprit des espaces de Sobolev*, **C. R. Acad. Sci. Paris Sr. A-B 287**, A113-A115 (1978).

[5] X. Y. Chen, H. Matano and L. Véron, *Anisotropic singularities of solutions of semilinear elliptic equations in* $\mathbb{R}^2$, **J. Funct. Anal. 83**, 50-97 (1989)

[6] L. Caffarelli, E. Fabes, S. Mortola and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, **Indiana Univ. Math. J. 30**, 621-640 (1981).

[7] B. E. Dalhberg, *Estimates on harmonic measures*, **Arch. Rat. Mech. Anal. 65**, 275-288 (1977).

[8] E. B. Dynkin, *Diffusions, superdiffusions and partial differential equations*, Amer. Math. Soc. Colloquium Publications, **50**, Providence, RI (2002).

[9] E. B. Dynkin, *Superdiffusions and positive solutions of nonlinear partial differential equations*, University Lecture Series, **34**, Amer. Math. Soc., Providence, RI, 2004.

[10] E. B. Dynkin and S. E. Kuznetsov, *Solutions of nonlinear differential equations on a Riemanian manifold and their trace on the boundary*, **Trans. Amer. Math. Soc. 350**, 4217-4552 (1998).

[11] J. Fabbri and L. Véron, *Singular boundary value problems for nonlinear elliptic equations in non smooth domains*, **Advances in Diff. Equ. 1**, 1075-1098 (1996).

[12] D. Feyel and A. de la Pradelle. *Topologies fines et compactifications associes certains espaces de Dirichlet*, **Ann. Inst. Fourier (Grenoble) 27**, 121146 (1977).

[13] N. Gilbarg and N. S. Trudinger. *Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin/New-York, 1983.

[14] A. Gmira and L. Véron, *Boundary singularities of solutions of nonlinear elliptic equations*, **Duke J. Math. 64**, 271-324 (1991).

[15] R. A. Hunt and R. L. Wheeden, *Positive harmonic functions on Lipschitz domains*, **Trans. Amer. Math. Soc. 147**, 507-527 (1970).

[16] D. S. Jerison and C. E. Kenig, *Boundary value problems on Lipschitz domains*, Studies in partial differential equations, 1-68, **MAA Stud. Math. 23**, (1982).

[17] D. S. Jerison and C. E. Kenig, *The Dirichlet problems in non-smooth domains*, **Annals of Math. 113**, 367-382 (1981).

[18] C. Kenig and J. Pipher, *The h-path distribution of conditioned Brownian motion for non-smooth domains*, **Proba. Th. Rel. Fields 82**, 615-623 (1989).

[19] J. B. Keller, *On solutions of* $\Delta u = f(u)$, **Comm. Pure Appl. Math. 10**, 503-510 (1957).

[20] J. F. Le Gall, *The Brownian snake and solutions of* $\Delta u = u^2$ *in a domain*, **Probab. Th. Rel. Fields 102**, 393-432 (1995).

[21] J. F. Le Gall, *Spatial branching processes, random snakes and partial differential equations* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

[22] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, **Arch. Rat. Mech. An. 144**, 201-231 (1998).

[23] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, **J. Math. Pures Appl. 77**, 481-521 (1998).

[24] M. Marcus and L. Véron, *Removable singularities and boundary traces*, **J. Math. Pures Appl. 80**, 879-900 (2001).

[25] M. Marcus and L. Véron *The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption*, **Comm. Pure Appl. Math. 56** (6), 689-731 (2003).

[26] M. Marcus, L. Véron, *On a new characterisation of Besov spaces with negative exponents*, Topics Around the Research of Vladimir Mazya /International Mathematical Series **11-13**, Springer Science+Business Media, 2009.

[27] M. Marcus, L. Véron, *The precise boundary trace of positive solutions of the equation* $\Delta u = u^q$ *in the supercritical case*, Perspectives in nonlinear partial differential equations, 345–383, **Contemp. Math., 446**, Amer. Math. Soc., Providence, RI, 2007.

[28] B. Mselati, *Classification and probabilistic representation of the positive solutions of a semilinear elliptic equation*, **Mem. Amer. Math. Soc. 168**, no. 798, (2004).

[29] R. Osserman, *On the inequality* $\Delta u \geq f(u)$, **Pacific J. Math. 7**, 1641-1647 (1957).

[30] E. Stein, *Singular Integral and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[31] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Pub. Co., 1978.

[32] N. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, **Comm. Pure App. Math. 20**, 721-747 (1967).

[33] L. Véron, *Singularities of Solutions of Second Order Quasilinear Equations*, Pitman Research Notes in Math. **353**, Addison-Wesley-Longman, 1996.