Test vectors for trilinear forms, given two unramified representations

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1 Introduction

Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$, and uniformizing parameter ϖ, whose residual field has q elements. For $G = \mathrm{GL}_2(F)$, let (π_1, V_1), (π_2, V_2) and (π_3, V_3) be three irreducible, admissible, infinite dimensional representations of G. Using the theory of Gelfand pairs, Dipendra Prasad proves in [P] that the space of G-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$ has dimension at most one. He gives a precise criterion for this dimension to be one, that we will explain now.

Let D_F^* be the group of invertible elements of the quaternion division algebra D_F over F. When (π_i, V_i) is a discrete series representation of G, denote by (π_i', V_i') the irreducible representation of D_F^* associated to (π_i, V_i) by the Jacquet-Langlands correspondence. Again, by the theory of Gelfand pairs, the space of D_F^*-invariant linear forms on $V_1' \otimes V_2' \otimes V_3'$ has dimension at most one.

Let σ_i be the two dimensional representations of the Weil-Deligne group of F associated to the irreducible representations π_i. The triple tensor product $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ is an eight dimensional symplectic representation of the Weil-Deligne group, and has local root number $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \pm 1$. When $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$, one can prove that the representations π_i's are all discrete series representations of G.

Theorem 1. *(Prasad, Theorem 1.4 of [P]) Let (π_1, V_1), (π_2, V_2), (π_3, V_3) be three irreducible, admissible, infinite dimensional representations of G such that the product of their central characters is trivial. If all the representations V_i's are cuspidal, assume that the residue characteristic of F is not 2. Then*

- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$ *if and only if there exists a non-zero G-invariant linear form on $V_1 \otimes V_2 \otimes V_3$*
- $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ *if and only if there exists a non-zero D_k^* invariant linear form on $V_1' \otimes V_2' \otimes V_3'$.*

Given a non zero G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$, or a non-zero D_k^*-invariant linear form ℓ' on $V_1' \otimes V_2' \otimes V_3'$, the goal is to find a vector in $V_1 \otimes V_2 \otimes V_3$ which is not in the kernel of ℓ, or a vector in $V_1' \otimes V_2' \otimes V_3'$ which is not in the kernel of ℓ'. Such a vector is called a test vector. At first sight, it appears to have strong connections with the new vectors v_1, v_2 and v_3 of the representations π_1, π_2 and π_3.

Theorem 2. *(Prasad, Theorem 1.3 of [P]) When all the π_i's are unramified principal series representations of G, $v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ.*

Theorem 3. *(Gross and Prasad, Proposition 6.3 of [G-P]) When all the π_i's are unramified twists of the special representation of G :*

- *if $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = 1$, then $v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ,*
- *if $\varepsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$, let R' be the unique maximal order in D_F. Then the open compact subgroup $R'^* \times R'^* \times R'^*$ fixes a unique line in $V_1' \otimes V_2' \otimes V_3'$. Any vector on this line is a test vector for ℓ'.*

The proof by Gross and Prasad of the first statement of this theorem actually contains another result:

Theorem 4. *When two of the π_i's are unramified twists of the special representation of G and the third one belongs to the unramified principal series of G, $v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ.*

But the paper [G-P] gives evidence that $v_1 \otimes v_2 \otimes v_3$ is not always a test vector for ℓ. Let $K = \mathrm{GL}(\mathcal{O}_F)$ be the maximal compact subgroup of G. If π_1 and π_2 are unramified and if π_3 has conductor $n \geq 1$, ℓ being G-invariant, v_1 and v_2 being K-invariant, one gets a K-invariant linear form

$$\begin{cases} V_3 & \longrightarrow & \mathbb{C} \\ v & \longmapsto & \ell(v_1 \otimes v_2 \otimes v) \end{cases}$$

which must be 0 since π_3 is ramified. Then $\ell(v_1 \otimes v_2 \otimes v_3) = 0$.

Now Gross and Prasad make the following suggestion. Let I_n be the congruence subgroup

$$I_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \quad | \quad c \equiv 0 \quad mod\, \varpi^n \quad \right\}$$

and R be a maximal order $\mathrm{M}_2(F)$ such that $R^* \cap K = I_n$. If v_1^* is a R^*-invariant vector in V_1, the linear form

$$\begin{cases} V_3 & \longrightarrow & \mathbb{C} \\ v & \longmapsto & \ell(v_1^* \otimes v_2 \otimes v) \end{cases}$$

is invariant under the action of $R^* \cap K = I_n$, and one can still hope that $v_1^* \otimes v_2 \otimes v_3$ is a test vector for ℓ.

The purpose of this paper is to prove that $v_1^* \otimes v_2 \otimes v_3$ actually *is* a test vector for ℓ. This is the object of Theorem 5. The case $n = 1$, together with Theorems 2, 3 and 4, complete the study of test vectors when the π_i's have ramification 0 or 1.

In the long term, the search for test vectors is motivated by the subconvexity problem for L-functions. Roughly speaking, one wants to bound some L-functions along the critical line $\Re(z) = \frac{1}{2}$. A recent and successful idea in this direction has been to relate triple products of automorphic forms to special values of L-functions on the critical line. In [B-R 1] and [B-R 2] Joseph Bernstein and Andre Reznikov established a so called *subconvexity bound* the for the L-function of a triple of representations : each representation is attached to the eigenvalue of a certain operator, and the eigenvalue of one representation varies. Philippe Michel and Akshay Venkatesh considered the case when the *level* of one representation varies. More details about subconvexity and those related techniques can be found in [V] or [M-V]. Test vectors are key ingredients. Bernstein and Reznikov use an explicit test vector. Venkatesh uses a theoretical one, but explains that the bounds would be better with an explicit one (see paragraph 5 of [V]). Unfortunately, the difficulty of finding them increases with the ramification of the representations involved.

There is an extension of Prasad's result in [H-S], where Harris and Scholl prove that the dimension of the space of G-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$ is one when π_1, π_2 and π_3 are principal series representations, either irreducible or reducible with their unique irreducible subspace, infinite dimensional. They apply the global setting of this to the construction of elements in the motivic cohomology of the product of two modular curves constructed by Beilinson.

I would like to thank Philippe Michel for suggesting this problem, Wen-Ching Winnie Li who invited me to spend one semester at PennState University where I wrote the first draft of this paper, and of course Benedict Gross and Dipendra Prasad for the inspiration. I would also like to thank Paul Broussous and Nicolas Templier for many interesting discussions, and Eric Bahuaud for his help with English.

In a previous version of this paper, I obtained Theorem 5 under an unpleasant technical condition. I am extremely grateful to Malden Dimitrov, because, thanks to our discussions on the subject, I found the way to remove the condition. In [D-N], we are working on a more general version of Theorem 5.

2 Statement of the result

2.1 About induced and contragredient representations

Let (ρ, W) be a smooth representation of a closed subgroup H of G. Let Δ_H be the modular function on H. The induction of ρ from H to G is a representation π whose space is the space $\mathrm{Ind}_H^G(\rho)$ of functions f from G to W satisfying the two following conditions :

(1) $\forall h \in H, \quad \forall g \in G, \quad f(hg) = {\Delta_H}^{-\frac{1}{2}}(h)\rho(h)f(g)$,

(2) there exists an open compact subgroup K_f of G such that

$$\forall k \in K_f, \quad \forall g \in G, \quad f(gk) = f(g)$$

where G acts by right translation. The resulting function will be denoted $\langle \pi(g), f \rangle$ that is

$$\forall g, g_0 \in G, \quad \langle \pi(g), f\rangle(g_0) = f(g_0 g).$$

With the additional condition that f must be compactly supported modulo H, one gets the *compact* induction denoted by ind_H^G. When G/H is compact, there is no difference between Ind_H^G and ind_H^G.

Let B the Borel subgroup of upper triangular matrices in G, and let T be the diagonal torus. The character Δ_T is trivial and we will use $\delta = {\Delta_B}^{-1}$ with $\delta\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = |\frac{a}{d}|$ where $|\cdot|$ is the normalised valuation of F. The quotient $B\backslash G$ is compact and can be identified with $\mathbb{P}^1(F)$.

For a smooth representation V of G, V^* is the space of linear forms on V. The contragredient representation $\widetilde{\pi}$ is given by the action of G on $\widetilde{V}$, the subspace of smooth vectors in V^*. If H is a subgroup of G, $\widetilde{V} \subset \widetilde{V_{|H}} \subset V^*$.

We refer the reader to [B-Z] for more details about induced and contragredient representations.

2.2 New vectors and ramification

Let (π, V) be an irreducible, admissible, infinite dimensional representation of G with central character ω. To the descending chain of compact subgroups of G

$$K \supset I_1 \supset \cdots \supset I_n \supset I_{n+1} \cdots$$

one can associate an ascending chain of vector spaces

$$V^0 = V^K \qquad \text{and} \qquad \forall n \geq 1, \quad V^n = \left\{ v \in V \quad | \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_n, \quad \pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})v = \omega(a)v \quad \right\}.$$

There exists a minimal n such that the vector space V^n is not $\{0\}$. It is necessarily one dimensional and any generator of V^n is called a new vector of (π, V). The integer n is the conductor of (π, V). The representation (π, V) is said to be unramified when $n = 0$. Else, it is ramified.

More information about new vectors can be found in [C].

2.3 The main result

Let (π_1, V_1), (π_2, V_2) and (π_3, V_3) be three irreducible, admissible, infinite dimensional representations of G such that the product of their central characters is trivial. Assume that π_1 and π_2 are unramified principal series, and that π_3 has conductor $n \geq 1$. According to Theorem 1, since π_1 and π_2 are not discrete series, there exists a non-zero, G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$. We are looking for a vector v in $V_1 \otimes V_2 \otimes V_3$ which is not in the kernel of ℓ. In order to follow the suggestion of Gross and Prasad we consider

$$\gamma = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R_n = \gamma^{-n} \mathrm{M}_2(\mathcal{O}_F) \gamma^n.$$

One can easily check that

$$R_n^* = \gamma^{-n} K \gamma^n \qquad \text{and} \qquad R_n^* \cap K = I_n.$$

If v_1, v_2 and v_3 denote the new vectors of π_1, π_2 and π_3, the vector

$$v_1^* = \pi_1(\gamma^{-n}) \cdot v_1$$

is invariant under the action of R_n^*. Hence we can write

$$v_1^* \in {V_1}^{R_n^*}, \qquad v_2 \in {V_2}^K \qquad \text{and} \qquad v_3 \in {V_3}^{R_n^* \cap K}.$$

Theorem 5. *Under those conditions, $v_1^* \otimes v_2 \otimes v_3$ is a test vector for ℓ.*

The proof will follow the same pattern as Prasad's proof of Theorem 2 in [P], with the necessary changes.

3 Going down Prasad's exact sequence

3.1 Central characters

Let ω_1, ω_2 and ω_3 be the central characters of π_1, π_2 and π_3. Notice that the condition $\omega_1\omega_2\omega_3 = 1$ derives from the G-invariance of ℓ. Since π_1 and π_2 are unramified, ω_1 and ω_2 are unramified too, and so is ω_3 because $\omega_1\omega_2\omega_3 = 1$. Let η_i, for $i \in \{1,2,3\}$ be unramified quasi-characters of F^* with $\eta_i^2 = \omega_i$ and $\eta_1\eta_2\eta_3 = 1$. Then

$$V_1 \otimes V_2 \otimes V_3 \simeq \left(V_1 \otimes \eta_1^{-1}\right) \otimes \left(V_2 \otimes \eta_2^{-1}\right) \otimes \left(V_3 \otimes \eta_3^{-1}\right)$$

as a representation of G. Hence it is enough to prove Theorem 5 when the central characters of the representations are trivial.

When $n = 1$, it is also enough to prove Theorem 5 when V_3 is the special representation Sp of G : take η_3 to be the unramified character such that $V_3 = \eta_3 \otimes \mathrm{Sp}$.

3.2 Prasad's exact sequence

Let us now explain how Prasad finds ℓ. It is equivalent to search for ℓ or to search for a non-zero element in $\mathrm{Hom}_G\left(V_1 \otimes V_2, \widetilde{V_3}\right)$. Since the central characters of π_1 and π_2 are trivial, there are unramified characters μ_1 and μ_2 such that for $i = 1$ and $i = 2$

$$\pi_i = \mathrm{Ind}_B^G \chi_i \qquad \text{with} \qquad \chi_i\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \mu_i\left(\frac{a}{d}\right).$$

Hence

$$V_1 \otimes V_2 = \mathrm{Res}_G \, \mathrm{Ind}_{B\times B}^{G\times G}\left(\chi_1 \times \chi_2\right)$$

where G is diagonally embedded in $G\times G$ for the restriction. The action of G on $B\times B\backslash G\times G = \mathbb{P}^1(F)\times\mathbb{P}^1(F)$ has precisely two orbits. The first is $\{(u,v) \in \mathbb{P}^1(F)\times\mathbb{P}^1(F) \quad | \quad u \neq v\}$ which is open and can be identified with $T\backslash G$. The second orbit is the diagonal embedding of $\mathbb{P}^1(F)$ in $\mathbb{P}^1(F) \times \mathbb{P}^1(F)$, which is closed and can be identified with $B\backslash G$. Then, we have a short exact sequence of G-modules

$$0 \to \mathrm{ind}_T^G\left(\frac{\chi_1}{\chi_2}\right) \xrightarrow{\mathbf{ext}} V_1 \otimes V_2 \xrightarrow{\mathbf{res}} \mathrm{Ind}_B^G\left(\chi_1\chi_2\delta^{\frac{1}{2}}\right) \to 0. \tag{1}$$

The surjection **res** is the restriction of functions from $G \times G$ to the diagonal part of $B\backslash G \times B\backslash G$, that is

$$\Delta_{B\backslash G} = \left\{(g, bg) \quad | \quad b \in B, \quad g \in G\right\}.$$

The injection **ext** takes a function $f \in \mathrm{ind}_T^G\left(\frac{\chi_1}{\chi_2}\right)$ to a function $F \in \mathrm{Ind}_{B\times B}^{G\times G}\left(\chi_1 \times \chi_2\right)$ vanishing on $\Delta_{B\backslash G}$, and is given by the relation

$$F\left(g, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g\right) = f(g),$$

on the other orbit. Applying the functor $\mathrm{Hom}_G\left(\cdot\,, \widetilde{V_3}\right)$, one gets a long exact sequence

$$0 \to \mathrm{Hom}_G\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big), \widetilde{V_3}\Big) \to \mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \to \mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_3}\Big)$$
$$\downarrow$$
$$\cdots \leftarrow \mathrm{Ext}^1_G\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big), \widetilde{V_3}\Big) \quad (2)$$

3.3 The simple case

The situation is easier when $n = 1$ and $\mu_1\mu_2|\cdot|^{\frac{1}{2}} = |\cdot|^{-\frac{1}{2}}$, as π_3 is special and there is a natural surjection

$$\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big) \longrightarrow \widetilde{V_3}$$

whose kernel is the one dimensional subspace of constant functions. Thanks to the exact sequence (1) one gets a surjection Ψ

$$\begin{array}{ccc} V_1 \otimes V_2 & \xrightarrow{\mathbf{res}} & \mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big) \\ & {}^{\Psi}\searrow \quad \swarrow & \\ & \widetilde{V_3} & \end{array}$$

which corresponds to

$$\ell \begin{cases} V_1 \otimes V_2 \otimes V_3 & \longrightarrow & \mathbb{C} \\ v \otimes v' \otimes v'' & \longmapsto & \Psi(v \otimes v').v'' \end{cases}$$

The surjection Ψ vanishes on $v_1^* \otimes v_2$ if and only if $\mathbf{res}(v_1^* \otimes v_2)$ has constant value on $\mathbb{P}^1(F) \simeq B\backslash G$. An easy computation proves that $\mathbf{res}(v_1^* \otimes v_2)$ is not constant : the new vectors v_1 and v_2 are functions from G to $\mathbb{C}$ such that

$$\forall i \in \{1,2\}, \quad \forall b \in B, \quad \forall k \in K, \qquad v_i(bk) = \chi_i(b) \cdot \delta(b)^{\frac{1}{2}}$$

and

$$\forall g \in G, \qquad v_1^*(g) = v_1(g\gamma^{-1}).$$

Then

$$(v_1^* \otimes v_2)\Big(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big) = v_1\Big(\gamma^{-1}\Big)v_2\Big(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big) = v_1\Big(\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\Big) = \mu_1(\varpi)^{-1}|\varpi|^{-\frac{1}{2}} = \frac{\sqrt{q}}{\mu_1(\varpi)}$$

and

$$(v_1^* \otimes v_2)\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) = v_1\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}\Big) = v_1\Big(\begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) = \frac{\mu_1(\varpi)}{\sqrt{q}}.$$

The representation π_1 is principal so $\frac{\sqrt{q}}{\mu_1(\varpi)} \neq \frac{\mu_1(\varpi)}{\sqrt{q}}$ and

$$(v_1^* \otimes v_2)\Big(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\Big) \neq (v_1^* \otimes v_2)\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big).$$

Hence Ψ does not vanish on $v_1^* \otimes v_2$. Now, v_1^* being R_1^*-invariant and v_2 being K-invariant, $\Psi(v_1^* \otimes v_2)$ is a non-zero I_1-invariant element of $\widetilde{V_3}$, that is, a new vector for $\widetilde{\pi_3}$. Consequently it does not vanish on v_3 :

$$\ell(v_1^* \otimes v_2 \otimes v_3) = \Psi(v_1^* \otimes v_2).v_3 \neq 0$$

and $v_1^* \otimes v_2 \otimes v_3$ is a test vector for ℓ.

3.4 The other case

If $n \geq 2$ or $\mu_1\mu_2|\cdot|^{\frac{1}{2}} \neq |\cdot|^{-\frac{1}{2}}$ then $\mathrm{Hom}_G\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big), \widetilde{V_3}\Big) = 0$ and by Corollary 5.9 of [P]

$$\mathrm{Ext}^1_G\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big), \widetilde{V_3}\Big) = 0.$$

Through the long exact sequence (2) we get an isomorphism

$$\mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \simeq \mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_3}\Big),$$

and by Frobenius reciprocity

$$\mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_3}\Big) \simeq \mathrm{Hom}_T\Big(\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_{3|T}}\Big),$$

where $\widetilde{V_{3|T}}$ is the space of the contragredient representation of $\pi_{3|T}$. By Lemmas 8 and 9 of [W], the latter space is one dimensional. Thus, we have a chain of isomorphic one dimensional vector spaces

$$\begin{array}{ccccccc} \mathrm{Hom}_G\Big(V_1 \otimes V_2 \otimes V_3, \mathbb{C}\Big) & \tilde{\rightarrow} & \mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) & \tilde{\rightarrow} & \mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_3}\Big) & \tilde{\rightarrow} & \mathrm{Hom}_T\Big(\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_{3|T}}\Big) \\ \ell & \mapsto & \Psi & \mapsto & \Phi & \mapsto & \varphi \end{array}$$

with generators ℓ, Ψ, Φ and φ corresponding via the isomorphisms. Notice that φ is a linear form on V_3 such that

$$\forall t \in T, \qquad \forall v \in V_3, \qquad \varphi\big(\pi_3(t)v\big) = \frac{\chi_2(t)}{\chi_1(t)}\varphi(v) \tag{3}$$

which is identified to the following element of $\mathrm{Hom}_T\Big(\Big(\frac{\chi_1}{\chi_2}\Big), \widetilde{V_{3|T}}\Big)$

$$\begin{cases} \mathbb{C} & \longrightarrow & \widetilde{V_{3|T}} \\ z & \longmapsto & z\varphi \end{cases}$$

Lemma 1. $\varphi(v_3) \neq 0$.

Proof : this is Proposition 2.6 of [G-P] with the following translation :

- the local field F is the same,

- the quadratic extension K/F of Gross and Prasad is $F \times F$ and their group K^* is our torus T,

- the infinite dimensional representation V_1 of Gross and Prasad is our π_3,

- the one dimensional, unramified representation V_2 of Gross and Prasad is $\frac{\chi_1}{\chi_2}$.

Then the representation that Gross and Prasad call V is $\frac{\chi_1}{\chi_2} \otimes \pi_3$ and their condition (1.3) is exactly our condition (3). In order to apply Gross and Prasad's Proposition, we need to check the equality

$$\varepsilon(\sigma \otimes \sigma_3) = \alpha_{K/F}(-1)\omega(-1).$$

Basically, it is true because K is not a field. Let us give some details.

- In [G-P], ω is the central character of the representation V_1 which is trivial for us.

- The character $\alpha_{K/F}$ is the quadratic character of F^* associated to the extension K/F by local class-field theory. Here, it is trivial because K is $F \times F$.

- To compute $\varepsilon(\sigma \otimes \sigma_3)$ we will use the first pages of [T].

$$\forall \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \in T \qquad \frac{\chi_1}{\chi_2}\Big(\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}\Big) = \frac{\mu_1}{\mu_2}(x)\frac{\mu_2}{\mu_1}(z) \Rightarrow \varepsilon(\sigma \otimes \sigma_3) = \varepsilon(\frac{\mu_1}{\mu_2} \otimes \sigma_3)\varepsilon(\frac{\mu_2}{\mu_1} \otimes \sigma_3)$$

Since the determinant of σ_3 is the central character of π_3 which is trivial, σ_3 is isomorphic to its own contragredient and the contragredient representation of $\frac{\mu_1}{\mu_2} \otimes \sigma_3$ is $\frac{\mu_2}{\mu_1} \otimes \sigma_3$. Formula (1.1.6) of [T] leads to

$$\varepsilon(\sigma \otimes \sigma_3) = \det\big(\sigma_3(-1)\big) = 1 = \alpha_{K/F}(-1)\omega(-1).$$

According to [G-P], the restriction of $\frac{\chi_1}{\chi_2} \otimes \pi_3$ to the group

$$M = \Big\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \quad | \quad x, y \in \mathcal{O}_F^* \Big\} \times I_n$$

fixes a unique line in V_3 : it is the line generated by the new vector v_3. Still according to Gross and Prasad, a non-zero linear form on V_3 which satisfies (3) cannot vanish on v_3. $\square$

We will deduce from lemma 1 that $\ell(v_1^* \otimes v_2 \otimes v_3) \neq 0$.

4 Going up Prasad's exact sequence

4.1 From $\varphi(v_3)$ to f

Let f be the element of $\mathrm{ind}_T^G\Big(\frac{\chi_1}{\chi_2}\Big)$ which is the characteristic function of the orbit of the unit in the decomposition of $T\backslash G$ under the action of I_n. This means :

$$f(g) = \begin{cases} \frac{\chi_1(t)}{\chi_2(t)} & \text{if} \quad g = tk \quad \text{with} \quad t \in T \quad \text{and} \quad k \in I_n \\ 0 & \text{else} \end{cases} \tag{4}$$

Then, the function

$$\begin{cases} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & f(g)\,\varphi\Big(\pi_3(g)v_3\Big) \end{cases}$$

is invariant by the action of T by left translation and we can do the following computation :

$$\begin{aligned} \Big(\Phi(f)\Big)(v_3) &= \int_{T\backslash G} f(g)\,\varphi\Big(\pi_3(g)v_3\Big)dg \\ &= \int_{(T\cap K)\backslash I_n} \varphi\Big(\pi_3(k)v_3\Big)dk \\ &= \lambda \cdot \varphi(v_3). \end{aligned}$$

where λ is a non-zero constant. Thanks to Lemma 1 we know that $\varphi(v_3) \neq 0$, so

$$\Big(\Phi(f)\Big)(v_3) \neq 0. \tag{5}$$

4.2 From f to F

Now, we are going to compute $F = \mathbf{ext}(f)$ in $V_1 \otimes V_2$. Let a and b be the numbers

$$a = \frac{\mu_1(\varpi)}{\sqrt{q}} \qquad\qquad b = \frac{\mu_2(\varpi)}{\sqrt{q}}$$

They verify

$$(a^2 - 1)(b^2 - 1) \neq 0$$

because π_1 and π_2 are principal series representations. For the sake of simplicity, we shall use the following notation : for any g in G

$$gv_1^* = \left\langle \pi_1(g), v_1^* \right\rangle \qquad \text{and} \qquad gv_2 = \left\langle \pi_2(g), v_2 \right\rangle$$

Lemma 2. *The function F is given by the formula*

$$F = A \cdot v_1' \otimes v_2'$$

with

$$A = \frac{a^n}{(a^2-1)(b^2-1)}, \qquad v_1' = a \cdot \gamma^{-(n-1)} v_1 - \gamma^{-n} v_1 \qquad \text{and} \qquad v_2' = b \cdot \gamma^{-1} v_2 - v_2.$$

Proof : the function f is described by formula (4), and $\mathbf{ext}(f)$ is described by the short exact sequence (1) using the orbits of the action of G on $B \times B\backslash G \times G$. The function F must vanish on the closed orbit

$$\Delta_{B\backslash G} = \Big\{ (g, bg) \quad | \quad b \in B, \quad g \in G \Big\}.$$

The open orbit can be identified with $T\backslash G$ via the bijection

$$\begin{cases} T\backslash G & \longrightarrow & \Big(B\backslash G \times B\backslash G \Big) \setminus \Delta_{B\backslash G} \\ Tg & \longmapsto & \Big(Bg, B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g \Big) \end{cases}$$

through which, the orbit of the unit in $T\backslash G$ under the action of I_n corresponds to

$$\Big\{ \Big(Bk, B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} k \Big) \quad | \quad k \in I_n \Big\}.$$

Now, pick any $(k, k') \in K \times K$. If $k' \in Bk$, then

$$k \in I_n \iff k' \in I_n \qquad \text{and} \qquad k \in I_1 \iff k' \in I_1.$$

When $k' \notin Bk$, write $k = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ and $k' = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$. There exists $(b_1, b_2) \in B \times B$ such that

$$k = b_1 k_0 \qquad \text{and} \qquad k' = b_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} k_0 \qquad \text{with} \qquad k_0 = \begin{pmatrix} z' & t' \\ z & t \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}_F).$$

Then (k,k') is in the orbit of the unit in $T\backslash G$ under the action of I_n if and only if k_0 is in TI_n. Because k and k' are in K, one can see that

$$k_0 \in TI_n \quad \Longleftrightarrow \quad k_0 \in I_n$$

and

$$\begin{aligned} k_0 \in I_n &\Longleftrightarrow z \equiv 0 \quad mod\, \varpi^n \quad \text{and} \quad z't \in \mathcal{O}_F^* \\ &\Longleftrightarrow z \equiv 0 \quad mod\, \varpi^n \quad \text{and} \quad z' \in \mathcal{O}_F^* \\ &\Longleftrightarrow k \in I_n \quad \text{and} \quad k' \notin I_1. \end{aligned}$$

It follows that (k,k') corresponds to an element of the orbit of the unit in the decomposition of $T\backslash G$ under the action of I_n if and only if $k \in I_n$ and $k' \notin I_1$. Then, it will be enough to check that

$$F(k,k') = \begin{cases} 1 & \text{if} \quad k \in I_n \quad \text{and} \quad k' \notin I_1 \\ 0 & \text{else} \end{cases} \tag{6}$$

This is mere calculation. With $a = \frac{\mu_1(\varpi)}{\sqrt{q}}$, we need

Lemma 3. *Let i be any element of $\mathbb{N}$. The values of the function $\gamma^{-i}v_1$ on K are given by the formula*

$$\forall k = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in K, \qquad \gamma^{-i}v_1(k) = \begin{cases} a^i & \text{if} \quad \mathrm{val}(\frac{z}{t}) \leq 0 \\ a^{i-2\,\mathrm{val}(\frac{z}{t})} & \text{if} \quad 1 \leq \mathrm{val}(\frac{z}{t}) \leq i-1 \\ a^{-i} & \text{if} \quad i \leq \mathrm{val}(\frac{z}{t}) \end{cases} \tag{7}$$

Proof : first, recall that the new vector $v_1$is a function from G to $\mathbb{C}$ such that

$$\forall b \in B, \quad \forall k \in K, \qquad v_1(bk) = \chi_1(b) \cdot \delta(b)^{\frac{1}{2}}.$$

Then, for $k = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ in K, either z or t is in $\mathcal{O}_F^*$. The other one is in $\mathcal{O}_F$. Write

$$k\gamma^{-i} = \begin{pmatrix} \frac{x}{\varpi^i} & y \\ \frac{z}{\varpi^i} & t \end{pmatrix}.$$

If $\mathrm{val}\,\frac{z}{t} \leq 0$, then $\mathrm{val}\, z = 0$, $\mathrm{val}(\frac{\varpi^i t}{z}) \geq 0$ and

$$k\gamma^{-i} = \begin{pmatrix} \frac{xt-yz}{z} & \frac{x}{\varpi^i} \\ 0 & \frac{z}{\varpi^i} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \frac{\varpi^i t}{z} \end{pmatrix}$$

with

$$\begin{pmatrix} 0 & -1 \\ 1 & \frac{\varpi^i t}{z} \end{pmatrix} \in K \quad \text{and} \quad (\chi_1 \cdot \delta^{\frac{1}{2}}) \begin{pmatrix} \frac{xt-yz}{z} & \frac{x}{\varpi^i} \\ 0 & \frac{z}{\varpi^i} \end{pmatrix} = \left(\frac{\mu_1(\varpi)}{\sqrt{q}}\right)^{i-2\mathrm{val}\, z} = a^{i-2\mathrm{val}\, z} = a^i.$$

If $1 \leq \mathrm{val}\,\frac{z}{t} \leq i-1$, then $\mathrm{val}\,t=0$, $\mathrm{val}(\frac{\varpi^i t}{z}) \geq 0$ and the computation is quite the same, except that

$$(\chi_1 \cdot \delta^{\frac{1}{2}})\begin{pmatrix} \frac{xt-yz}{z} & \frac{x}{\varpi^i} \\ 0 & \frac{z}{\varpi^i} \end{pmatrix} = \Big(\frac{\mu_1(\varpi)}{\sqrt{q}}\Big)^{i-2\mathrm{val}\,z} = a^{i-2\mathrm{val}\,\frac{z}{t}}.$$

If $i \leq \mathrm{val}\,\frac{z}{t}$, then $\mathrm{val}\,t=0$, $\mathrm{val}(\frac{\varpi^i t}{z}) \leq 0$ and

$$k\gamma^{-i} = \begin{pmatrix} \frac{xt-yz}{t\,\varpi^i} & y \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{z}{t\,\varpi^i} & 1 \end{pmatrix}$$

with

$$\begin{pmatrix} 1 & 0 \\ \frac{z}{t\,\varpi^i} & 1 \end{pmatrix} \in K \quad \text{and} \quad (\chi_1 \cdot \delta^{\frac{1}{2}})(\begin{pmatrix} \frac{xt-yz}{t\,\varpi^i} & y \\ 0 & t \end{pmatrix}) = \Big(\frac{\mu_1(\varpi)}{\sqrt{q}}\Big)^{-i-2\mathrm{val}\,t} = a^{-i}.$$

□

NB : the case $t=0$ is included in $\mathrm{val}\,\frac{z}{t} \leq 0$.

We can now finish the proof of Lemma 2. The same computation, with v_2 and b instead of v_1 and a, gives the values of $\gamma^{-i}v_2$ for any $i \in \mathbb{N}$. It is then easy to compute $F(k,k')$ as given in Lemma 2, for k and k' in K, and to check formula (6).

4.3 Some test-vectors

On the one hand, from the expression of F given by Lemma 2 we deduce

$$\Big(\Psi(F)\Big)(v_3) = A \cdot \ell(v_1' \otimes v_2' \otimes v_3)$$

On the other hand, from the relation $F = \mathbf{ext}(f)$ we deduce

$$\Big(\Psi(F)\Big)(v_3) = \Big(\Phi(f)\Big)(v_3)$$

and from equation 5 in Section 4.1 we get

$$\Big(\Psi(F)\Big)(v_3) \neq 0$$

Then

$$\ell(v_1' \otimes v_2' \otimes v_3) \neq 0$$

and $v_1' \otimes v_2' \otimes v_3$ is a test vector for ℓ. We are going to simplify it. We can deduce from lemma 2 that

$$F = A \cdot \Big\{ab \cdot \gamma^{-(n-1)}v_1 \otimes \gamma^{-1}v_2 - a \cdot \gamma^{-(n-1)}v_1 \otimes v_2 - b \cdot \gamma^{-n}v_1 \otimes \gamma^{-1}v_2 + \gamma^{-n}v_1 \otimes v_2\Big\}.$$

If $n \geq 2$, we write

$$\Big(\Psi(F)\Big)(v_3) = A \cdot \big(ab \cdot \langle \gamma^{-1}\psi_{n-2}, v_3\rangle - a \cdot \langle \psi_{n-1}, v_3\rangle - b \cdot \langle \gamma^{-1}\psi_{n-1}, v_3\rangle + \langle \psi_n, v_3\rangle\big)$$

where, for m in $\{n-1, n-2, n\}$

$$\psi_m \quad \begin{cases} V_3 & \longrightarrow \quad \mathbb{C} \\ v & \longmapsto \quad \ell(\gamma^{-m}v_1 \otimes v_2 \otimes v) \end{cases}$$

Since ℓ is G invariant, ψ_m is an element of $\widetilde{V_3}$ which is invariant by the action of

$$R_m \cap K = I_m$$

But $\widetilde{\pi_3}$ has conductor n so

$$\psi_{n-2} = \psi_{n-1} = 0$$

and

$$\Big(\Psi(F)\Big)(v_3) = A \cdot \psi_3(v_3) = A \cdot \ell(\gamma^{-n} v_1 \otimes v_2 \otimes v_3)$$

then

$$\ell(\gamma^{-n} v_1 \otimes v_2 \otimes v_3) \neq 0$$

and $\gamma^{-n} v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ.

If $n = 1$, only the two terms in the middle vanish and we get

$$\Big(\Psi(F)\Big)(v_3) = A \cdot \big(ab \cdot \ell(v_1 \otimes \gamma^{-1} v_2 \otimes v_3) + \ell(\gamma^{-1} v_1 \otimes v_2 \otimes v_3)\big)$$

Now, take

$$g = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$

On the one hand

$$g\gamma^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and this matrix is in K so

$$g\gamma^{-1} v_1 = v_1$$

On the other hand

$$\begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi \end{pmatrix} \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The first matrix belongs to the center of G, the second one is precisely γ^{-1} and the third one is in K, so

$$gv_2 = \gamma^{-1} v_2.$$

Then

$$\begin{aligned}
\Big(\Psi(F)\Big)(v_3) =& A \cdot \big\{ab \cdot \ell(g\gamma^{-1} v_1 \otimes gv_2 \otimes v_3) + \ell(\gamma^{-1} v_1 \otimes v_2 \otimes v_3)\big\} \\
=& A \cdot \big\{ab \cdot \ell(\gamma^{-1} v_1 \otimes v_2 \otimes g^{-1} v_3) + \ell(\gamma^{-1} v_1 \otimes v_2 \otimes v_3)\big\} \\
=& A \cdot \ell\big(\gamma^{-1} v_1 \otimes v_2 \otimes v_3'\big)
\end{aligned}$$

with

$$v_3' = ab \cdot (g^{-1} v_3) + v_3$$

The linear form

$$\psi_1 \quad \begin{cases} V_3 & \longrightarrow \quad \mathbb{C} \\ v & \longmapsto \quad \ell(\gamma^{-1} v_1 \otimes v_2 \otimes v) \end{cases}$$

is a non zero element of $\widetilde{V_3}^{I_1}$, so it is a new vector in $\widetilde{V_3}$. It is known from [B] that a new vector in $\widetilde{V_3}$ does not vanish on on $V_3^{I_1}$ that is, it does not vanish on a new vector of V_3 :

$$\ell(\gamma^{-1}v_1 \otimes v_2 \otimes v_3) \neq 0$$

and $\gamma^{-1}v_1 \otimes v_2 \otimes v_3$ is a test vector for ℓ. This concludes the proof of Theorem 5.

NB : it is easy to deduce from Theorem 5 that $v_1 \otimes \gamma^{-n}v_2 \otimes v_3$ also is a test vector for ℓ. Take

$$g = \begin{pmatrix} 0 & 1 \\ \varpi^n & 0 \end{pmatrix}.$$

Then

$$g\gamma^{-n}v_1 = v_1 \quad , \quad gv_2 = \gamma^{-n}v_2$$

and

$$\ell(v_1 \otimes \gamma^{-n}v_2 \otimes gv_3) = \ell(g\gamma^{-n}v_1 \otimes gv_2 \otimes gv_3) = \ell(\gamma^{-n}v_1 \otimes v_2 \otimes v_3) \neq 0.$$

So the linear form

$$\begin{cases} V_3 & \longrightarrow & \mathbb{C} \\ v & \longmapsto & \ell(v_1 \otimes \gamma^{-n}v_2 \otimes v) \end{cases}$$

is not zero. Being I_n-invariant, it is a new vector in $\widetilde{V_3}$, which does not vanish on v_3 :

$$\ell(v_1 \otimes \gamma^{-n}v_2 \otimes v_3) \neq 0.$$

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