Distinguished dihedral representations of $GL(2)$ over a p-adic field

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1 Introduction

Let F be a finite extension of a p-adic field, K a quadratic extension of F. The principal series representations of $GL_2(K)$ distingushed for $GL_2(F)$ are well known, it's also known that the Steinberg representation is distingushed (cf.[A-T] for a summary of these results due to Y.Flicker, J.Hakim and D.Prasad,).
Moreover there is a complete characterisation of distinguishedness in terms of the epsilon factor (due to J.Hakim) and in terms of base change from representations of the unitary group U(2, K/F) (due to Y.Flicker, cf.[A-T] for a local proof).
Using those, we give here a description of dihedral representations on the parameter's side (those which the Langlands correpondance associate with 2 dimension induced representations of the Weil group of K, cf.(2) p.122 in [G-L]) distinguished with respect to $GL_2(F)$.
Every distinguished representation of $GL_2(K)$ is paramatrised by a regular multiplicative character ω of a quadratic extension L of K.
We show (theorem 5.1) that such a representation is distinguished for $GL_2(F)$ if and only if one can choose L to be biquadratic over F and ω trivial one the invertible elements of the two other quadratic extensions of F in L.
The results we prove here are theorems 3.3, 5.1 and 6.1. The method is to isolate first representations distingushed for $GL_2(F)_+$ using theorem 3.1, then to determinate those who are $GL_2(F)$-distinguished using theorems 4.1 and proposition 4.1.
Thus, in the case of odd residual caracteristic, we obtain every supercuspidal distinguished representations.
We also observe (see proposition 5.3) that if we consider the principal series as paramatrised by a multiplicative character of a two dimensional semi-simple commutative algebra over K, the statement for distinguishedness is the same as for the supercuspidal dihedral representations.

We then give a generalisation of theorem 5.1 to dihedral representations (non necessarily supercuspidal) in theorem 6.1.

2 Preliminaries

2.1 Generalities

We consider F a finite extension of $\mathbb{Q}_p$, and K a quadratic extension of F in an algebraic closure $\bar{F}$ of $\mathbb{Q}_p$.
If L is a quadratic extension of K in $\bar{F}$, then to every character ω of L^*, we associate a representation of $GL_2(K)$ via the Weil representation (cf.[J-L] p.144).
Such a representation is called dihedral.
We note θ the conjugation of F/K.
For A is a ring, we note A^* the group of its invertible elements.
For E_2 a finite extension of a local field E_1, we note respectively Tr_{E_2/E_1} and N_{E_2/E_1} the trace and norm of E_2 over E_1.
We also note $Gal(E_2/E_1)$ the Galois group of E_2 over E_1 when E_2/E_1 is Galois, otherwise we note $Aut_{E_1}(E_2/E_1)$ the group of automorphisms of the algebra E_2 over E_1.
Moreover if E_2 is quadratic over E_1, we note η_{E_2/E_1} the nontrivial character of ${E_1}^*$ with kernel $N_{E_2/E_1}({E_1}^*)$.
For n a positive integer, we note $GL_n(K)_+$, the subgroup with index two of $GL_n(K)$, of matrices whose determinant is a norm of K over F.
For Π a representation of a group G, we note π its class, and $\Pi^\vee$ its smooth contragedient when Π is a smooth representation of a totally disconnected locally compact group.
If ϕ is an automorphism of G, we note Π^ϕ the representation of G given by $\Pi \circ \phi$.
If H is a subgroup of G, and μ is a character of G, we say that a representation Π of G is μ-distinguished for (with respect to) H if there exists on the space of Π a linear functionnal L verifying for h in H, $L \circ \Pi(h) = \mu(h)L$.
If μ is trivial, we say that Π is distinguished for H.

2.2 Quadratic extensions of K

For L a quadratic extension of K, three cases arise:

1. L/F is biquadratic (hence Galois), it contains K and two other quadratic extensions F, K' and K''.

   Its Galois group is isomorphic with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, its non trivial elements are conjugations of L over K, K' and K''.

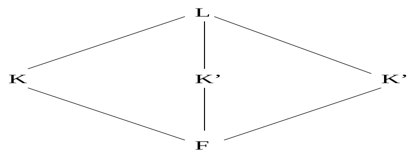

Figure 1:

The conjugation L over K extend those of K' and K'' over F.

2. L/F is cyclic with Galois group isomorphic with $\mathbb{Z}/4\mathbb{Z}$.

3. L/F non Galois. Then its Galois Closure M is quadratic over L and the Galois group of M over F is dihedral with order 8.
To see this, we consider a morphism $\tilde{\theta}$ from L to $\bar{F}$ which extends θ. Then if $L' = \tilde{\theta}(L)$, L and L' are distinct, quadratic over K and generate M biquadratic over K. M is the Galois closure of L because any morphism from L into $\bar{F}$, either extends θ, or the identity map of K, sothat its image is either L or L', so it is always included in M.
Finally the Galois group M over F cannot be abelian (for L is not Galois), it is of order 8, and it's not the quaternion group which only has one element of order 2, whereas here the conjugations of M over L and L' are of order 2. Hence it is the dihedral group of order 8.
We deduce from this the folowing lattice:

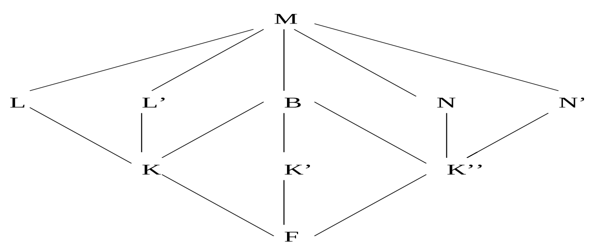

Figure 2:

Here M/K' is cyclic of degree 4, M/K and B/F are biquadratics.

In the case p odd, F has exactly three quadratic extensions which generate its unique biquadratic extension. If there exists L non Galois over F, then it implies that the cardinal q of the residual field F verifies $q \equiv 3[4]$, and M is generated over L by a primitive fourth root of unity in $\bar{F}$.

2.3 Quadratic characters

We wish to calculate how $\eta_{L/K}$ restricts to F^* in the following two cases.

1. If L is biquadratic over F, then $\eta_{L/K}$ has a trivial restriction to F^*.

Indeed, we have $N_{L/K}(K'^*) = N_{K'/F}(K'^*)$ and $N_{L/K}(K''^*) = N_{K''/F}(K''^*)$ because the conjugation of L over K extend those of K' and K'' over F.

Both these groups are distict from local class field theory and of index 2 in F^*, sothat they generate this latter, but both are contained in $N_{L/K}(L^*)$ which therefore contains F^*.

In other words $\eta_{L/K}$ restricts trivially to F^*.

2. If L is cyclic over F, then ${\eta_{L/K}}_{|F^*}$ is non trivial.

   If it wasn't the case, F^* would be contained in $N_{L/K}(L^*)$, and composing with $N_{K/F}$ on both sides, F^{*2} would be a subgroup of $N_{L/F}(L^*)$. But F^{*2} and $N_{L/F}(L^*)$ have both index 4 in F^* and give different quotients ($\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for the first and $\mathbb{Z}/4\mathbb{Z}$ for the second), sothat one cannot be contained in the other.

2.4 Weil's representation

Let L be a quadratic extension of K, then for any character ω of L^* , we associate an irreducible representation $\Pi(\omega)$ of $GL_2(K)$ (cf.[J-L]), with central character $\omega_{|K^*}\eta_{L/K}$.
If ω is regular for $N_{L/K}$, then $\Pi(\omega)$ is supercuspidal, otherwise there exists a character μ of K^* sothat $\omega = \mu \circ N_{L/K}$, and then $\pi(\omega)$ is the the principal series $\pi(\mu, \mu\eta_{L/K})$.

The conjugation of K over F naturally extends to an involutive automorphism of $GL_2(K)$ which we also note θ.

Here we want to determinate $\Pi(\omega)^\theta$.
Suppose there exists $\tilde{\theta}$ an element of $Aut_F(L/F)$ which extends θ, we then have:

Proposition 2.1 *If θ extends to an element $\tilde{\theta}$ of $Aut_F(L/F)$, then $\Pi(\omega)^\theta$ is isomorphic with $\Pi(\omega^{\tilde{\theta}})$.*

Proof:
Following [J-L], $\Pi(\omega)$ is the induced of $r(\omega, \psi_F)$ from $Gl_2(K)_+$ the group $GL_2(K)$.

The space of this representation $\mathbf{S}(L, \omega)$ is constitued by the continuous functions f with compact support from L^* to complex numbers verifying, for x in L and y in $Ker(N_{L/K})$, $f(xy) = \omega^{-1}(y)f(x)$.

Then the mapping which associates to f in $\mathbf{S}(L, \omega)$ the function $f^\theta = f \circ \tilde{\theta}$ is an equivariant morphism between $r(\omega, \psi_F)^\theta$ and $r(\omega^{\tilde{\theta}}, \psi_F)$.

We then see $\pi(\omega)^\theta = [Ind_{GL_2(K)_+}^{GL_2(K)}(r(\omega,\psi_F))]^\theta \approx [Ind_{GL_2(K)_+}^{GL_2(K)}(r(\omega,\psi_F)^\theta)] \approx Ind_{GL_2(K)_+}^{GL_2(K)}(r(\omega^{\tilde\theta},\psi_F)) = \pi(\omega^{\tilde\theta})$ where $Ind_{GL_2(K)_+}^{GL_2(K)}$ designs the induced representation from $GL_2(K)_+$ to $GL_2(K)$.

Remark :
It is not always true that θ extends to an element $\tilde\theta$ of $Aut_F(L/F)$.
For instance, take F local with residual characteristic $q \equiv 3[4]$, and let π_F be a prime element (generating the maximal ideal of the integers ring).
We choose $K = F({\pi_F}^{1/2})$ and $L = F({\pi_F}^{1/4})$.
Let $\tilde\theta$ be a F linear morphism extending θ to L, with values in $\bar F$. Then $\tilde\theta({\pi_F}^{1/4}) = i{\pi_F}^{1/4}$, where i is a primitive fourth root of unity.
Indeed $i^2 = -1$, because $\tilde\theta({\pi_F}^{1/2}) = \theta({\pi_F}^{1/2}) = -{\pi_F}^{1/2}$.
Moreover, i cannot be in L: indeed, this element is a root of unity with order prime to q, thus it would implie that the residual field of L, which is the one of F as L/F is totally ramified, contains a primitive fourth root of unity.
This cannot happen because 4 does not divide $q-1$.
We conclude that any F linear morphism extending θ to L, sends L onto $F(i{\pi_F}^{1/4})$ which is distinct from L , and hence cannot be in $Aut_F(L/F)$.

3 Representations distinguished by a character

3.1 Definitions and preliminary results

The following theorem due to Y.Flicker [A-T] (th. 1.3) will be of constant use.

Theorem 3.1 *Let Π be an irreducible admissible representation of $GL_2(K)$, such that c_π is trivial on F^*. Then $\pi^\theta = \pi^\vee$ if and only if π is distingushed or $\eta_{K/F}$-distinguished for $GL_2(F)$.*

Let $GL_2(F)_+$ be the subgroup of index two in $GL_2(F)$, it is clear that if a representation of $GL_2(K)$ is distinguished or $\eta_{K/F}$-distinguished for $GL_2(F)$, it is distinguished for $GL_2(F)_+$, the reverse is true.

Proposition 3.1 *(cf.[P], p.71)*
A representation of $GL_2(K)$ is distinguished or $\eta_{K/F}$-distinguished for $GL_2(F)$ if and only if it is distinguished for $GL_2(F)_+$.

Proof:
We show the non trivial implication.

Let s be an element of $GL_2(F)$ whose determinant is not a norm and let Π be a $GL_2(F)_+$-distingushed representation.

Let L_+ be the $GL_2(F)_+$-invariant linear form on the space of Π, two cases arise:

1. If $\Pi^\vee(s)L_+ = -L_+$, then for h in $GL_2(F)\backslash GL_2(F)_+$, we have $\Pi^\vee(h)L_+ = \Pi^\vee(h)\Pi^\vee(s^2)L_+ = \Pi^\vee(hs)\Pi^\vee(s)L_+ = -L_+$ because hs is in $GL_2(F)_+$ (here we also note $\Pi^\vee$ the non smooth contragedient). Π is therefore $\eta_{K/F}$-distinguished for $GL_2(F)$.
2. Otherwise $\Pi^\vee(s)L_+ \neq -L_+$, and $L_+ + \Pi^\vee(s)L_+$ is fixed under the action of $GL_2(F)$.

Theorem 3.1 takes the following form:

Theorem 3.2 *Let Π be an irreducible admissible representation of $GL_2(K)$. Then Π is $GL_2(F)_+$-distinguished if and only if $\pi^\theta = \pi^\vee$ and c_Π restricts trivially to F^*.*

3.2 Description of the $GL_2(F)_+$-distinguished representations

Theorem 3.3 *A supercuspidal dihedral representation Π of $GL_2(K)$ is $GL_2(F)_+$-distinguished if and only if there exists a quadratic extension L of K biquadratic on F, and a multiplicative character ω of L trivial on $N_{L/K'}(K'^*)$ or on $N_{L/K''}(K''^*)$, such that $\pi = \pi(\omega)$.*

Proof:

Let L be a quadratic extension of K and ω a regular multiplicative of L such that $\pi = \pi(\omega)$, we note σ the conjugation of L over K, three cases show up:

1. L/F si biquadratic.
   we note σ' the conjugation of L over K' and σ'' the conjugation of L over K'', σ' and σ'' both extend θ, and thus can play $\tilde{\theta}$'s role in proposition 1.1.
   The condition $\pi^{\vee} = \pi^{\theta}$ which one can also read $\pi(\omega^{-1}) = \pi(\omega^{\tilde{\theta}})$, is then equivalent to $\omega^{\sigma'} = \omega^{-1}$ or $\omega^{\sigma''} = \omega^{-1}$.
   This is equivalent to ω trivial on $N_{L/K'}(K'^*)$ and on $N_{L/K''}(K''^*)$.
   As $\eta_{L/K}$, $\eta_{L/K'}$, and $\eta_{L/K''}$ are trivial F^*, we have $c_{\pi|F^*} = \omega_{|F^*}\eta_{L/K}{}_{|F^*} = 1$ for such a representation .

2. L/F is cyclic, the regularity of ω makes the condition $\pi(\omega^{-1}) = \pi(\omega^{\tilde{\theta}})$ impossible.
   Indeed one would have $\omega^{\tilde{\theta}} = \omega^{-1}$, which would implie $\omega^{\sigma} = \omega$ for $\sigma^2 = \theta$, and so ω would be trivial on the kernel of $N_{L/K}$ from Hilbert's theorem 90.
   Π can therefore not be $GL_2(F)_+$-distniguished.

3. L/K is not Galois (which implies $q \equiv 3[4]$ in the case p odd), we note again θ the conjugation of B over K' which extends the one of K over F.
   Let $\pi_{B/K}$ be the representation of $GL_2(B)$ which is the base change lifting of π to B. As $\pi_{B/K} = \pi(\omega \circ N_{M/L})$, if $\omega \circ N_{M/L} = \mu \circ N_{M/B}$ for a character μ of B^*, then $\pi(\omega) = \pi(\mu)$ (cf.[G-L], (3) p.123) and we are brought back to case 1.
   Otherwise $\omega \circ N_{M/L}$ is regular for $N_{M/B}$.
   If π was $GL_2(F)_+$-distinguished, taht is $\pi^{\theta} = \pi^{\vee}$ and $c_{\pi|F^*}$, we would have $\pi^{\theta}_{B/K} = \pi^{\vee}_{B/K}$ and $c_{\pi_{B/K}} = c_{\pi} \circ N_{B/K}$ from theorem 1 of [G-L].
   As $N_{B/K}(K'^*) = N_{K'/F}(K'^*)$ for the conjugation of B over K extends that of K' over F, one would deduce that $c_{\pi_{B/K}}$ would be trivial on K'^* and theorem 2.2 would implie that $\pi_{B/K}$ would be $GL_2(K')_+$-distinguished.
   That would contradict case 2 because M/K' is cyclic.

4 Distingushed representations

We described in the previous section the supercuspidal dihedral representations of $GL_2(K)$ which are $GL_2(F)_+$ distinguished.
We want to characterize those who are $GL_2(F)$-distinguished among them.

4.1 Definitions and useful results

We refer to [J-L] for definitions and basic properties of ϵ factors attached to an irreducible admissible representation of $GL_2(K)$, and to [T] for those of ϵ factors attached to a multplicative character of a local field.
The ϵ used here for representations of $GL_2(K)$ is the one described in [J-L] evaluated at $s = 1/2$ and the ϵ attached to a multplicative character of a local field is Langlands'ϵ_L described in [T].

We will use the three following results.

The first, due to J.Hakim can be found in [H], page 8.
Here we repaced γ with ϵ because both are equal for supercuspidal representations:

Theorem 4.1 *Let Π be a supercuspidal irreducible representation of $GL_2(K)$, and ψ a nontrivial character of K trivial on F. Then Π is distinguished if and only if $\epsilon(\Pi \otimes \chi, \psi) = 1$ for every character χ of K^* trivial on F^*.*

The second, due to Fröhlich and Queyrut, is in [F-Q], page 130 :

Theorem 4.2 *Let L_2 be a quadratic extension of L_1 which is a quadratic extension of $\mathbb{Q}_p$, then if ψ_{L_2} is the standard character of L_2 and if Δ is an element of ${L_2}^*$ with $Tr_{L_2/L_1}(\Delta) = 0$, we then have $\epsilon(\chi, \psi_{L_2}) = \chi(\Delta)$ for every character χ of L_2^* trivial on L_1^* .*

The third is a corollary of proposition 3.1 of [A-T]:

Proposition 4.1 *There exists no supercuspidal representation of $GL_2(K)$ which is distinguished and $\eta_{K/F}$-distinguished at the same time.*

5 Description of distingushed representations

Theorem 5.1 *A dihedral supercuspidal representation Π of $GL_2(K)$ is $GL_2(F)$-distinguished if and only if there exists a quadratic extension L of K biquadratic over F, and a regular multiplicative character ω of L trivial on K'^* or on K''^*, such that $\pi = \pi(\omega)$.*

Proof:

From the second section, we can suppose that $\pi = \pi(\omega)$, for ω a regular multiplicative character of a quadratic extension L of K biquadratic over F, with ω trivial on $N_{L/K'}(K'^*)$ or on $N_{L/K''}(K''^*)$.

Let ψ_K be the standard character of K, ψ_L the one of L, and a a non null element of K such that $Tr_{K/F}(a) = 0$, which implies $Tr_{L/K'}(a) = Tr_{L/K''}(a) = 0$.

we note $(\psi_K)_a$ the character trivial on F given by $(\psi_K)_a(x) = (\psi_K)(ax)$.

To see if $\pi(\omega)$ is distinguished, we use Hakim's criterion (th.3.1).

So let χ be a character of K^* trivial on F^*, we have $\pi(\omega) \otimes \chi = \pi(\omega \times \chi \circ N_{L/K})$ and we note $\mu = \omega \times \chi \circ N_{L/K}$.

i) if $\omega_{|K'^*} = 1$: on a $\epsilon(\pi(\omega)\otimes\chi, (\psi_K)_a) = \epsilon(\pi(\mu), (\psi_K)_a) = \epsilon(\pi(\mu), \psi_K)\mu(a)\eta_{L/K}(a)$. Now $\epsilon(\pi(\mu), \psi_K) = \lambda(L/K, \psi_K)\epsilon(\mu, \psi_L)$ (cf.[J-L] p.150), where the Langlands-Deligne factor $\lambda(L/K, \psi_K)$ equals $\epsilon(\eta_{L/K}, \psi_K)$ divided by its module.

As $\eta_{L/K_{|F^*}} = 1$ et $\mu_{|K'^*} = 1$, from theorem 4.2, we have that $\epsilon(\mu, \psi_L) = \mu(a)$ and $\epsilon(\eta_{L/K}, \psi_K) = \eta_{L/K}(a)$.

We deduce that $\epsilon(\pi(\omega) \otimes \chi, (\psi_K)_a) = \mu(a)^2\eta_{L/K}(a)^2 = 1$ for a^2 is in F. $\pi(\omega)$ is therefore distinguished.

ii) If $\omega_{|K'^*} = \eta_{L/K'}$: Let χ' be a character of K^* which extends $\eta_{K/F}$, then $\chi' \circ N_{L/K}$ equals $\eta_{K/F} \circ N_{K'/F}$ on K' because the conjugation of L over K extends the one of K' over F.
But $\eta_{K/F} \circ N_{K'/F}$ is trivial on the image of $N_{L/K'}$ from the identity $N_{K'/F} \circ N_{L/K'} = N_{K/F} \circ N_{L/K}$, but not trivial for $N_{K'/F}$ is not the kernel $N_{K/F}(K^*)$ of $\eta_{K/F}$ from local class field theory.
Thus $\omega \times \chi' \circ N_{L/K}$ is trivial on K', and we deduce that $\pi(\omega) \otimes \chi' = \pi(\omega \times \chi' \circ N_{L/K})$ is distinguished from i).
This implies that $\pi(\omega)$ is $\eta_{K/F}$-distinguished and thus not distinguished from proposition 3.1.

The cases $\omega_{|K''^*} = 1$ and $\omega_{|K''^*} = \eta_{L/K''}$ are handled as well.

6 The principal series

Representations of the principal series of $GL_2(K)$ distinguished for $GL_2(F)$ are well known, and described for example in proposition 4.2 of [A-T].

The result is the following:

Proposition 6.1 *Let λ and μ be two characters of K^*, whose quotient is not the module of K or its inverse. The principal series representation $\Pi(\lambda, \mu)$ of $GL_2(K)$ is $GL_2(F)$-distinguished either when $\lambda = \mu^{-\theta}$ or when λ and μ have a trivial restriction to F^*.*

Now one can construct the principal series $\Pi(\lambda, \mu)$ via the Weil representation (cf.[B] p.523 à 557), in this case (λ, μ) identifies with a character of $K^* \times K^*$.

This way of parametrising irreducible representations of $GL_2(K)$ with multiplicatice characters of two-dimensional semi-simple commutative algebras over K, includes the principal series (for the algebra $K \times K$) and the dihedral representations (for quadratic extensions of K).

Let L be a quadratic extension of K biquadratic over F, as we are here interested with $GL_2(F)$-distinguishedness, we consider the following F-algebras.

1. **the algebra** $K \times K$

   One note $Aut_F(K \times K)$ its automorphisms group. The elements of this group are $(x, y) \mapsto (x, y)$, $(x, y) \mapsto (y, x)$, $(x, y) \mapsto (x^\theta, y^\theta)$, $(x, y) \mapsto (y^\theta, x^\theta)$, and $Aut_F(K \times K)$ is isomorphic with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   The three sub-algebras fixed by non trivial elements of $Aut_F(K \times K)$ are K via the natural diagonal inclusion, the twisted form $\tilde{K}$ of K given by $x \mapsto (x, x^\theta)$, and $F \times F$.

2. **l'algèbre** L

   The group $Gal(B_F/K)$ of its automorphisms is isomorphic with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   The three sub-algebras fixed by non trivial elements of $Gal(B_F/K)$ are K, K' et K''.

We then observe that proposition 5.2 for the principal series has the same statement that the one for theorem 4.1:

Proposition 6.2 *A principal series representation $\Pi(\lambda, \mu)$ of $GL_2(K)$ is $GL_2(F)$-distinguished if and only if the multiplicative character (λ, μ) is trivial one the invertible elements of one of the two intermediate sub-algebras of $K \times K$ distinct from K.*

We now study dihedral non supercuspidal representations.
Let Π be such a representation, there exists a quadratic extension L over K and a non regular multiplicative character ω of L such that $\pi = \pi(\omega)$.
If μ is a character of K^* such that $\omega = \mu \circ N_{L/K}$, then $\pi = \pi(\mu, \mu\eta_{L/K})$.

Three cases arise:

1. For L biquadratic over F, we show that ω restricts trivially to K'^* or K''^* if and only if $(\mu, \mu\eta_{L/K})$ restricts trivially to $\tilde{K}^*$ or to $F^* \times F^*$.

   We have the following equivalences:

   - $\omega(K'^*) = 1 \Leftrightarrow \mu(N_{L/K}(K'^*)) = 1 \Leftrightarrow \mu(N_{K'/F}(K'^*)) = 1$ because the conjugation of L/K extends the one of K'/F, and so $\omega(K'^*) = 1 \Leftrightarrow \mu_{|F^*} = 1$ or $\eta_{K'/F}$.
   - $\omega(K''^*) = 1 \Leftrightarrow \mu(N_{L/K}(K''^*)) = 1 \Leftrightarrow \mu(N_{K''/F}(K''^*)) = 1$ because the conjugation of L/K extends the one of K''/F, and so $\omega(K''^*) = 1 \Leftrightarrow \mu_{|F^*} = 1$ or $\eta_{K''/F}$.
   - $(\mu, \mu\eta_{L/K})$ trivial on $\tilde{K}^* \Longleftrightarrow \mu^\theta\mu\eta_{L/K} = 1 \Longleftrightarrow \mu\circ N_{K/F} = \eta_{L/K}$.
     We deduce that $\mu \circ N_{L/F} = 1$, which implies that $\mu_{|F^*} = 1$, $\eta_{K/F}$, $\eta_{K'/F}$ or $\eta_{K''/F}$, but the first two possibilities are excluded by the identity $\mu \circ N_{K/F} = \eta_{L/K}$.
     Conversely if $\mu_{|F^*} = \eta_{K'/F}$ or $\eta_{K''/F}$, then $\mu\circ N_{K/F}$ is a character of order two of K^* which cannot be trivial from local class field theory. As the equalities $N_{L/F} = N_{L/K'} \circ N_{K'/K} = N_{L/K''} \circ N_{K''/K}$ implie that $\mu\circ N_{K/F}$ is trivial on $N_{L/K}(L^*)$, it is therefore $\eta_{L/K}$.
     Eventually $(\mu, \mu\eta_{L/K})$ trivial on $\tilde{K}^* \Longleftrightarrow \mu_{|F^*} = \eta_{K'/F}$ or $\eta_{K''/F}$.
   - Also $(\mu, \mu\eta_{L/K})$ trivial on $F^* \times F^* \Longleftrightarrow \mu_{|F^*} = 1$ because we have already seen that $\eta_{L/K}$ is trivial on F^*.
   - Finally these equivalences show that $\omega(K'^*) = 1$ or $\omega(K''^*) = 1$ $\Longleftrightarrow (\mu, \mu\eta_{L/K})$ trivial on $\tilde{K}^*$ or on $F^* \times F^*$.

2. If L is cyclic over F. One shows that $\pi(\omega) = \pi(\mu, \mu\eta_{L/K})$ is distinguished if and only if $\mu_{|F^*}$ generates the cyclic group of the characters of $F^*/N_{L/F}(L^*)$.

   - It is not possible for $(\mu, \mu\eta_{L/K})$ to be trivial on $F^* \times F^*$ because we saw in the preliminaries that $\eta_{L/K}$ is not trivial on F^*.
   - $(\mu, \mu\eta_{L/K})$ trivial on $\tilde{K}^* \Longleftrightarrow \mu \circ N_{K/F} = \eta_{L/K}$.
     We deduce as before that μ is a character of $F^*/N_{L/F}(L^*)$.
     As $F^*/N_{L/F}(L^*)$ is cyclic of order four, the same is true for its characters group.

As $\mu \circ N_{K/F} = \eta_{L/K}$, we deduce that $\mu_{|F^*}$ is non trivial, moreover if $\mu_{|F^*}$ was of order 2, it would be equal to $\eta_{K/F}$ which is the unique element with order 2 of the characters group of $F^*/N_{L/F}(L^*)$, which contradicts $\mu \circ N_{K/F} = \eta_{L/K}$.
We deduce that $\mu \circ N_{K/F} = \eta_{L/K} \Longrightarrow \mu_{|F^*}$ generates the dual group of $F^*/N_{L/F}(L^*)$.

- Conversely if $\mu_{|F^*}$ is of order four in the dual group of $F^*/N_{L/F}(L^*)$, we deduce that $\mu \circ N_{K/F}$ is a character of K^* trivial on $N_{L/K}(L^*)$, but not trivial because it would implie $\mu_{|F^*} = 1$ or $\eta_{K/F}$, namely $\mu_{|F^*}$ with order less than 2.
On conclude that $\mu \circ N_{K/F} = \eta_{L/K}$.

3. If L is not Galois over F. As $\omega \circ N_{M/K} = \mu \circ N_{M/F} = \mu' \circ N_{M/B}$, where $\mu' = \mu \circ N_{B/F}$, we conclude as in the case 3. of the proof of theorem 2.3 that $\pi(\omega) = \pi(\mu')$ and we are brought back to case 1. because B is biquadratic over F.

Thus we have the following general theorem:

Theorem 6.1 *A dihedral representation* Π *of* $GL_2(K)$ *is* $GL_2(F)$*-distinguished if and only if* $\pi = \pi(\omega)$ *for some multiplicative character* ω *of a quadratic extension* L *over* K *verifying i) or ii):*
i) L/F *is biquadratic , and* $\omega_{|K'^*} = 1$ *or* $\omega_{|K''^*} = 1$,
ii) L/F *is cyclic and* $\omega = \mu \circ N_{L/K}$ *for* μ *a character of* K^* *whose restriction to* F^* *generates the dual group of* $F^*/N_{L/F}(L^*)$.

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