# A CLT FOR INFORMATION-THEORETIC STATISTICS OF NON-CENTERED GRAM RANDOM MATRICES

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Abstract. In this article, we study the fluctuations of the random variable:

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log\det\left(\Sigma_n\Sigma_n^* + \rho I_N\right), \quad (\rho > 0)$$

where $\Sigma_n = n^{-1/2} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n$, as the dimensions of the matrices go to infinity at the same pace. Matrices $X_n$ and $A_n$ are respectively random and deterministic $N \times n$ matrices; matrices $D_n$ and $\tilde{D}_n$ are deterministic and diagonal, with respective dimensions $N \times N$ and $n \times n$; matrix $X_n = (X_{ij})$ has centered, independent and identically distributed entries with unit variance, either real or complex.

We prove that when centered and properly rescaled, the random variable $\mathcal{I}_n(\rho)$ satisfies a Central Limit Theorem and has a Gaussian limit. The variance of $\mathcal{I}_n(\rho)$ depends on the moment $\mathbb{E}X_{ij}^2$ of the variables $X_{ij}$ and also on its fourth cumulant $\kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\mathbb{E}X_{ij}^2|^2$.

The main motivation comes from the field of wireless communications, where $\mathcal{I}_n(\rho)$ represents the mutual information of a multiple antenna radio channel. This article closely follows the companion article "A CLT for Information-theoretic statistics of Gram random matrices with a given variance profile", *Ann. Appl. Probab. (2008)* by Hachem et al., however the study of the fluctuations associated to non-centered large random matrices raises specific issues, which are addressed here.

**Key words and phrases:** Random Matrix, empirical distribution of the eigenvalues, Stieltjes Transform.
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## 1. Introduction

**The model, the statistics, and the literature.** Consider a $N \times n$ random matrix $\Sigma_n = (\xi_{ij}^n)$ which writes:

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{\frac{1}{2}} X_n \tilde{D}_n^{\frac{1}{2}} + A_n \ , \tag{1.1}$$

where $A_n = (a_{ij}^n)$ is a deterministic $N \times n$ matrix with uniformly bounded spectral norm, $D_n$ and $\tilde{D}_n$ are diagonal deterministic matrices with nonnegative entries, with respective dimensions $N \times N$ and $n \times n$; $X_n = (X_{ij})$ is a $N \times n$ matrix with the entries $X_{ij}$'s being centered, independent and identically distributed (i.i.d.) random variables with unit variance $\mathbb{E}|X_{ij}|^2 = 1$ and finite $16^{\text{th}}$ moment.

Consider the following linear statistics of the eigenvalues:

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log\det\left(\Sigma_n\Sigma_n^* + \rho I_N\right) = \frac{1}{N} \sum_{i=1}^{N} \log(\lambda_i + \rho) \ ,$$

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where $I_N$ is the $N \times N$ identity matrix, $\rho > 0$ is a given parameter and the $\lambda_i$'s are the eigenvalues of matrix $\Sigma_n \Sigma_n^*$ ($\Sigma_n^*$ stands for the Hermitian adjoint of $\Sigma_n$). This functional, known as the mutual information for multiple antenna radio channels, is fundamental in wireless communication as it characterizes the performance of a (coherent) communication over a wireless Multiple-Input Multiple-Output (MIMO) channel with gain matrix $\Sigma_n$. Channels with non-centered gain matrix $\Sigma_n = n^{-1/2} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n$ are known as Rician channels. The deterministic matrix $A_n$ accounts for the so-called line-of-sight component, while $D_n$ and $\tilde{D}_n$ account for the correlations at the receiving and emitting sides, respectively.

Since the seminal work of Telatar [36], the study of the mutual information $\mathcal{I}_n(\rho)$ of a MIMO channel (and other performance indicators) in the regime where the dimensions of the gain matrix grow to infinity at the same pace has turned to be extremely fruitful. However, Rician channels have been comparatively less studied from this point of view, as their analysis is more difficult due to the presence of the deterministic matrix $A_n$. First order results can be found in Girko [14, 15]; Dozier and Silverstein [10, 11] established convergence results for the spectral measure; and the systematic study of the convergence of $\mathcal{I}_n(\rho)$ for a correlated Rician channel has been undertaken by Hachem et al. in [20, 12], etc. The fluctuations of $\mathcal{I}_n$ are important as well, for the computation of the outage probability of a MIMO channel for instance. With the help of the replica method, Taricco [34, 35] provided a closed-form expression for the asymptotic variance of $\mathcal{I}_n$ for the Rician channel.

The purpose of this article is to establish a Central Limit Theorem (CLT) for $\mathcal{I}_n(\rho)$ in the following regime

$$N, n \to \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty \ ,$$

(simply denoted by $N, n \to \infty$ in the sequel) under mild assumptions for matrices $X_n$, $A_n$, $D_n$ and $\tilde{D}_n$.

The contributions of this article are twofold. From a wireless communication perspective, the fluctuations of $\mathcal{I}_n$ are established, regardless of the gaussianity of the entries and the CLT conjectured by Tarrico is fully proved. Also, this article concludes a series of studies devoted to Rician MIMO channels, initiated in [20] where a deterministic equivalent of the mutual information was provided, and continued in [12] where the computation of the ergodic capacity was addressed and an iterative algorithm proposed.

From a mathematical point of view, the study of the fluctuations of $\mathcal{I}_n$ is the first attempt (up to our knowledge) to establish a CLT for a linear statistics of the eigenvalues of a Gram non-centered matrix (so-called signal plus noise model in [10, 11]). It complements (but does not supersede) the CLT established in [21] for a centered Gram matrix with a given variance profile. The fact that matrix $\Sigma_n$ is non-centered ($\mathbb{E}\,\Sigma_n = A_n$) raises specific issues, from a different nature than those addressed in close-by results [1, 4, 21], etc. These issues arise from the presence in the computations of bilinear forms $u_n^* Q_n(z)\, v_n$ where at least one of the vectors $u_n$ or $v_n$ is deterministic. Often, the deterministic vector is related to the columns of matrix $A_n$, and has to be dealt with in such a way that the assumption over the spectral norm of $A_n$ is exploited.

Another important contribution of this paper is to establish the CLT regardless of specific assumptions on the real or complex nature of the underlying random variables. It is in particular *not* assumed that the random variables are gaussian, neither that whenever the random variables $X_{ij}$ are complex, their second moment $\mathbb{E}X_{ij}^2$ is zero; nor is assumed that

the random variables are circular[1]. As we shall see, all these assumptions, if assumed, would have resulted in substantial simplifications. As a reward however, we obtain a variance expression which smoothly depends upon $\mathbb{E}X_{ij}^2$ whose value is 1 in the real case, and zero in the complex case where the real and imaginary parts are not correlated.

Interestingly, the mutual information $\mathcal{I}_n$ has a strong relationship with the Stieltjes transform $f_n(z) = \frac{1}{N}\text{Trace}(\Sigma_n\Sigma_n^* - zI_N)^{-1}$ of $\Sigma_n\Sigma_n^*$:

$$\mathcal{I}_n(\rho) = \log\rho + \int_\rho^\infty \left(\frac{1}{w} - f_n(-w)\right) dw \ .$$

Accordingly, the study of the fluctuations of $\mathcal{I}_n$ is also an important step toward the study of general linear statistics of $\Sigma_n\Sigma_n^*$'s eigenvalues which can be expressed via the Stieltjes transform:

$$\frac{1}{N}\text{Trace } h(\Sigma_n\Sigma_n) = \frac{1}{N}\sum_{i=1}^N h(\lambda_i) = -\frac{1}{2\mathbf{i}\pi}\oint_C h(z)f_n(z)\,dz \ ,$$

for some well-chosen contour $C$ (see for instance [4]).

Fluctuations for particular linear statistics (and general classes of linear statistics) of large random matrices have been widely studied: CLTs for Wigner matrices can be traced back to Girko [13] (see also [16]). Results for this class of matrices have also been obtained by Khorunzhy et al. [27], Boutet de Monvel and Khorunzhy [6], Johansson [24], Sinai and Sochnikov [32], Soshnikov [33], Cabanal-Duvillard [7], Guionnet [17], Anderson and Zeitouni [1], Mingo and Speicher [29], Chatterjee [8], Lytova and Pastur [28], etc. The case of Gram matrices has been studied in Arharov [2], Jonsson [25], Bai and Silverstein [4], Hachem et al. [21], and also in [28, 29, 8]. Fluctuation results dedicated to wireless communication applications have been developed in the centered case ($A_n = 0$) by Debbah and Müller [9] and Tulino and Verdù [37] (based on Bai and Silverstein [4]), Hachem et al. [19] (for gaussian entries) and [21]. Other fluctuation results either based on the replica method or on saddle-point analysis have been developed by Moustakas, Sengupta and coauthors [30, 31], and Tarrico [34, 35].

**Presentation of the results.** We first introduce the fundamental equations needed to express the deterministic approximation of the mutual information and the variance in the CLT.

*Fundamental equations, deterministic equivalents.* We collect here resuls from [20]. The following system of equations

$$\left\{\begin{array}{rcl} \delta_n(z) &=& \frac{1}{n}\text{Tr}\, D_n\left(-z(I_N + \tilde\delta_n(z)D_n) + A_n(I_n + \delta_n(z)\tilde D_n)^{-1}A_n^*\right)^{-1} \\ \tilde\delta_n(z) &=& \frac{1}{n}\text{Tr}\,\tilde D_n\left(-z(I_n + \delta_n(z)\tilde D_n) + A_n^*(I_N + \tilde\delta_n(z)D_n)^{-1}A_n\right)^{-1} \end{array}\right. , \quad z\in\mathbb{C}-\mathbb{R}^+ \tag{1.2}$$

[1]A random variable $X\in\mathbb{C}$ is circular if the distribution of $X$ is equal to the distribution of $\rho X$ for every $\rho\in\mathbb{C}$, $|\rho| = 1$. This assumption is very often relevant in wireless communication and has an important consequence; it implies that all the cross moments $\mathbb{E}|X|^kX^\ell$ ($\ell\geq 1$) are zero.

admits a unique solution $(\delta_n, \tilde{\delta}_n)$ in the class of Stieltjes transforms of nonnegative measures[2] with support in $\mathbb{R}^+$. Matrices $T_n(z)$ and $\tilde{T}_n(z)$ defined by

$$\left\{ \begin{array}{rcl} T_n(z) & = & \Big(-z(I_N + \tilde{\delta}_n(z) D_n) + A_n(I_n + \delta_n \tilde{D}_n)^{-1} A_n^* \Big)^{-1} \\ \tilde{T}_n(z) & = & \Big(-z(I_n + \delta_n(z) \tilde{D}_n) + A_n^*(I_N + \tilde{\delta}_n D_n)^{-1} A_n \Big)^{-1} \end{array} \right. \tag{1.3}$$

are approximations of the resolvent $Q_n(z) = (\Sigma_n \Sigma_n^* - zI_N)^{-1}$ and the co-resolvent $\tilde{Q}_n(z) = (\Sigma_n^* \Sigma_n - zI_N)^{-1}$ in the sense that ($\xrightarrow{a.s.}$ stands for the almost sure convergence):

$$\frac{1}{N} \mathrm{Tr}\, (Q_n(z) - T_n(z)) \xrightarrow[N,n\to\infty]{a.s.} 0 \ ,$$

which readily gives a deterministic approximation of the Stieltjes transform $N^{-1}\mathrm{Tr}\, Q_n(z)$ of the spectral measure of $\Sigma_n \Sigma_n^*$ in terms of $T_n$ (and similarly for $\tilde{Q}_n$ and $\tilde{T}_n$). Also proved in [22] is the convergence of bilinear forms

$$u_n^*(Q_n(z) - T_n(z)) v_n \xrightarrow[N,n\to\infty]{a.s.} 0 \ , \tag{1.4}$$

where $(u_n)$ and $(v_n)$ are sequences of $N \times 1$ deterministic vectors with uniformily bounded euclidian norm, which complements the picture of $T_n$ approximating $Q_n$.

Matrices $T_n = (t_{ij}\ ; 1 \le i,j \le N)$ and $\tilde{T}_n = (\tilde{t}_{ij}\ ; 1 \le i,j \le n)$ will play a fundamental role in the sequel and enable us to express a deterministic equivalent to $\mathbb{E}\mathcal{I}_n(\rho)$. Define $V_n(\rho)$ by:

$$V_n(\rho) = \frac{1}{N} \log\det \Big( \rho(I_N + \tilde{\delta}_n D_n) I_N + A_n(I_n + \delta_n \tilde{D}_n)^{-1} A_n^* \Big) + \frac{1}{N} \log(I_n + \delta_n \tilde{D}_n) - \frac{\rho n}{N} \delta_n \tilde{\delta}_n \ , \tag{1.5}$$

where $\delta_n$ and $\tilde{\delta}_n$ are evaluated at $z = -\rho$. Then the difference $\mathbb{E}\,\mathcal{I}_n(\rho) - V_n(\rho)$ goes to zero as $N, n \to \infty$.

In order to study the fluctuations $N(\mathcal{I}_n(\rho) - V_n(\rho))$ and to establish a CLT, we study separately the quantity $N(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho))$ from which the fluctuations arise and the quantity $N(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho))$ which yields a bias.

*The fluctuations.* In every case where the fluctuations of the mutual information have been studied, the variance of $N\left(\mathcal{I}_n(\rho) - V_n(\rho)\right)$ always proved to take a (somehow unexpected) remarkably simple closed-form expression (see for instance [30, 35, 37] and in a more mathematical flavour [19, 21]). The same phenomenon again occurs for the matrix model $\Sigma_n$ under consideration. Drop the subscripts $N, n$ and let

$$\gamma = \frac{1}{n} \mathrm{Tr}\, DTDT \ , \ \tilde{\gamma} = \frac{1}{n} \mathrm{Tr}\, \tilde{D}\tilde{T}\tilde{D}\tilde{T} \ , \ \underline{\gamma} = \frac{1}{n} \mathrm{Tr}\, DTD\bar{T} \ , \ \underline{\tilde{\gamma}} = \frac{1}{n} \mathrm{Tr}\, \tilde{D}\tilde{T}\tilde{D}\bar{\tilde{T}} \ , \tag{1.6}$$

where $\bar{M}$ stands for the (elementwise) conjugate of matrix $M$. Let

$$\vartheta = \mathbb{E}(X_{ij})^2 \quad \text{and} \quad \kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\vartheta|^2 \ .$$

[2] In fact, $\delta_n$ is the Stieltjes transform of a measure with total mass equal to $n^{-1}\mathrm{Tr} D_n$ while $\tilde{\delta}_n$ is the Stieltjes transform of a measure with total mass equal to $n^{-1}\mathrm{Tr}\tilde{D}_n$.

Let

$$\Theta_n = -\log\left(\left(1-\frac{1}{n}\mathrm{Tr}\, D^{\frac12}TA(I+\delta\tilde D)^{-1}\tilde D(I+\delta\tilde D)^{-1}A^*TD^{\frac12}\right)^2-\rho^2\gamma\tilde\gamma\right)$$
$$-\log\left(\left|1-\vartheta\frac{1}{n}\mathrm{Tr}\, D^{\frac12}\bar T\bar A(I+\delta\tilde D)^{-1}\tilde D(I+\delta\tilde D)^{-1}A^*TD^{\frac12}\right|^2-|\vartheta|^2\rho^2\underline{\gamma}\,\underline{\tilde\gamma}\right)$$
$$+\kappa\frac{\rho^2}{n^2}\sum_i d_i^2t_{ii}^2\sum_j\tilde d_j^2\tilde t_{jj}^2\ , \quad (1.7)$$

where $d_i=[D_n]_{ii}$, $\tilde d_j=[\tilde D_n]_{jj}$, and all the needed quantities are evaluated at $z=-\rho$. The CLT then expresses as:

$$\frac{N}{\sqrt{\Theta_n}}\left(\mathcal{I}_n-\mathbb{E}\mathcal{I}_n\right)\xrightarrow[N,n\to\infty]{\mathcal{D}}\mathcal{N}(0,1)\ ,$$

where $\xrightarrow{\mathcal{D}}$ stands for the convergence in distribution. Although complicated at first sight, variance $\Theta_n$ encompasses the case of standard real random variables ($\vartheta=1$), standard complex random variables ($\vartheta=0$) and all the intermediate cases $0<|\vartheta|<1$. Moreover, $\Theta_n$ often takes simpler forms if the variables are gaussian, real, etc. (see for instance Remark 2.2).

*The bias.* In the case where the entries are complex gaussian and $\kappa=0$, it has already been proved in [12] that $\mathbb{E}\,\mathcal{I}_n(\rho)-V_n(\rho)=\mathcal{O}(n^{-2})$. In the case where matrices $D_n$ and $\tilde D_n$ are equal to the identity, we establish that there exists a deterministic quantity $\mathcal{B}_n(\rho)$ (described in Theorem 2.3) such that:

$$N\left(\mathbb{E}\mathcal{I}_n(\rho)-V_n(\rho)\right)-\mathcal{B}_n(\rho)\xrightarrow[N,n\to\infty]{}0\ .$$

**Outline of the article.** In Section 2, we provide the main assumptions and state the main results of the paper: Definition of the variance $\Theta_n$ and asymptotic fluctuations of $N\left(\mathcal{I}_n(\rho)-\mathbb{E}\mathcal{I}_n(\rho)\right)$ (Theorem 2.2), asymptotic bias of $N\left(\mathbb{E}\mathcal{I}_n(\rho)-V_n(\rho)\right)$ (Theorem 2.3). Notations, important estimates and classical results are provided in Section 3. Sections 4, 5 and 6 are devoted to the proof of Theorem 2.2. In Section 4, the general framework of the proof is exposed; in Section 5, the central part of the CLT and of the identification of the variance are established; remaining proofs are provided in Section 6. Finally, proof of Theorem 2.3 (bias) is provided in Section 7.

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## 2. The Central Limit Theorem for $\mathcal{I}_n(\rho)$

2.1. **Notations, assumptions and first-order results.** The indicator function of the set $\mathcal{A}$ will be denoted by $\mathbf{1}_{\mathcal{A}}(x)$, its cardinality by $\#\mathcal{A}$. If $z\in\mathbb{C}$, then $\bar z$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively stand for its complex conjugate, real and imaginary part; denote by $\mathbf{i}=\sqrt{-1}$. As usual, $\mathbb{R}^+=\{x\in\mathbb{R}\ :\ x\geq 0\}$, $\mathbb{C}^+=\{z\in\mathbb{C}\ :\ \mathrm{Im}(z)>0\}$. Denote by $\xrightarrow{\mathcal{P}}$ the convergence in probability of random variables and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution of probability measures. Denote by $\mathrm{diag}(a_i;\,1\leq i\leq k)$ the $k\times k$ diagonal matrix whose diagonal entries are the $a_i$'s. Element $(i,j)$ of matrix $M$ will be either denoted $m_{ij}$ or $[M]_{ij}$ depending on the

notational context. if $M$ is a $n \times n$ square matrix, $\mathrm{diag}(M) = \mathrm{diag}(m_{ii}; 1 \leq i \leq n)$. Denote by $M^T$ the matrix transpose of $M$, by $M^*$ its Hermitian adjoint, by $\bar{M}$ the (elementwise) conjugate of matrix $M$, by $\mathrm{Tr}(M)$ its trace and $\det(M)$ its determinant (if $M$ is square). When dealing with vectors, $\|\cdot\|$ will refer to the Euclidean norm, and $\|\cdot\|_\infty$, to the max (or $\ell_\infty$) norm. In the case of matrices, $\|\cdot\|$ will refer to the spectral norm. If $(u_n)$ is a sequence of real numbers, then $u_n = \mathcal{O}(v_n)$ stands for $|u_n| \leq K|v_n|$ where constant $K$ does not depend on $n$.

Recall that

$$\Sigma_n = \frac{1}{\sqrt{n}} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n \ , \tag{2.1}$$

denote $D_n = \mathrm{diag}(d_i,\ 1 \leq i \leq N)$ and $\tilde{D}_n = \mathrm{diag}(\tilde{d}_j,\ 1 \leq j \leq n)$. When no confusion can occur, we shall often drop subscripts and superscripts $n$ for readability. Recall also that the asymptotic regime of interest is:

$$N, n \to \infty \quad \text{and} \quad 0 \ < \ \liminf \frac{N}{n} \ \leq \ \limsup \frac{N}{n} \ < \infty \ ,$$

and will be simply denoted by $N, n \to \infty$ in the sequel. We can assume without loss of generality that there exist nonnegative real numbers $\boldsymbol{\ell}^-$ and $\boldsymbol{\ell}^+$ such that:

$$0 \ < \ \boldsymbol{\ell}^- \ \leq \ \frac{N}{n} \ \leq \ \boldsymbol{\ell}^+ \ < \ \infty \quad \text{as} \quad N, n \to \infty \ . \tag{2.2}$$

**Assumption A-1.** *The random variables* $(X_{ij}^n \ ; \ 1 \leq i \leq N, 1 \leq j \leq n \, , \, n \geq 1)$ *are complex, independent and identically distributed. They satisfy*

$$\mathbb{E}X_{11}^n = 0, \quad \mathbb{E}|X_{11}^n|^2 = 1 \quad \text{and} \quad \mathbb{E}|X_{11}^n|^{16} < \infty \ .$$

Associated to these variables are the quantities:

$$\vartheta = \mathbb{E}(X_{11})^2 \quad \text{and} \quad \kappa = \mathbb{E}|X_{11}|^4 - 2 - |\vartheta|^2 \ .$$

*Remark* 2.1. (Gaussian distributions) If $X_{11}$ is a standard complex or real Gaussian random variable, then $\kappa = 0$. More precisely, in the complex case, $\mathrm{Re}(X_{11})$ and $\mathrm{Im}(X_{11})$ are independent real Gaussian random variables, then $\vartheta = \kappa = 0$; in the real case, then $\vartheta = 1$ while $\kappa = 0$.

**Assumption A-2.** *The family of deterministic $N \times n$ complex matrices $(A_n, n \geq 1)$ is uniformily bounded for the spectral norm:*

$$\boldsymbol{a}_{\max} = \sup_{n \geq 1} \|A_n\| < \infty \ .$$

**Assumption A-3.** *The families of real deterministic $N \times N$ and $n \times n$ matrices $(D_n)$ and $(\tilde{D}_n)$ are diagonal with non-negative diagonal elements, and are bounded for the spectral norm as $N, n \to \infty$:*

$$\boldsymbol{d}_{\max} = \sup_{n \geq 1} \|D_n\| < \infty \quad \textit{and} \quad \tilde{\boldsymbol{d}}_{\max} = \sup_{n \geq 1} \|\tilde{D}_n\| < \infty \ .$$

*Moreover,*

$$\boldsymbol{d}_{\min} = \inf_n \frac{1}{n} \mathrm{Tr}\, D_n > 0 \quad \textit{and} \quad \tilde{\boldsymbol{d}}_{\min} = \inf_n \frac{1}{n} \mathrm{Tr}\, \tilde{D}_n > 0 \ .$$

**Theorem 2.1** (First order results - [20, 12])**.** *Consider the $N \times n$ matrix $\Sigma_n$ given by* (2.1) *and assume that* **A-1**, **A-2** *and* **A-3** *hold true. Then, the system* (1.2) *admits a unique solution $(\delta_n, \tilde{\delta}_n)$ in the class of Stieltjes transforms of nonnegative measures.*

2.2. **The Central Limit Theorem.** In this section, we state the CLT then provide the asymptotic bias in some particular cases.

**Theorem 2.2** (The CLT)**.** *Consider the $N \times n$ matrix $\Sigma_n$ given by* (2.1) *and assume that* **A-1**, **A-2** *and* **A-3** *hold true. Recall the definitions of $\delta$ and $\tilde{\delta}$ given by* (1.2)*, $T$ and $\tilde{T}$ given by* (1.3)*, $\gamma$, $\tilde{\gamma}$, $\underline{\gamma}$ and $\underline{\tilde{\gamma}}$ given by* (1.6)*. Let $\rho > 0$. All the considered quantities are evaluated at $z = -\rho$. Define $\Delta_n$ and $\underline{\Delta}_n$ as*

$$\Delta_n = \left(1 - \frac{1}{n} \operatorname{Tr} D^{\frac{1}{2}} T A (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}}\right)^2 - \rho^2 \gamma \tilde{\gamma}$$

*and*

$$\underline{\Delta}_n = \left|1 - \vartheta \frac{1}{n} \operatorname{Tr} D^{\frac{1}{2}} \bar{T} \bar{A} (I + \delta \tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}}\right|^2 - |\vartheta|^2 \rho^2 \underline{\gamma}\, \underline{\tilde{\gamma}}.$$

*Then the real numbers*

$$\Theta_n = -\log \Delta_n - \log \underline{\Delta}_n + \kappa \frac{\rho^2}{n^2} \sum_{i=1}^{N} d_i^2 t_{ii}^2 \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \tag{2.3}$$

*are well-defined and satisfy:*

$$0 \quad < \quad \liminf_n \Theta_n \quad \leq \quad \limsup_n \Theta_n \quad < \quad \infty \tag{2.4}$$

*as $N, n \to \infty$. Let*

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log \det \left(\Sigma_n \Sigma_n^* + \rho I_N\right) \ ,$$

*then the following convergence holds true:*

$$\frac{N}{\sqrt{\Theta_n}} \left(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho)\right) \xrightarrow[N,n\to\infty]{\mathcal{D}} \mathcal{N}(0,1) \ .$$

*Remark* 2.2. (Simpler forms for the variance) We consider here special cases where the variance $\Theta_n$ takes a simpler form.

(1) The standard complex Gaussian case. Assume that the $X_{ij}$'s are standard complex gaussian random variables, i.e. that both the real and imaginary parts of $X_{ij}$ are independent real gaussian random variables, each with variance $1/2$. In this case, $\vartheta = \kappa = 0$ and $\Theta_n$ is equal to the first term of the right hand side (r.h.s.) of (1.7) - we in particular recover the variance formula given in [35].
(2) The standard real case. Assume that the $X_{ij}$'s are standard real random variables, assume also that $A$ has real entries. Then the two first terms of the r.h.s. of (1.7) are equal.
(3) The 'signal plus noise' model. In this case, $D_n = I_N$ and $\tilde{D}_n = I_n$, which already yields simplifications in (1.7). In the case where $\vartheta = 0$, the variance writes:

$$\Theta_n = -\log\left(\left(1 - \tfrac{\operatorname{Tr} T A A^* T}{n(1+\delta)^2}\right)^2 - \rho^2 \gamma \tilde{\gamma}\right) + \frac{\kappa \rho^2}{n^2} \sum_i d_i^2 t_{ii}^2 \sum_j \tilde{d}_j^2 \tilde{t}_{jj}^2 \ .$$

As one may easily check, the first term of the variance only depends upon the spectrum of $AA^*$. The second term however also depends on the eigenvectors of $AA^*$ (see for instance [26]).

The asymptotic bias is described in the following theorem in two important cases.

**Theorem 2.3** (The bias)**.** *Assume that the setting of Theorem 2.2 holds true. Recall that:* $\kappa = \mathbb{E}|X_{ij}|^4 - 2 - |\mathbb{E}X_{ij}^2|^2$.

(i) *If the random variables* $(X_{ij}^n; i, j, n)$ *are moreover complex gaussian with* $\mathrm{Re}(X_{ij}^n)$ *and* $\mathrm{Im}(X_{ij}^n)$ *independent, both with distribution* $\mathcal{N}(0, 1/2)$*, then:*

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) = \mathcal{O}\left(\frac{1}{N}\right) .$$

(ii) *If* $D_n = I_N$ *and* $\tilde{D}_n = I_n$ *(signal plus noise model), let the quantities* $\delta, \tilde{\delta}, T, \tilde{T}, S$ *and* $\tilde{S}$ *be evaluated at* $z = -\omega$ *and consider:*

$$\mathcal{B}_n(\omega) = \kappa \frac{A(\omega)}{B(\omega)} , \tag{2.5}$$

*where*

$$\begin{aligned} A(\omega) &= \omega^2(1+\tilde{\delta})\frac{1}{n}\,\mathrm{Tr}\,\tilde{S}^2\frac{1}{n}\,\mathrm{Tr}\,ST^2 + \omega^2(1+\delta)\frac{1}{n}\,\mathrm{Tr}\,S^2\frac{1}{n}\,\mathrm{Tr}\,\tilde{S}\tilde{T}^2 - \omega\frac{1}{n}\,\mathrm{Tr}\,S^2\frac{1}{n}\,\mathrm{Tr}\,\tilde{S}^2 , \\ B(\omega) &= 1 + \omega(1+\delta)\tilde{\gamma} + \omega(1+\tilde{\delta})\gamma . \end{aligned}$$

*Then,*

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) - \int_\rho^\infty \mathcal{B}_n(\omega)d\omega \xrightarrow[N,n\to\infty]{} 0 .$$

Proof of Theorem 2.3-(i) can be found in [12, Theorem 2]; proof of Theorem 2.3-(ii) is postponed to Section 7.

## 3. Notations and classical results

3.1. **Further notations.** Denote by $Y$ the $N \times n$ matrix $n^{-1/2}D^{1/2}X\tilde{D}^{1/2}$; by $(\eta_j)$, $(a_j)$ and $(y_j)$ the columns of matrices $\Sigma$, $A$ and $Y$. Denote by $\Sigma_j$, $A_j$, $Y_j$ and $\tilde{D}_j$, the matrices $\Sigma$, $A$, $Y$ and $\tilde{D}$ where column $j$ has been removed. The associated resolvent is $Q_j(z) = (\Sigma_j\Sigma_j^* - zI_N)^{-1}$. We also denote by $A_{1:j}$ and $\Sigma_{1:j}$ the $N \times j$ matrices $A_{1:j} = [a_1, \cdots, a_j]$ and $\Sigma_{1:j} = [\eta_1, \cdots, \eta_j]$. Denote by $\mathbb{E}_j$ the conditional expectation with respect to the $\sigma$-field $\mathcal{F}_j$ generated by the vectors $(y_\ell, 1 \le \ell \le j)$. By convention, $\mathbb{E}_0 = \mathbb{E}$.

We introduce here intermediate quantities of constant use in the rest of the paper. Let $1 \leq j \leq n$, denote by:

$$\tilde{b}_j(z) = \frac{1}{-z\left(1 + a_j^* Q_j(z) a_j + \frac{\tilde{d}_j}{n} \operatorname{Tr} D Q_j(z)\right)} , \tag{3.1}$$

$$\tilde{c}_j(z) = \frac{1}{-z\left(1 + a_j^* \mathbb{E} Q_j(z) a_j + \frac{\tilde{d}_j}{n} \operatorname{Tr} D \mathbb{E} Q(z)\right)} , \tag{3.2}$$

$$\begin{aligned} e_j(z) &= \eta_j^* Q_j(z) \eta_j - \left(\frac{\tilde{d}_j}{n} \operatorname{Tr} D Q_j(z) + a_j^* Q_j(z) a_j\right) , \\ &= \left(y_j^* Q_j(z) y_j - \frac{\tilde{d}_j}{n} \operatorname{Tr} D Q_j(z)\right) + a_j^* Q_j(z) y_j + y_j^* Q_j(z) a_j , \end{aligned} \tag{3.3}$$

$$\alpha(z) = \frac{1}{n} \operatorname{Tr} D \mathbb{E} Q(z) , \qquad \tilde{\alpha}(z) = \frac{1}{n} \operatorname{Tr} \tilde{D} \mathbb{E} \tilde{Q}(z) , \tag{3.4}$$

$$C(z) = \left(-z\left(I_N + \tilde{\alpha}(z) D\right) + A\left(I_n + \alpha(z)\tilde{D}\right)^{-1} A^*\right)^{-1} . \tag{3.5}$$

Using the well-known characterization of Stieltjes transforms (see for instance [20, Proposition 2.2-(2)]), one can easily prove that $\tilde{b}_j$ is the Stieltjes transform of a probability measure. In particular $|\tilde{b}_j(z)| \leq (\operatorname{dist}(z, \mathbb{R}^+))^{-1}$ for $z \in \mathbb{C} - \mathbb{R}^+$. The same estimate holds true for $\tilde{c}_j$.

3.2. **Important identities.** Recall the following classical identities.

- The inverse of a partitionned matrix (see for instance [23, Section 0.7.3]):

$$\text{If} \quad A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} , \qquad \text{then} \quad \left(A^{-1}\right)_{11} = \left(a_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} . \tag{3.6}$$

- The inverse of a perturbated matrix (see [23, Section 0.7.4]):

$$(A + XRY)^{-1} = A^{-1} - A^{-1} X \left(R^{-1} + Y A^{-1} X\right)^{-1} Y A^{-1} . \tag{3.7}$$

*Identities involving the resolvents.* The following identity expresses the diagonal elements of the co-resolvent; the two following ones are obtained from (3.7).

$$\tilde{q}_{jj}(z) = -\frac{1}{z(1 + \eta_j^* Q_j(z) \eta_j)} , \tag{3.8}$$

$$\begin{aligned} Q(z) &= Q_j(z) - \frac{Q_j(z) \eta_j \eta_j^* Q_j(z)}{1 + \eta_j^* Q_j(z) \eta_j} \\ &= Q_j(z) + z \tilde{q}_{jj}(z) Q_j(z) \eta_j \eta_j^* Q_j(z) \end{aligned} \tag{3.9}$$

$$Q_j(z) = Q(z) + \frac{Q(z) \eta_j \eta_j^* Q(z)}{1 - \eta_j^* Q(z) \eta_j} \tag{3.10}$$

$$1 + \eta_j^* Q_j(z) \eta_j = \frac{1}{1 - \eta_j^* Q(z) \eta_j} \tag{3.11}$$

Note that:

$$\tilde{q}_{jj} = \tilde{b}_j + z \tilde{q}_{jj} \tilde{b}_j e_j . \tag{3.12}$$

A useful consequence of (3.9) is:

$$\eta_j^* Q(z) = \frac{\eta_j^* Q_j(z)}{1+\eta_j^* Q_j(z)\eta_j} = -z\tilde{q}_{jj}(z)\eta_j^* Q_j(z)\ . \tag{3.13}$$

*Identities involving the deterministic equivalents $T$ and $\tilde{T}$.* Define the $N \times N$ matrix $\mathcal{T}_j$ as

$$\mathcal{T}_j = \left(-z(I_N + \tilde{\delta}D) + A_j(I_{n-1} + \delta\tilde{D}_j)^{-1}A_j^*\right)^{-1}\ , \tag{3.14}$$

where $\delta$ and $\tilde{\delta}$ are defined in (1.2). Notice that matrix $\mathcal{T}_j$ is not obtained in general by solving the analogue of system (1.3) where $A$ is replaced with $A_j$. This matrix naturally pops up when expressing the diagonal elements of $\tilde{T}$. Indeed after some algebra (see for instance [18, Appendix B], we obtain:

$$\tilde{t}_{jj}(z) = -\frac{1}{z\left(1 + a_j^*\mathcal{T}_j(z)a_j + \tilde{d}_j\delta(z)\right)}\ . \tag{3.15}$$

Let $b$ be a given $N \times 1$ vector. The following identity holds true:

$$-z\tilde{t}_{\ell\ell}(z)a_{\square}^*\mathcal{T}_\ell(z)b = \frac{a_\ell^* T(z)b}{1+\tilde{d}_\ell\delta(z)} \tag{3.16}$$

Thanks to (3.7), we also have

$$\tilde{T}(z) = -z^{-1}(I+\delta(z)\tilde{D})^{-1} + z^{-1}(I+\delta(z)\tilde{D})^{-1}A^*T(z)A(I+\delta(z)\tilde{D})^{-1}. \tag{3.17}$$

3.3. **Important estimates.** We gather in this section matrix estimates which will be of constant use in the sequel.

Let $A$ and $B$ be two square matrices. Then

$$|\mathrm{Tr}(AB)| \le \sqrt{\mathrm{Tr}(AA^*)}\sqrt{\mathrm{Tr}(BB^*)} \tag{3.18}$$

When $B$ is Hermitian non negative, then a consequence of Von Neumann's trace theorem is

$$|\,\mathrm{Tr}(AB)| \quad \le \quad \|A\|\ \mathrm{Tr}\,B\ . \tag{3.19}$$

The following lemma gives an estimate for a rank-one perturbation of the resolvent.

**Lemma 3.1.** *The resolvents $Q$ and the perturbed resolvent $Q_j$ satisfy:*

$$|\mathrm{Tr}\ A\,(Q-Q_j)| \le \frac{\|A\|}{\mathrm{Im}(z)}$$

*for any $N \times N$ matrix $A$ with bounded spectral norm.*

The following results describe the asymptotic behaviour of quadratic forms based on the resolvent.

**Lemma 3.2** (Bai and Silverstein, Lemma 2.7 in [3])**.** *Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ be a $n \times 1$ vector where the $x_i$ are centered i.i.d. complex random variables with unit variance. Let $M$ be a $n \times n$ deterministic complex matrix. Then for any $p \ge 2$, there exists a constant $K_p$ for which*

$$\mathbb{E}|\boldsymbol{x}^* M\boldsymbol{x} - \mathrm{Tr}\,M|^p \le K_p\left(\left(\mathbb{E}|x_1|^4\,\mathrm{Tr}\,MM^*\right)^{p/2} + \mathbb{E}|x_1|^{2p}\,\mathrm{Tr}(MM^*)^{p/2}\right)$$

*Remark* 3.1. There are some important consequences of the previous lemma. Let $(M_n)$ be a sequence of $n \times n$ deterministic matrices with bounded spectral norm and $(\boldsymbol{x}_n)$ be a sequence of random vectors as in the statement of Lemma 3.2. Then for any $p \geq 2$,

$$\max\left(\mathbb{E}\left|\frac{\boldsymbol{x}_n^* M_n \boldsymbol{x}_n}{n} - \frac{\operatorname{Tr} M_n}{n}\right|^p, \mathbb{E}|e_j|^p\right) \leq \frac{K}{n^{p/2}} \tag{3.20}$$

where $e_j$ is given by (3.3) (the estimate $\mathbb{E}|e_j|^p = \mathcal{O}(n^{-p/2})$ is proved in [18, Appendix B]).

We gather in the following theorem some of the results proved in [22].

**Theorem 3.3.** *Assume that the setting of Theorem 2.2 holds true. Let $(u_n)$ and $(v_n)$ be two sequences of deterministic complex $N \times 1$ vectors bounded in the Euclidian norm:*

$$\sup_{n \geq 1} \max\left(\|u_n\|, \|v_n\|\right) < \infty \ .$$

*Then,*

(1) *For every $z \in \mathbb{C} - \mathbb{R}^+$, there exist nonnegative constants $K_1(z), K_2(z) < \infty$, which do not depend on $N, n$, such that:*

$$\sum_{j=1}^n \mathbb{E}|u_n^* Q_j a_j|^2 \ \leq \ K_1(z) \quad \text{and} \quad \mathbb{E}\left(\sum_{j=1}^n \mathbb{E}_j|u_n^* Q_j a_j|^2\right)^2 \ \leq \ K_2(z) \ .$$

(2) *For every $z \in \mathbb{C} - \mathbb{R}^+$ and for $p \geq 1$, there exists a nonnegative constant $K(z) < \infty$, which does not depend on $N, n$, such that:*

$$\mathbb{E}\left|u_n^*(Q(z) - T(z))v_n\right|^{2p} \leq \frac{K(z)}{n^p} \ .$$

(3) *For every $\rho \in \mathbb{R}_+^*$ and every sequence of deterministic matrices $(U_n)$ with bounded spectral norms, we have:*

$$\left|\frac{1}{n}\operatorname{Tr} U(T(-\rho) - \mathbb{E}Q(-\rho))\right| \leq \frac{K}{n} \ .$$

(4) *For every $z \in \mathbb{C} - \mathbb{R}^+$ and for $p \geq 1$, there exists a nonnegative constant $K(z) < \infty$, which does not depend on $N, n$, such that:*

$$\mathbb{E}\left|u_n^*(Q_j(z) - \mathcal{T}_j(z))v_n\right|^{2p} \leq \frac{K(z)}{n^p} \ . \tag{3.21}$$

Items (3) and (4) of Theorem 3.3 are not direct consequences of results in [22]; therefore elements of proof are provided in [18, Appendix B].

The following results stem from lemma 3.2 and theorem 3.3 and will be of constant use in the sequel. Recalling (3.1), (3.3), and (3.8), it is clear that $\tilde{q}_{jj} = \tilde{b}_j + z\tilde{q}_{jj}\tilde{b}_j e_j$. Using (3.20) and bounding $|q_{jj}|$ and $|\tilde{b}_j|$ (see for instance [20]), we have

$$\mathbb{E}\left|\tilde{q}_{jj} - \tilde{b}_j\right|^2 \leq \frac{K}{n}. \tag{3.22}$$

Of course, the counterpart of Theorem 3.3 for the co-resolvent $\tilde{Q}$ and matrix $\tilde{T}$ holds true. In particular, taking the vectors $u_n$ and $v_n$ as the $j$th canonical vector of $\mathbb{C}^n$ yields the following estimate:

$$\mathbb{E}\left|\tilde{q}_{jj} - \tilde{t}_{jj}\right|^2 \leq \frac{K}{n}. \tag{3.23}$$

The following two lemmas, proved in Appendices A.1 and A.2, provide some important bounds:

**Lemma 3.4.** *Assume that the setting of theorem 2.2 holds true. Then, the following quantities satisfy:*

$$\begin{array}{rcccl}
\delta_{\min} \stackrel{\triangle}{=} \frac{\boldsymbol{d}_{\min}}{\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}} & \leq & \delta_n & \leq & \delta_{\max} \stackrel{\triangle}{=} \frac{\boldsymbol{\ell}^+\boldsymbol{d}_{\max}}{\rho} \ , \\
\tilde{\delta}_{\min} \stackrel{\triangle}{=} \frac{\tilde{\boldsymbol{d}}_{\min}}{\rho+\boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}} & \leq & \tilde{\delta}_n & \leq & \tilde{\delta}_{\max} \stackrel{\triangle}{=} \frac{\tilde{\boldsymbol{d}}_{\max}}{\rho} \ , \\
\frac{\boldsymbol{d}^2_{\min}}{\boldsymbol{\ell}^+(\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max})^2} & \leq & \gamma_n & \leq & \frac{\boldsymbol{\ell}^+\boldsymbol{d}^2_{\max}}{\rho^2} \ , \\
\frac{\tilde{\boldsymbol{d}}^2_{\min}}{\left(\rho+\boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}\right)^2} & \leq & \tilde{\gamma}_n & \leq & \frac{\tilde{\boldsymbol{d}}^2_{\max}}{\rho^2} \\
\frac{\boldsymbol{d}^2_{\min}}{\boldsymbol{\ell}^+(\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max})^2} & \leq & n^{-1}\sum_{i=1}^N d_i^2 t_{ii}^2 & \leq & \frac{\boldsymbol{\ell}^+\boldsymbol{d}^2_{\max}}{\rho^2} \ , \\
\frac{\tilde{\boldsymbol{d}}^2_{\min}}{\left(\rho+\boldsymbol{\ell}^+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}^2_{\max}\right)^2} & \leq & n^{-1}\sum_{j=1}^n \tilde{d}_j^2 \tilde{t}_{jj}^2 & \leq & \frac{\tilde{\boldsymbol{d}}^2_{\max}}{\rho^2} \ .
\end{array}$$

**Lemma 3.5.** *Assume that the setting of theorem 2.2 holds true. Then*

$$\sup_n \frac{1}{n} \operatorname{Tr} D^{1/2} T A (I+\delta\tilde{D})^{-2} \tilde{D} A^* T D^{\frac{1}{2}} \ < \ 1.$$

*As a consequence, the sequence $(\Delta_n)$ as defined in Theorem 2.2 satisfies:*

$$\liminf_n \Delta_n > 0 \ .$$

3.4. **Other important results.** The main result we shall rely on to establish the Central Limit theorem is the following CLT for martingales:

**Theorem 3.6** (CLT for martingales, Th. 35.12 in [5])**.** *Let $\gamma_1^{(n)}, \gamma_2^{(n)}, \ldots, \gamma_n^{(n)}$ be a martingale difference sequence with respect to the increasing filtration $\mathcal{F}_1^{(n)}, \ldots, \mathcal{F}_n^{(n)}$. Assume that there exists a sequence of real positive numbers $\Upsilon_n^2$ such that*

$$\frac{1}{\Upsilon_n^2} \sum_{j=1}^n \mathbb{E}_{j-1} {\gamma_j^{(n)}}^2 \xrightarrow[n\to\infty]{\mathcal{P}} 1 \ . \tag{3.24}$$

*Assume further that the Lyapounov condition (*[5, Section 27]*) holds true:*

$$\exists \delta > 0, \quad \frac{1}{\Upsilon_n^{2(1+\delta)}} \sum_{j=1}^n \mathbb{E} \left| \gamma_j^{(n)} \right|^{2+\delta} \xrightarrow[n\to\infty]{} 0 \ .$$

*Then $\Upsilon_n^{-1} \sum_{j=1}^n \gamma_j^{(n)}$ converges in distribution to $\mathcal{N}(0,1)$.*

*Remark* 3.2. Note that if moreover $\liminf_n \Upsilon_n^2 > 0$, it is sufficient to prove:

$$\sum_{j=1}^n \mathbb{E}_{j-1} {\gamma_j^{(n)}}^2 - \Upsilon_n^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0 \ , \tag{3.25}$$

instead of (3.24).

We now state a covariance identity (the proof of which is straightforward and therefore omitted) for quadratic forms based on non-centered vectors. This identity explains to some extent the various terms obtained in the variance.

Let $\boldsymbol{x} = (x_1, \cdots, x_n)^T$ be a $n \times 1$ vector where the $x_i$ are centered i.i.d. complex random variables with unit variance. Let $\boldsymbol{y} = n^{-1/2} D^{1/2} \boldsymbol{x}$ where $D$ is a $n \times n$ diagonal nonnegative deterministic matrix. Let $M = (m_{ij})$ and $P = (p_{ij})$ be $n \times n$ deterministic complex matrices and let $\boldsymbol{u}$ be a $n \times 1$ deterministic vector.

If $M$ is a $n \times n$ matrix, vdiag$(M)$ stands for the $n \times 1$ vector $[M_{11}, \cdots, M_{nn}]^T$.

Denote by $\Upsilon(M)$ the random variable:

$$\Upsilon(M) = (\boldsymbol{y} + \boldsymbol{u})^* M (\boldsymbol{y} + \boldsymbol{u}) \ .$$

Then $\mathbb{E}\Upsilon(M) = \frac{1}{n} \operatorname{Tr} DM + \boldsymbol{u}^* M \boldsymbol{u}$ and the covariance between $\Upsilon(M)$ nad $\Upsilon(P)$ writes:

$$\begin{aligned}
&\mathbb{E}\left[ (\Upsilon(M) - \mathbb{E}\Upsilon(M)) (\Upsilon(P) - \mathbb{E}\Upsilon(P)) \right] \\
&= \frac{1}{n^2} \operatorname{Tr}(MDPD) + \frac{1}{n} \left( \boldsymbol{u}^* MDP\boldsymbol{u} + \boldsymbol{u}^* PDM\boldsymbol{u} \right) \\
&\quad + \frac{\mathbb{E}[x_1^2]}{n^2} \operatorname{Tr}(MDP^T D) + \frac{\mathbb{E}[x_1^2]}{n} \boldsymbol{u}^* PDM^T \bar{\boldsymbol{u}} + \frac{\mathbb{E}[\bar{x}_1^2]}{n} \boldsymbol{u}^T M^T DP\boldsymbol{u} \\
&\quad + \frac{\mathbb{E}[|x_1|^2 x_1]}{n^{3/2}} \left( \boldsymbol{u}^* PD^{3/2} \text{vdiag}(M) + \boldsymbol{u}^* MD^{3/2} \text{vdiag}(P) \right) \\
&\quad + \frac{\mathbb{E}[|x_1|^2 \bar{x}_1]}{n^{3/2}} \left( \text{vdiag}(P)^T D^{3/2} M\boldsymbol{u} + \text{vdiag}(M)^T D^{3/2} P\boldsymbol{u} \right) \\
&\quad + \frac{\kappa}{n^2} \sum_{i=1}^n d_{ii}^2 m_{ii} p_{ii} \ , \qquad (3.26)
\end{aligned}$$

where $\kappa = \mathbb{E}|x_1|^4 - 2 - |\mathbb{E}x_1^2|^2$.

*Remark* 3.3. Identity (3.26) is the cornerstone for the proof of the CLT; it is the counterpart of Identity (1.15) in [4]. The complexity of Identity (3.26) with respect to [4, Identity (1.15)] lies in 8 extra terms and stems from two elements:

(1) The fact that matrix $\Sigma$ is non-centered.
(2) The fact that the random variables $X_{ij}$'s are either real and complex with no further assumption (in particular, $\mathbb{E}X_{ij}^2 \neq 0$ a priori in the complex case).

It is this identity which induces to a large extent all the computations in the present article.

## 4. Proof of Theorem 2.2 (part I)

*Decomposition of $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$, Cumulant and cross-moments terms in the variance*

### 4.1. Decomposition of $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$ as a sum of martingale differences.

Denote by

$$\Gamma_j = \frac{\eta_j^* Q_j \eta_j - \left( \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j + a_j^* Q_j a_j \right)}{1 + \frac{\tilde{d}_j}{n} \operatorname{Tr} DQ_j + a_j^* Q_j a_j} \ .$$

With this notation at hand, the decomposition of $\mathcal{I}_n - \mathbb{E}\mathcal{I}_n$ as

$$\mathcal{I}_n - \mathbb{E}\mathcal{I}_n = \sum_{j=1}^{n} (\mathbb{E}_j - \mathbb{E}_{j-1}) \log(1 + \Gamma_j) \tag{4.1}$$

follows verbatim from [21, Section 6.2]. Moreover, it is a matter of bookkeeping to establish the following (cf. [21, Section 6.4]):

$$\sum_{j=1}^{n} \mathbb{E}_{j-1} \left( (\mathbb{E}_j - \mathbb{E}_{j-1}) \log(1 + \Gamma_j) \right)^2 - \sum_{j=1}^{n} \mathbb{E}_{j-1} (\mathbb{E}_j \Gamma_j)^2 \xrightarrow[N,n\to\infty]{\mathcal{P}} 0 \ . \tag{4.2}$$

Hence, the details are omitted. In view of Theorem 3.6, Eq. (3.25), (4.1) and (4.2), the CLT will be established if one proves the following 3 results:

(1) (Lyapounov condition)

$$\exists \delta > 0, \quad \sum_{j=1}^{n} \mathbb{E} \left| \mathbb{E}_j \Gamma_j \right|^{2+\delta} \xrightarrow[n\to\infty]{} 0 \ ,$$

(2) (Martingale increments and variance)

$$\sum_{j=1}^{n} \mathbb{E}_{j-1} (\mathbb{E}_j \Gamma_j)^2 - \Theta_n \xrightarrow[N,n\to\infty]{\mathcal{P}} 0 \ .$$

(3) (estimates over the variance)

$$0 < \liminf_n \Theta_n \leq \limsup_n \Theta_n < \infty$$

It is straightforward (and hence omitted) to verify Lyapounov condition. The convergence toward the variance is the cornerstone of the proof of the CLT: The rest of this section together with much of Section 5 are devoted to establish it. The estimates over the variance $\Theta_n$, also central to apply Theorem 3.6, are established in Section 6.2.

Notice that $\mathbb{E}_{j-1}(\mathbb{E}_j \Gamma_j)^2 = \mathbb{E}_{j-1}(\mathbb{E}_j \rho \tilde{b}_j e_j)^2$. We prove hereafter that

$$\sum_{j=1}^{n} \mathbb{E}_{j-1} (\mathbb{E}_j \rho \tilde{b}_j e_j)^2 - \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \mathbb{E}_{j-1} (\mathbb{E}_j e_j)^2 \xrightarrow[N,n\to\infty]{\mathcal{P}} 0 \ . \tag{4.3}$$

The triangular inequality together with Estimates (3.22) and (3.23) yield $\mathbb{E}|\tilde{b}_j - \tilde{t}_{jj}|^2 = \mathcal{O}(n^{-1})$. Now this estimate, together with (3.20), readily implies that:

$$\mathbb{E} \left| \mathbb{E}_{j-1} (\mathbb{E}_j \rho \tilde{b}_j e_j)^2 - \rho^2 \tilde{t}_{jj}^2 \mathbb{E}_{j-1} (\mathbb{E}_j e_j)^2 \right| = \mathcal{O}(n^{-3/2}) \ ,$$

hence (4.3). Let $\varsigma = \mathbb{E}(|X_{11}^2| X_{11})$. Using Identity (3.26), we develop the quantity $\mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2$:

$$
\begin{aligned}
&\sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \mathbb{E}_{j-1}(\mathbb{E}_j e_j)^2 \\
&\quad = \frac{\kappa}{n^2} \sum_{j=1}^{n} \rho^2 \tilde{d}_j^2 \tilde{t}_{jj}^2 \sum_{i=1}^{N} d_i^2 [\mathbb{E}_j Q_j]_{ii}^2 \\
&\qquad + \frac{4}{n} \sum_{j=1}^{n} \rho^2 \tilde{d}_j^{3/2} \tilde{t}_{jj}^2 \operatorname{Re}\left( \varsigma \frac{a_j^*(\mathbb{E}_j Q_j) D^{3/2} \operatorname{vdiag}(\mathbb{E}_j Q_j)}{\sqrt{n}} \right) \\
&\qquad + \frac{1}{n} \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \left( \frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) D + 2 \tilde{d}_j a_j^*(\mathbb{E}_j Q_j) D (\mathbb{E}_j Q_j) a_j \right) \\
&\qquad + \frac{1}{n} \sum_{j=1}^{n} \rho^2 \tilde{t}_{jj}^2 \left( |\vartheta|^2 \frac{\tilde{d}_j^2}{n} \operatorname{Tr}(\mathbb{E}_j Q_j) D (\mathbb{E}_j \bar{Q}_j) D + 2 \operatorname{Re}\left( \vartheta \tilde{d}_j a_j^*(\mathbb{E}_j Q_j) D (\mathbb{E}_j \bar{Q}_j) \bar{a}_j \right) \right) \\
&\quad \triangleq \sum_{j=1}^{n} \chi_{1j} + \sum_{j=1}^{n} \chi_{2j} + \sum_{j=1}^{n} \chi_{3j} + \sum_{j=1}^{n} \chi_{4j} \ .
\end{aligned}
$$

4.2. **Key lemmas for the identification of the variance.** The remainder of the proof of Theorem 2.2 is devoted to find deterministic equivalents for the terms $\sum_{j=1}^{n} \chi_{\ell j}$ for $\ell = 1, 2, 3, 4$.

**Lemma 4.1.** *Assume that the setting of Theorem 2.2 holds true, then:*

$$
\sum_{j=1}^{n} \chi_{1j} - \frac{\kappa \rho^2}{n^2} \sum_{i=1}^{N} \sum_{j=1}^{n} d_i^2 t_{ii}^2 \tilde{d}_j^2 \tilde{t}_{jj}^2 \xrightarrow[N,n\to\infty]{\mathcal{P}} 0 \ .
$$

*Proof.* Write

$$
\begin{aligned}
&\frac{1}{n} \sum_{i=1}^{N} d_i^2 [\mathbb{E}_j Q_j]_{ii}^2 - \frac{1}{n} \sum_{i=1}^{N} d_i^2 [\mathbb{E}_j Q_j]_{ii} t_{ii} = \\
&\qquad \frac{1}{n} \sum_{i=1}^{N} d_i^2 [\mathbb{E}_j Q_j]_{ii} \mathbb{E}_j([Q_j]_{ii} - [Q]_{ii}) + \frac{1}{n} \sum_{i=1}^{N} d_i^2 [\mathbb{E}_j Q_j]_{ii} ([\mathbb{E}_j Q]_{ii} - t_{ii}) = \varepsilon_{1,j} + \varepsilon_{2,j}.
\end{aligned}
$$

The term $|\varepsilon_{1,j}| = n^{-1} |\mathbb{E}_j[\operatorname{Tr} D^2 \operatorname{diag}(\mathbb{E}_j Q_j)(Q_j - Q)]|$ is of order $\mathcal{O}(n^{-1})$ thanks to Lemma 3.1. Moreover, $\mathbb{E}|\varepsilon_{2,j}| = \mathcal{O}(n^{-1/2})$ by (3.23). Hence,

$$
\sum_{j=1}^{n} \chi_{1j} - \frac{\kappa \rho^2}{n} \sum_{j=1}^{n} \tilde{d}_j^2 \tilde{t}_{jj}^2 \left( \frac{1}{n} \sum_{i=1}^{N} d_i^2 [\mathbb{E}_j Q_j]_{ii} t_{ii} \right) \xrightarrow[N,n\to\infty]{\mathcal{P}} 0 \ .
$$

Iterating the same arguments, we can replace the remaining term $\mathbb{E}_j[Q_j]_{ii}$ by $t_{ii}$ to obtain the desired result. □

**Lemma 4.2.** *Assume that the setting of Theorem 2.2 holds true. Then:*

$$
\sum_{j=1}^{n} \chi_{2j} \xrightarrow[N,n\to\infty]{\mathcal{P}} 0 \ .
$$

*Proof.* Recall that

$$\sum_{j=1}^{n}\chi_{2j}=\frac{4}{n}\sum_{j=1}^{n}\rho^2\tilde{d}_j^{3/2}\tilde{t}_{jj}^2\,\mathrm{Re}\left(\varsigma\frac{a_j^*(\mathbb{E}_jQ_j)D^{3/2}\mathrm{vdiag}(\mathbb{E}_jQ_j)}{\sqrt{n}}\right)$$

Taking the expectation, we obtain:

$$\begin{aligned}\mathbb{E}\left|\sum_{j=1}^{n}\chi_{2j}\right| &\leq \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\mathrm{vdiag}(Q_j)}{\sqrt{n}}\right|\\ &\leq \frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*Q_jD^{3/2}\frac{\mathrm{vdiag}(T)}{\sqrt{n}}\right|+\frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\mathrm{vdiag}(Q-T)}{\sqrt{n}}\right|\\ &\quad+\frac{K}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\mathrm{vdiag}(Q_j-Q)}{\sqrt{n}}\right|. \qquad (4.4)\end{aligned}$$

The first term satisfies

$$\sum_{j=1}^{n}\mathbb{E}\left|a_j^*Q_jD^{3/2}\frac{\mathrm{vdiag}(T)}{\sqrt{n}}\right|\leq\sqrt{n}\left(\sum_{j=1}^{n}\mathbb{E}\left|a_j^*Q_jD^{3/2}\frac{\mathrm{vdiag}(T)}{\sqrt{n}}\right|^2\right)^{1/2}.$$

As $\|n^{-1/2}D^{3/2}\mathrm{vdiag}(T)\|=(n^{-1}\sum_{i=1}^{N}d_i^3t_{ii}^2)^{1/2}\leq K$, Theorem 3.3-(1) can be applied, and the first term in the right handside (r.h.s.) of (4.4) is of order $n^{-1/2}$. We now deal with the second term of the r.h.s.

$$\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\mathrm{vdiag}(Q-T)}{\sqrt{n}}\right|\leq K\,\mathbb{E}\left\|\frac{\mathrm{vdiag}(Q-T)}{\sqrt{n}}\right\|\leq\left(\frac{K}{n}\sum_{i=1}^{N}\mathbb{E}|q_{ii}-t_{ii}|^2\right)^{1/2}\leq\frac{K}{\sqrt{n}}$$

by (3.23). We now consider the third term. As $\|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\|$ is uniformly bounded,

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left|a_j^*(\mathbb{E}_jQ_j)D^{3/2}\frac{\mathrm{vdiag}(Q_j-Q)}{\sqrt{n}}\right|\\ &\qquad=\frac{1}{n^{3/2}}\sum_{j=1}^{n}\mathbb{E}\left|\mathrm{Tr}\left(\mathrm{diag}(a_j^*(\mathbb{E}_jQ_j)D^{3/2})(Q_j-Q)\right)\right|=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\end{aligned}$$

by Lemma 3.1. □

**Lemma 4.3.** *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^{n}\chi_{3j}+\log\left(\left(1-\frac{1}{n}\mathrm{Tr}\,D^{\frac{1}{2}}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right)^2-\rho^2\gamma\tilde{\gamma}\right)\xrightarrow[n\to\infty]{\mathcal{P}}0.$$

**Lemma 4.4.** *Assume that the setting of Theorem 2.2 holds true, then:*

$$\sum_{j=1}^{n}\chi_{4j}+\log\left(\left|1-\vartheta\frac{1}{n}\mathrm{Tr}\,D^{\frac{1}{2}}\bar{T}\bar{A}(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right|^2-|\vartheta|^2\rho^2\underline{\gamma}\,\underline{\tilde{\gamma}}\right)\xrightarrow[n\to\infty]{\mathcal{P}}0\ .$$

The core of the paper is devoted to the proof of Lemma 4.3. This proof is provided in Section **??**. The proof of Lemma 4.4 follows the same canvas with minor differences.

## 5. Proof of Theorem 2.2 (part II)

This section is devoted to the proof of Lemma 4.3. We begin with the following lemma which implies that $\sum_{j=1}^n \chi_{3j}$ can be replaced by its expectation.

**Lemma 5.1.** *For any $N \times 1$ vector $a$ with bounded Euclidean norm, we have,*

$$\max_j \operatorname{var}(a^*(\mathbb{E}_j Q)D(\mathbb{E}_j Q)a) \;=\; \mathcal{O}(n^{-1}) \quad \textit{and} \quad \max_j \operatorname{var}\left(\operatorname{Tr}\left(\mathbb{E}_j Q\right)D(\mathbb{E}_j Q)D\right) \;=\; \mathcal{O}(1).$$

Proof of Lemma 5.1 is postponed to Appendix A.4. Recall that:

$$\begin{aligned}\sum_{i=1}^j \chi_{3j} &= \frac{1}{n}\sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \left(\frac{\tilde{d}_j^2}{n}\operatorname{Tr}(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)a_j\right)\\ &= \frac{1}{n}\sum_{j=1}^n \rho^2 \tilde{t}_{jj}^2 \left(\frac{\tilde{d}_j^2}{n}\operatorname{Tr}(\mathbb{E}_j Q)D(\mathbb{E}_j Q)D + 2\tilde{d}_j a_j^*(\mathbb{E}_j Q_j)D(\mathbb{E}_j Q_j)a_j\right) + \mathcal{O}(n^{-1}) \;,\end{aligned}$$

due to Lemma 3.1. Consider the following notations:

$$\begin{aligned}\psi_j &= \frac{1}{n}\operatorname{Tr}\mathbb{E}\left[(\mathbb{E}_j Q)D(\mathbb{E}_j Q)D\right] = \frac{1}{n}\operatorname{Tr}\mathbb{E}\left[(\mathbb{E}_j Q)DQD\right] \;,\\ \zeta_{kj} &= \mathbb{E}\left[a_k^*(\mathbb{E}_j Q)D(\mathbb{E}_j Q)a_k\right] = \mathbb{E}\left[a_k^*(\mathbb{E}_j Q)DQa_k\right] \;,\\ \theta_{kj} &= \mathbb{E}\left[a_k^*(\mathbb{E}_j Q_k)D(\mathbb{E}_j Q_k)a_k\right] = \mathbb{E}\left[a_k^*(\mathbb{E}_j Q_k)DQ_k a_k\right] \;,\\ \varphi_j &= \frac{1}{n}\sum_{k=1}^j \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj} \;.\end{aligned}$$

Thanks to Lemma 5.1, we only need to show that

$$\frac{1}{n}\sum_{j=1}^n \left(\rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj}\right) \quad + \quad \log\Delta_n \quad \xrightarrow[n\to\infty]{} 0 \;. \tag{5.1}$$

There are structural links between the various quantities $\psi_j$, $\zeta_{kj}$, $\theta_{kj}$ and $\varphi_j$. The idea behind the proof is to establish the equations between these quantities. Solving these equations will yield explicit expressions which will enable to identify $\frac{1}{n}\sum_{j=1}^n \left(\rho^2 \tilde{t}_{jj}^2 \tilde{d}_j^2 \psi_j + 2\rho^2 \tilde{d}_j \tilde{t}_{jj}^2 \theta_{jj}\right)$ as the deterministic quantity $-\log\Delta_n$.

Proof of (5.1) is broken down into four steps. In the first step, we establish an equation between $\zeta_{kj}$, $\psi_j$ and $\varphi_j$ (up to $\mathcal{O}(n^{-1/2})$): Eq. (5.7). In the second step, we establish an equation between $\psi_j$ and $\varphi_j$: Eq. (5.11). In the third step, we establish an equation between $\zeta_{kj}$, $\psi_j$ and $\theta_{kj}$: Eq. (5.12). Gathering these results, we obtain a $2 \times 2$ linear system (5.15) whose solutions are $\psi_j$ and $\varphi_j$. In the fourth step, we solve this system and finally establish (5.1).

### 5.1. Step 1: Expression of $\zeta_{kj} = \mathbb{E}[a_k^*(\mathbb{E}_j Q)DQa_k]$.

Writing

$$Q = T + T(T^{-1} - Q^{-1})Q = T + T\left(\rho\tilde{\delta}D + A(I + \delta\tilde{D})^{-1}A^* - \Sigma\Sigma^*\right)Q \;, \tag{5.2}$$

we have:

$$\begin{aligned}\zeta_{kj} &= \mathbb{E}\left[a_k^* \mathbb{E}_j\left[T + T\left(\rho\tilde{\delta}D + A(I+\delta\tilde{D})^{-1}A^* - \Sigma\Sigma^*\right)Q\right]DQa_k\right] \ , \\ &= \mathbb{E}[a_k^* TDQa_k] + \rho\tilde{\delta}\,\mathbb{E}[a_k^* TD(\mathbb{E}_j Q)DQa_k] \\ &\quad + \mathbb{E}[a_k^* TA(I+\delta\tilde{D})^{-1}A^*(\mathbb{E}_j Q)DQa_k] - \mathbb{E}[a_k^* T(\mathbb{E}_j \Sigma\Sigma^* Q)DQa_k] \ , \qquad (5.3) \\ &\triangleq a_k^* TDTa_k + \rho\tilde{\delta}\,\mathbb{E}[a_k^* TD(\mathbb{E}_j Q)DQa_k] + X + Z + \varepsilon \ , \qquad (5.4)\end{aligned}$$

where $X$ and $Z$ are the last two terms at the r.h.s. of (5.3) and where $|\varepsilon| = \mathcal{O}(n^{-1/2})$ by Theorem 3.3-(2). Beginning with $X$, we have

$$\begin{aligned}X &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* Ta_\ell a_\ell^*(\mathbb{E}_j Q)DQa_k]}{1+\delta\tilde{d}_\ell} \ , \\ &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* Ta_\ell a_\ell^*(\mathbb{E}_j Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} - \sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell} a_k^* Ta_\ell a_\ell^*(\mathbb{E}_j Q_\ell\eta_\ell\eta_\ell^* Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} + \varepsilon_1 \ , \\ &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* Ta_\ell a_\ell^*(\mathbb{E}_j Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} - \sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell} a_k^* Ta_\ell a_\ell^*(\mathbb{E}_j Q_\ell a_\ell\eta_\ell^* Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} + \varepsilon_1 + \varepsilon_2 \ , \\ &= \sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* Ta_\ell a_\ell^*(\mathbb{E}_j Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} - \sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell} a_k^* Ta_\ell a_\ell^*\mathcal{T}_\ell a_\ell(\mathbb{E}_j \eta_\ell^* Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \ , \\ &\triangleq X_1 + X_2 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \ ,\end{aligned}$$

where

$$\begin{aligned}\varepsilon_1 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[a_k^* Ta_\ell(\mathbb{E}_j(\rho\tilde{q}_{\ell\ell} - \rho\tilde{t}_{\ell\ell})a_\ell^* Q_\ell\eta_\ell\eta_\ell^* Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} \ , \qquad (5.5) \\ \varepsilon_2 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell} a_k^* Ta_\ell a_\ell^*(\mathbb{E}_j Q_\ell y_\ell\eta_\ell^* Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} \ , \\ \varepsilon_3 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell} a_k^* Ta_\ell(\mathbb{E}_j a_\ell(Q_\ell - \mathcal{T}_\ell)a_\ell\,\eta_\ell^* Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} \ .\end{aligned}$$

Using (3.8) and (3.13), $\varepsilon_1$ writes:

$$\varepsilon_1 = \mathbb{E}\left[\mathbb{E}_j\left(a_k^* TA\ \mathrm{diag}(\xi_\ell)\,(I+\delta\tilde{D})^{-1}\Sigma^* Q\right)DQa_k\right] \ ,$$

where $\xi_\ell = \rho(\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell})(1+\eta_\ell^* Q_\ell\eta_\ell)a_\ell^* Q_\ell\eta_\ell$. Recalling that $\|\Sigma^* Q\|$ is bounded, we obtain $|\varepsilon_1| \le K\mathbb{E}\|a_k^* TA\,\mathrm{diag}(\xi_\ell)\| \le K(\sum_{\ell=1}^n |[a_k^* TA]_\ell|^2\mathbb{E}\xi_\ell^2)^{1/2} \le K/\sqrt{n}$ by (3.23). We show similarly that $\varepsilon_2$ and $\varepsilon_3$ (with the help of Theorem 3.3-(4)) are of order $\mathcal{O}(n^{-1/2})$. We now develop $X_2$ as:

$$\begin{aligned}X_2 &= -\sum_{\ell=1}^n \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_k^* Ta_\ell\, a_\ell^*\mathcal{T}_\ell a_\ell\, a_\ell^*(\mathbb{E}_j Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} - \sum_{\ell=1}^j \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_k^* Ta_\ell\, a_\ell^*\mathcal{T}_\ell a_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQa_k]}{1+\delta\tilde{d}_\ell} \ , \\ &\triangleq U_1 + U_2 \ .\end{aligned}$$

The term $U_2$ writes:

$$U_2 = \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}\tilde{q}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell\, \eta_\ell^* Q_\ell a_k]}{1+\delta\tilde{d}_\ell} ,$$
$$= \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell\, \eta_\ell^* Q_\ell a_k]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2}) .$$

Write $\eta_\ell\eta_\ell^* = a_\ell a_\ell^* + a_\ell y_\ell^* + y_\ell y_\ell^* + y_\ell a_\ell^*$. The term in $a_\ell a_\ell^*$ is zero. Applying Cauchy-Schwarz inequality to $\mathbb{E}_\ell$, we have:

$$\sum_{\ell=1}^{j} \frac{\left|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell\, y_\ell^* Q_\ell a_k]\right|}{1+\delta\tilde{d}_\ell} \;\le\; \frac{K}{n}\sum_{\ell=1}^{j} |a_k^* T a_\ell| \;\le\; \frac{K}{\sqrt{n}} ,$$

and

$$\sum_{\ell=1}^{j} \frac{\left|\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell\, y_\ell^* Q_\ell a_k]\right|}{1+\delta\tilde{d}_\ell}$$
$$= \sum_{\ell=1}^{j} \frac{\left|\mathbb{E}\left[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell \left(y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell - \tilde{d}_\ell n^{-1} \operatorname{Tr} D(\mathbb{E}_j Q_\ell) D Q_\ell\right) y_\ell^* Q_\ell a_k\right]\right|}{1+\delta\tilde{d}_\ell}$$
$$\le \frac{K}{n}\sum_{\ell=1}^{j} |a_k^* T a_\ell| \quad = \quad \mathcal{O}(n^{-1/2}) .$$

The term in $y_\ell a_\ell^*$ writes

$$\sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell\, a_\ell^* Q_\ell a_k]}{1+\delta\tilde{d}_\ell} = \psi_j \sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^* Q_\ell a_k]}{1+\delta\tilde{d}_\ell} + \varepsilon$$

where $\varepsilon = \mathcal{O}(n^{-1})$ by Lemmas 5.1 and 3.1. The remaining term in the r.h.s. can be handled by the following lemma which is proved in appendix A.3:

**Lemma 5.2.** *Let $(u) = (u_n)_{n\in\mathbb{N}}$ be a sequence of vectors with uniformly bounded Euclidean norm. For $j \le n$, let $(\alpha_\ell)_{1\le\ell\le j} = (\alpha_{\ell,j,n})_{1\le\ell\le j}$ be a triangular array of uniformly bounded real numbers. Then:*

$$\sum_{\ell=1}^{j} \alpha_\ell\, u^* T a_\ell\, \mathbb{E}\left[a_\ell Q_\ell u\right] = \sum_{\ell=1}^{j} \frac{\alpha_\ell\, u^* T a_\ell\, a_\ell^* T u}{\rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)} + \mathcal{O}(n^{-1/2}) .$$

As a consequence of this lemma, $U_2$ writes:

$$U_2 = \psi_j \sum_{\ell=1}^{j} \frac{\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^* T a_k}{(1+\delta\tilde{d}_\ell)^2} + \mathcal{O}(n^{-1/2}).$$

Gathering these results, and using the identity $(1-\rho\tilde{t}_{\ell\ell} a_\ell^* \mathcal{T}_\ell a_\ell) = \rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)$ (see (3.15)), we obtain

$$X = \sum_{\ell=1}^{n} \rho\tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q a_k] + \psi_j \sum_{\ell=1}^{j} \frac{\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^* T a_k}{(1+\tilde{d}_\ell\delta)^2} + \mathcal{O}(n^{-1/2}). \quad (5.6)$$

We now turn to the term $Z$ in (5.4).

$$Z = -\sum_{\ell=1}^{n} \mathbb{E}[a_k^* T(\mathbb{E}_j \rho\tilde{q}_{\ell\ell}\eta_\ell\eta_\ell^* Q_\ell) DQa_k] = -\sum_{\ell=1}^{n} \rho\tilde{t}_{\ell\ell}\mathbb{E}[a_k^* T(\mathbb{E}_j \eta_\ell\eta_\ell^* Q_\ell) DQa_k] + \varepsilon \ ,$$

where

$$\varepsilon = \sum_{\ell=1}^{n} \mathbb{E}[a_k^* T \left(\mathbb{E}_j \, \rho(\tilde{q}_{\ell\ell} - \tilde{t}_{\ell\ell})\eta_\ell\eta_\ell^* Q_\ell\right) DQa_k]$$

satisfies $\varepsilon = \mathcal{O}(n^{-1/2})$ (same arguments as for $\varepsilon_1$ in (5.5)). Writing $\eta_\ell\eta_\ell^* = a_\ell a_\ell^* + y_\ell y_\ell^* + a_\ell y_\ell^* + y_\ell a_\ell^*$, we obtain:

$$\begin{aligned}
Z = &-\sum_{\ell=1}^{n} \rho\tilde{t}_{\ell\ell}\, a_k^* T a_\ell \, \mathbb{E}[a_\ell^*(\mathbb{E}_j Q_\ell) DQa_k] \\
&- \left( \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\, \mathbb{E}[a_k^* T y_\ell \, y_\ell^*(\mathbb{E}_j Q_\ell) DQa_k] + \frac{1}{n} \sum_{\ell=j+1}^{n} \rho\tilde{t}_{\ell\ell}\tilde{d}_\ell \, \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) DQa_k] \right) \\
&- \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\, a_k^* T a_\ell \, \mathbb{E}[y_\ell^*(\mathbb{E}_j Q_\ell) DQa_k] - \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\, \mathbb{E}[a_k^* T y_\ell \, a_\ell^*(\mathbb{E}_j Q_\ell) DQa_k] + \mathcal{O}(n^{-1/2}) \\
\triangleq\ & Z_1 + Z_2 + Z_3 + Z_4 + \mathcal{O}(n^{-1/2}).
\end{aligned}$$

The term $Z_1$ cancels with the first term in the decomposition of $X$ (first term at the r.h.s. of (5.6)). The term $Z_2$ writes:

$$\begin{aligned}
Z_2 \ = \ & - \left( \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\, \mathbb{E}[a_k^* T y_\ell \, y_\ell^*(\mathbb{E}_j Q_\ell) DQ_\ell a_k] + \frac{1}{n} \sum_{\ell=j+1}^{n} \rho\tilde{t}_{\ell\ell}\tilde{d}_\ell \, \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) DQa_k] \right) \\
& + \sum_{\ell=1}^{j} \rho^2\tilde{t}_{\ell\ell}^2\, \mathbb{E}[a_k^* T y_\ell \, y_\ell^*(\mathbb{E}_j Q_\ell) DQ_\ell\eta_\ell \, \eta_\ell^* Q_\ell a_k] + \varepsilon \\
\triangleq \ & W_1 + W_2 + \varepsilon \ ,
\end{aligned}$$

where $\varepsilon$ follows from the substitution of $\rho\tilde{q}_{\ell\ell}$ with $\rho\tilde{t}_{\ell\ell}$ and satisfies $\varepsilon = \mathcal{O}(n^{-1/2})$ as in (5.5). Consider first $W_1$:

$$W_1 = -\left( \frac{1}{n} \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\tilde{d}_\ell \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) DQ_\ell a_k] + \frac{1}{n} \sum_{\ell=j+1}^{n} \rho\tilde{t}_{\ell\ell}\tilde{d}_\ell \mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) DQa_k] \right).$$

Write:

$$(\mathbb{E}_j Q_\ell) DQ_\ell - (\mathbb{E}_j Q) DQ = (\mathbb{E}_j Q_\ell) D(Q_\ell - Q) + (\mathbb{E}_j Q_\ell - \mathbb{E}_j Q) DQ \ .$$

Using (3.10) and (3.11),

$$\begin{aligned}
\frac{1}{n} \sum_{\ell=1}^{j} \rho\tilde{t}_{\ell\ell}\tilde{d}_\ell \, |\mathbb{E}[a_k^* T D(\mathbb{E}_j Q_\ell) D(Q_\ell - Q) a_k]| \ &\leq \ \frac{K}{n} \sum_{\ell=1}^{j} \left(\mathbb{E}|(1 + \eta_\ell^* Q_\ell \eta_\ell)|^2\right)^{1/2} \left(\mathbb{E}|\eta_\ell^* Q a_k|^2\right)^{1/2} \\
&\leq \ \frac{K}{\sqrt{n}} (\mathbb{E} a_k^* Q \Sigma_{1:j} \Sigma_{1:j}^* Q a_k)^{1/2} \ = \ \mathcal{O}(n^{-1/2}) \ ,
\end{aligned}$$

and the same arguments apply to the term $(\mathbb{E}_j Q_\ell - \mathbb{E}_j Q) DQ$. Hence,

$$W_1 = -\rho\tilde{\delta} \, \mathbb{E}[a_k^* T D(\mathbb{E}_j Q) DQa_k] + \mathcal{O}(n^{-1/2}).$$

We now develop the term $W_2$ writing $\eta_\ell = y_\ell + a_\ell$. We have:

$$\sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \, \mathbb{E}[a_k^* T y_\ell \, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell \, a_\ell^* Q_\ell a_k]$$

$$= \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \, \mathbb{E}\Big[a_k^* T y_\ell \Big( y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell - \frac{\tilde{d}_\ell}{n} \operatorname{Tr} D (\mathbb{E}_j Q_\ell) D Q_\ell \Big) a_\ell^* Q_\ell a_k \Big]$$

whose module is of order $\mathcal{O}(n^{-1/2})$. The term

$$\sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell y_\ell^* Q_\ell a_k]$$

can be handled similarly.

The term

$$\sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_k^* T y_\ell y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell a_\ell^* Q_\ell a_k] = \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell \mathbb{E}[a_k^* T D (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell a_\ell^* Q_\ell a_k]$$

is bounded by $K n^{-1/2}$. Finally,

$$\begin{aligned} W_2 &= \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \mathbb{E}[y_\ell^* Q_\ell a_k a_k^* T y_\ell \, y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell y_\ell] + \mathcal{O}(n^{-1/2}) \ , \\ &\overset{(a)}{=} \psi_j \, a_k^* T D T a_k \, \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \mathcal{O}(n^{-1/2}) \ , \end{aligned}$$

where $(a)$ follows by standard arguments as those already developed.

The term $Z_3$ satisfies

$$Z_3 = \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 a_k^* T a_\ell \mathbb{E}[y_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \, \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}).$$

Writing $\eta_\ell \eta_\ell^* = y_\ell y_\ell^* + a_\ell y_\ell^* + a_\ell a_\ell^* + y_\ell a_\ell^*$ and relying arguments as those already developed, one can check that the only non-negligible contribution stems from the term containing $y_\ell a_\ell^*$. Hence,

$$\begin{aligned} Z_3 &= \psi_j \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell a_k^* T a_\ell \mathbb{E}[a_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}) \ , \\ &= \psi_j \sum_{\ell=1}^{j} \frac{\rho \tilde{t}_{\ell\ell} \tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1 + \tilde{d}_\ell \delta} + \mathcal{O}(n^{-1/2}) \ , \end{aligned}$$

by Lemma (5.2). Similarly,

$$\begin{aligned} Z_4 &= \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \, \mathbb{E}[a_k^* T y_\ell \, a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell \eta_\ell \, \eta_\ell^* Q_\ell a_k] + \mathcal{O}(n^{-1/2}) \ , \\ &= a_k^* T D T a_k \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell \tilde{t}_{\ell\ell}^2 \mathbb{E}[a_\ell^* (\mathbb{E}_j Q_\ell) D Q_\ell a_\ell] + \mathcal{O}(n^{-1/2}) \ , \\ &= a_k^* T D T a_k \varphi_j + \mathcal{O}(n^{-1/2}) \ . \end{aligned}$$

Gathering these results, we obtain

$$Z = -\sum_{\ell=1}^{n} \rho\tilde{t}_{\ell\ell}\, a_k^* T a_\ell\, \mathbb{E}[a_\ell^*(\mathbb{E}_j Q_\ell) DQ a_k] - \rho\tilde{\delta}\, \mathbb{E}[a_k^* TD(\mathbb{E}_j Q) DQ a_k]$$
$$+ \psi_j\, a_k^* TDT a_k\, \frac{1}{n}\sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 + \psi_j \sum_{\ell=1}^{j} \frac{\rho\tilde{t}_{\ell\ell}\tilde{d}_\ell a_k^* T a_\ell a_\ell^* T a_k}{1+\tilde{d}_\ell\delta} + a_k^* TDT a_k\, \varphi_j + \mathcal{O}(n^{-1/2}).$$

Plugging this and Eq. (5.6) into (5.4), and noticing that $\rho\tilde{t}_{\ell\ell}(a_\ell \mathcal{T}_\ell a_\ell (1+\tilde{d}_\ell\delta)^{-1} + 1) = (1+\tilde{d}_\ell\delta)^{-1}$, we obtain:

$$\zeta_{kj} = a_k^* TDT a_k + \psi_j \left( \sum_{\ell=1}^{j} \frac{a_k^* T a_\ell\, \tilde{d}_\ell\, a_\ell^* T a_k}{(1+\tilde{d}_\ell\delta)^2} \;+\; a_k^* TDT a_k\, \frac{1}{n}\sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 \right)$$
$$+ a_k^* TDT a_k\, \varphi_j + \mathcal{O}(n^{-1/2})\ . \quad (5.7)$$

5.2. **Step 2: Expression of $\psi_j = n^{-1}\operatorname{Tr}\mathbb{E}[(\mathbb{E}_j Q)DQD]$.** Using Identity (5.2), we obtain:

$$\psi_j = \frac{1}{n}\operatorname{Tr}\mathbb{E}[TDQD] + \frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD]$$
$$+ \frac{1}{n}\operatorname{Tr}\mathbb{E}[TA(I+\delta\tilde{D})^{-1}A^*(\mathbb{E}_j Q)DQD] - \frac{1}{n}\operatorname{Tr}\mathbb{E}[T(\mathbb{E}_j \Sigma\Sigma^* Q)DQD]\ , \quad (5.8)$$
$$= \frac{1}{n}\operatorname{Tr} DTDT + \frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_j Q)DQD] + X + Z + \varepsilon\ , \quad (5.9)$$

where $X$ and $Z$ are the last two terms of the r.h.s. of (5.8), and where $\varepsilon = \mathcal{O}(n^{-1})$ by Theorem 3.3-(3). Due to the presence of the multiplying factor $n^{-1}$, the treatment of $X$ and $Z$ is simpler here than the treatment of their analogues for $\zeta_{kj}$. We skip hereafter the details related to the bounds over the $\varepsilon$'s. The term $X$ satisfies

$$X = \frac{1}{n}\sum_{\ell=1}^{n} \frac{\mathbb{E}[a_\ell^*(\mathbb{E}_j Q)DQDTa_\ell]}{1+\delta\tilde{d}_\ell}\ ,$$
$$= \frac{1}{n}\sum_{\ell=1}^{n} \frac{\mathbb{E}[a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} - \frac{1}{n}\sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\,(\mathbb{E}_j a_\ell^* Q_\ell \eta_\ell\, \eta_\ell^* Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} + \varepsilon\ ,$$
$$= \frac{1}{n}\sum_{\ell=1}^{n} \frac{\mathbb{E}[a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} - \frac{1}{n}\sum_{\ell=1}^{n} \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^* \mathcal{T}_\ell a_\ell\, a_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell}$$
$$- \frac{1}{n}\sum_{\ell=1}^{j} \frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\, a_\ell^* \mathcal{T}_\ell a_\ell\, y_\ell^*(\mathbb{E}_j Q_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} + \varepsilon'\ ,$$

where $\max(|\varepsilon|,|\varepsilon'|) = \mathcal{O}(n^{-1/2})$. As $1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell = \rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)$,

$$
\begin{aligned}
X &= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}\,a_\ell^*(\mathbb{E}_jQ_\ell)DQDTa_\ell] - \frac{1}{n}\sum_{\ell=1}^{j}\frac{\mathbb{E}[\rho\tilde{t}_{\ell\ell}\,a_\ell^*\mathcal{T}_\ell a_\ell\,y_\ell^*(\mathbb{E}_jQ_\ell)DQDTa_\ell]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2})\ ,\\
&= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}\,a_\ell^*(\mathbb{E}_jQ_\ell)DQDTa_\ell] + \frac{1}{n}\sum_{\ell=1}^{j}\frac{\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2\,a_\ell^*\mathcal{T}_\ell a_\ell\,y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell\eta_\ell\,\eta_\ell^*Q_\ell DTa_\ell]}{1+\delta\tilde{d}_\ell} + \mathcal{O}(n^{-1/2})\ ,\\
&= \frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}\,a_\ell^*(\mathbb{E}_jQ_\ell)DQDTa_\ell] + \frac{\psi_j}{n}\sum_{\ell=1}^{j}\frac{\tilde{d}_\ell\,a_\ell^*Ta_\ell\,a_\ell^*TDTa_\ell}{(1+\delta\tilde{d}_\ell)^3} + \mathcal{O}(n^{-1/2})\ , \qquad (5.10)
\end{aligned}
$$

where (3.16) is used to obtain the last equation. The term $Z$ writes:

$$
\begin{aligned}
Z &= -\frac{1}{n}\sum_{\ell=1}^{n}\operatorname{Tr}\mathbb{E}[\rho\tilde{t}_{\ell\ell}T(\mathbb{E}_j\eta_\ell\eta_\ell^*Q_\ell)DQD] + \mathcal{O}(n^{-1/2})\ ,\\
&= -\frac{1}{n}\sum_{\ell=1}^{n}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_jQ_\ell)DQDTa_\ell]\\
&\quad -\left(\frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}y_\ell^*(\mathbb{E}_jQ_\ell)DQDTy_\ell] + \frac{1}{n}\sum_{\ell=j+1}^{n}\frac{1}{n}\operatorname{Tr}\mathbb{E}[\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}TD(\mathbb{E}_jQ_\ell)DQD]\right)\\
&\quad -\frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}y_\ell^*(\mathbb{E}_jQ_\ell)DQDTa_\ell] - \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho\tilde{t}_{\ell\ell}a_\ell^*(\mathbb{E}_jQ_\ell)DQDTy_\ell] + \mathcal{O}(n^{-1/2})\ ,\\
&\triangleq Z_1+Z_2+Z_3+Z_4+\mathcal{O}(n^{-1/2}).
\end{aligned}
$$

The term $Z_1$ cancels with the first term in the r.h.s. of $X$'s decomposition (5.10). The terms $Z_2$, $Z_3$ and $Z_4$ satisfy:

$$
\begin{aligned}
Z_2 &= -\frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_jQ)DQD] + \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell\eta_\ell\,\eta_\ell^*Q_\ell DTy_\ell] + \mathcal{O}(n^{-1/2})\\
&= -\frac{\rho\tilde{\delta}}{n}\operatorname{Tr}\mathbb{E}[TD(\mathbb{E}_jQ)DQD] + \psi_j\frac{1}{n}\operatorname{Tr}DTDT\frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde{d}_\ell^2\tilde{t}_{\ell\ell}^2 + \mathcal{O}(n^{-1/2})\ ,\\
Z_3 &= \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2y_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell\eta_\ell\,\eta_\ell^*Q_\ell DTa_\ell] + \mathcal{O}(n^{-1/2})\\
&= \psi_j\frac{1}{n}\sum_{\ell=1}^{j}\rho\tilde{d}_\ell\tilde{t}_{\ell\ell}\,\frac{a_\ell^*TDTa_\ell}{1+\tilde{d}_\ell\delta} + \mathcal{O}(n^{-1/2})\ ,\\
Z_4 &= \frac{1}{n}\sum_{\ell=1}^{j}\mathbb{E}[\rho^2\tilde{t}_{\ell\ell}^2a_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell\eta_\ell\,\eta_\ell^*Q_\ell DTy_\ell] + \mathcal{O}(n^{-1/2})\\
&= \frac{1}{n}\operatorname{Tr}DTDT\frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde{d}_\ell\tilde{t}_{\ell\ell}^2\mathbb{E}[a_\ell^*(\mathbb{E}_jQ_\ell)DQ_\ell a_\ell] + \mathcal{O}(n^{-1/2}).
\end{aligned}
$$

Plugging these terms in (5.9), we obtain:

$$\psi_j = \gamma + \psi_j \left( \frac{1}{n} \sum_{\ell=1}^{j} a_\ell^* T D T a_\ell \left( \frac{\tilde{d}_\ell \, a_\ell^* T a_\ell}{(1+\delta\tilde{d}_\ell)^3} + \frac{\rho \tilde{d}_\ell \tilde{t}_{\ell\ell}}{1+\tilde{d}_\ell\delta} \right) + \frac{\gamma}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell^2 \tilde{t}_{\ell\ell}^2 \right) + \gamma\varphi_j + \mathcal{O}(n^{-1/2}) \, ,$$

$$= \gamma + \psi_j \left( \frac{1}{n} \sum_{\ell=1}^{j} \frac{\tilde{d}_\ell \, a_\ell^* T D T a_\ell}{(1+\delta\tilde{d}_\ell)^2} + \frac{\gamma}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{d}_\ell^2 \tilde{t}_{\ell\ell}^2 \right) + \gamma\varphi_j + \mathcal{O}(n^{-1/2}) \, , \tag{5.11}$$

using (3.15) and (3.16).

5.3. **Step 3: Relation between $\zeta_{kj}$ and $\theta_{kj}$ for $k \leq j$.** The term $\zeta_{kj}$ writes

$$\begin{aligned}
\zeta_{kj} &= \mathbb{E}[a_k^* \mathbb{E}_j (Q_k - \rho\tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) D (Q_k - \rho\tilde{q}_{kk} Q_k \eta_k \eta_k^* Q_k) a_k] \\
&= \theta_{kj} - \rho\tilde{t}_{kk} \mathbb{E}[a_k^* \mathbb{E}_j (Q_k \eta_k \eta_k^* Q_k) D Q_k a_k - \rho\tilde{t}_{kk} \, \mathbb{E}[a_k^* \mathbb{E}_j (Q_k) D Q_k \eta_k \, \eta_k^* Q_k a_k] \\
&\quad + \rho^2 \tilde{t}_{kk}^2 \, \mathbb{E}[a_k^* \mathbb{E}_j (Q_k \eta_k \eta_k^* Q_k) D Q_k \eta_k \, \eta_k^* Q_k a_k] + \mathcal{O}(n^{-1/2}) \\
&\triangleq \theta_{kj} + X_1 + X_2 + X_3 + \mathcal{O}(n^{-1/2}) \, .
\end{aligned}$$

Using similar arguments as those developed previously, we get:

$$\begin{aligned}
X_1 &= -\rho\tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \, \mathbb{E}[a_k^* \mathbb{E}_j (Q_k) D Q_k a_k] + \mathcal{O}(n^{-1/2}) \; = \; -\rho\tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \, \theta_{kj} + \mathcal{O}(n^{-1/2}), \\
X_2 &= -\rho\tilde{t}_{kk} a_k^* \mathcal{T}_k a_k \, \theta_{kj} + \mathcal{O}(n^{-1/2}) \, .
\end{aligned}$$

As $k \leq j$,

$$\begin{aligned}
X_3 &= \rho^2 \tilde{t}_{kk}^2 \, (a_k^* \mathcal{T}_k a_k)^2 \, \mathbb{E}[\eta_k^* \mathbb{E}_j (Q_k) D Q_k \eta_k] + \mathcal{O}(n^{-1/2}) \, , \\
&= \rho^2 \tilde{t}_{kk}^2 \, (a_k^* \mathcal{T}_k a_k)^2 \, \left( \theta_{kj} + \tilde{d}_k \psi_j \right) + \mathcal{O}(n^{-1/2}) \, .
\end{aligned}$$

Using (3.15) and (3.16), we finally obtain:

$$\zeta_{kj} = \rho^2 \tilde{t}_{kk}^2 (1+\tilde{d}_k\delta)^2 \, \theta_{kj} + \tilde{d}_k \left( \frac{a_k^* T a_k}{1+\tilde{d}_k\delta} \right)^2 \, \psi_j + \mathcal{O}(n^{-1/2}) \, . \tag{5.12}$$

5.4. **Step 4: A system of perturbed linear equations in $(\psi_j, \varphi_j)$. Expression of the r.h.s. of** (5.1)**.** Combining (5.12) with (5.7), we obtain

$$\begin{aligned}
\rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj} &= \tilde{d}_k \frac{a_k^* T D T a_k}{(1+\tilde{d}_k\delta)^2} \\
&\quad + \left( \sum_{\ell=1}^{j} \frac{\tilde{d}_k \, a_k^* T a_\ell \, \tilde{d}_\ell \, a_\ell^* T a_k}{(1+\tilde{d}_k\delta)^2 (1+\tilde{d}_\ell\delta)^2} \; + \; \frac{\tilde{d}_k \, a_k^* T D T a_k}{(1+\tilde{d}_k\delta)^2} \, \frac{1}{n} \sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 - \tilde{d}_k^2 \frac{(a_k^* T a_k)^2}{(1+\tilde{d}_k\delta)^4} \right) \psi_j \\
&\qquad + \frac{\tilde{d}_k \, a_k^* T D T a_k}{(1+\tilde{d}_k\delta)^2} \, \varphi_j + \mathcal{O}(n^{-1/2})
\end{aligned} \tag{5.13}$$

which implies that $\varphi_j = \frac{1}{n} \sum_{k=1}^{j} \rho^2 \tilde{d}_k \tilde{t}_{kk}^2 \theta_{kj}$ satisfies

$$(1 - F_j)\varphi_j - (G_j + F_j M_j) \, \psi_j = F_j + \mathcal{O}(n^{-1/2})$$

where

$$\begin{aligned} F_j &= \frac{1}{n}\sum_{k=1}^{j} \frac{a_k^* T D T a_k \, \tilde{d}_k}{(1+\tilde{d}_k\delta)^2} \ , \\ M_j &= \frac{1}{n}\sum_{\ell=1}^{j} \rho^2 \tilde{t}_{\ell\ell}^2 \tilde{d}_\ell^2 \ , \\ G_j &= \frac{1}{n}\sum_{k=1}^{j}\sum_{\substack{\ell=1\\ \ell\neq k}}^{j} \frac{\tilde{d}_k\tilde{d}_\ell \, |a_k^* T a_\ell|^2}{(1+\tilde{d}_k\delta)^2(1+\tilde{d}_\ell\delta)^2} \ . \end{aligned} \tag{5.14}$$

With these new notations, equation (5.11) is rewritten

$$-\gamma\varphi_j + (1-F_j-\gamma M_j)\psi_j = \gamma + \mathcal{O}(n^{-1/2}),$$

and we end up with a system of two perturbed linear equations in $(\varphi_j, \psi_j)$:

$$\left\{ \begin{array}{lcl} (1-F_j)\varphi_j - (G_j + F_j M_j)\,\psi_j &=& F_j + \mathcal{O}(n^{-1/2}) \\ -\gamma\varphi_j + (1-F_j-\gamma M_j)\psi_j &=& \gamma + \mathcal{O}(n^{-1/2}) \end{array} \right. . \tag{5.15}$$

The determinant of this system is $\mathbf{\Delta}_j = (1-F_j)^2 - \gamma M_j - \gamma G_j$. The following lemma establishes the link between the $\mathbf{\Delta}_j$'s and $\Delta_n$ as defined in Theorem 2.2.

**Lemma 5.3.** *Recall the definition of $\Delta_n$ :*

$$\Delta_n = \left(1 - \frac{1}{n}\operatorname{Tr} D^{\frac{1}{2}} T A (I+\delta\tilde{D})^{-2}\tilde{D} A^* T D^{\frac{1}{2}}\right)^2 - \rho^2\gamma\tilde{\gamma} \ .$$

*The determinants $\mathbf{\Delta}_j$ decrease as $j$ goes from 1 to $n$; moreover, $\mathbf{\Delta}_n$ coincides with $\Delta_n$.*

Proof of Lemma 5.3 is postponed to Appendix A.5.

Solving this system of equations and using the lemma in conjunction with the fact $\liminf \Delta_n > 0$, established in Lemma 3.5, we obtain:

$$\begin{bmatrix} \varphi_j \\ \psi_j \end{bmatrix} = \frac{1}{\mathbf{\Delta}_j} \begin{bmatrix} F_j(1-F_j) + \gamma G_j \\ \gamma \end{bmatrix} + \boldsymbol{\varepsilon}_j \ ,$$

where $\|\boldsymbol{\varepsilon}_j\| = \mathcal{O}(n^{-1/2})$. Replacing into (5.13), we obtain

$$\begin{aligned} \frac{2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}}{n} &= 2(F_j - F_{j-1}) + (G_j - G_{j-1} + 2M_j(F_j - F_{j-1}))\psi_j + 2(F_j - F_{j-1})\varphi_j + \mathcal{O}(n^{-3/2}) \\ &= 2(F_j - F_{j-1}) + \frac{\gamma(G_j - G_{j-1}) + 2\gamma M_j(F_j - F_{j-1})}{\mathbf{\Delta}_j} \\ &\qquad + \frac{2(F_j - F_{j-1})(F_j(1-F_j) + \gamma G_j)}{\mathbf{\Delta}_j} + \mathcal{O}(n^{-3/2}) \end{aligned}$$

which leads to

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n}\left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}\right) \\ &\qquad = \sum_{j=1}^{n} \frac{2(F_j - F_{j-1})(1-F_j) + \gamma(M_j - M_{j-1}) + \gamma(G_j - G_{j-1})}{\mathbf{\Delta}_j} + \mathcal{O}(n^{-1/2}). \end{aligned}$$

On the other hand, $\boldsymbol{\Delta}_{j-1}-\boldsymbol{\Delta}_j = 2(F_j-F_{j-1})(1-F_j)+\gamma(M_j-M_{j-1})+\gamma(G_j-G_{j-1})+\mathcal{O}(n^{-2})$, hence, due to lemma 5.3 and to $\liminf \Delta_n > 0$,

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2\psi_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\theta_{jj}\right) &= \sum_{j=1}^{n}\frac{\boldsymbol{\Delta}_{j-1}-\boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2}) \\ &= \sum_{j=1}^{n}\log\left(1+\frac{\boldsymbol{\Delta}_{j-1}-\boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j}\right) + \mathcal{O}(n^{-1/2}) \\ &= \sum_{j=1}^{n}\log\frac{\boldsymbol{\Delta}_{j-1}}{\boldsymbol{\Delta}_j} + \mathcal{O}(n^{-1/2}) = -\log(\Delta_n) + \mathcal{O}(n^{-1/2})\end{aligned}$$

which proves (5.1). Lemma 4.3 is proved.

## 6. Proof of Theorem 2.2 (part III)

In this section, we complete the proof of Theorem 2.2. Proof of Lemma 4.4 is very close to the proof of Lemma 4.3; we therefore only provide its main landmarks. We finally establish the main estimates over $(\Theta_n)$.

6.1. **Elements of proof for Lemma 4.4.** Proof of Lemma 4.4 relies on the following counterpart Lemma 3.2:

**Lemma 6.1.** *Assume that the setting of Lemma 3.2 holds true; and let $\mathbb{E}x^2 = \vartheta$. Then for any $p \geq 2$,*

$$\mathbb{E}|\boldsymbol{x}^T M\boldsymbol{x} - \vartheta\,\mathrm{Tr}\,M|^p \leq K_p\left(\left(\mathbb{E}|x_1|^4\,\mathrm{Tr}\,MM^*\right)^{p/2} + \mathbb{E}|x_1|^{2p}\,\mathrm{Tr}(MM^*)^{p/2}\right)\ .$$

*Proof.* The result is obtained upon noticing that

$$\boldsymbol{x}^T M\boldsymbol{x} = \frac{1}{4}\sum_{k=0}^{3}\mathbf{i}^k\left(\mathbf{i}^k\bar{\boldsymbol{x}}+\boldsymbol{x}\right)^* M\left(\mathbf{i}^k\bar{\boldsymbol{x}}+\boldsymbol{x}\right)$$

and using Lemma 3.2. □

Here are the main steps of the proof. Introducing the notations

$$\begin{aligned}\underline{\psi}_j &= \frac{1}{n}\mathrm{Tr}\,\mathbb{E}\left[(\mathbb{E}_jQ)D\bar{Q}D\right]\ , \\ \underline{\zeta}_{kj} &= \mathbb{E}\left[a_k^*(\mathbb{E}_jQ)D\bar{Q}\bar{a}_k\right]\ , \\ \underline{\theta}_{kj} &= \mathbb{E}\left[a_k^*(\mathbb{E}_jQ_k)D\bar{Q}_k\bar{a}_k\right]\ , \\ \underline{\varphi}_j &= \frac{1}{n}\sum_{k=1}^{j}\rho^2\tilde{d}_k\tilde{t}_{kk}^2\underline{\theta}_{kj}\ ,\end{aligned}$$

and adapting Lemma 5.1, we only need to prove that:

$$\frac{1}{n}\sum_{j=1}^{n}\left(\rho^2\tilde{t}_{jj}^2\tilde{d}_j^2|\vartheta|^2\underline{\psi}_j + 2\rho^2\tilde{d}_j\tilde{t}_{jj}^2\,\mathrm{Re}\left(\vartheta\underline{\theta}_{jj}\right)\right) \quad + \quad \log\underline{\Delta}_n \quad \xrightarrow[n\to\infty]{} 0\ .$$

Similar derivations as those performed in Steps 1-3 in Section 5 yield the perturbed system:

$$\left\{\begin{array}{rcl}(1-\vartheta\underline{F}_j)\underline{\varphi}_j-(\bar\vartheta\underline{G}_j+|\vartheta|^2\underline{F}_jM_j)\underline{\psi}_j &=& \underline{F}_j+\mathcal{O}(n^{-1/2})\\ -\vartheta\underline{\gamma}\underline{\varphi}_j+(1-\bar\vartheta\underline{\bar F}_j-\underline{\gamma}|\vartheta|^2M_j)\underline{\psi}_j &=& \underline{\gamma}+\mathcal{O}(n^{-1/2})\end{array}\right. ,$$

where

$$\underline{F}_j = \frac{1}{n}\sum_{k=1}^{j}\frac{a_k^*TD\bar T\bar a_k\,\tilde d_k}{(1+\tilde d_k\delta)^2}\in\mathbb{C},\qquad \underline{G}_j = \frac{1}{n}\sum_{k=1}^{j}\sum_{\substack{\ell=1\\ \ell\neq k}}^{j}\frac{\tilde d_k\tilde d_\ell\,(a_k^*Ta_\ell)^2}{(1+\tilde d_k\delta)^2(1+\tilde d_\ell\delta)^2}\in\mathbb{R}\ ,$$

$$M_j = \frac{1}{n}\sum_{\ell=1}^{j}\rho^2\tilde t_{\ell\ell}^2\tilde d_\ell^2\ .$$

The determinant of this system is:

$$\underline{\boldsymbol{\Delta}}_j=\left|1-\vartheta\underline{F}_j\right|^2-|\vartheta|^2\underline{\gamma}\left(M_j+\underline{G}_j\right).$$

By (3.18), $0\le\underline{\gamma}\le\gamma$; furthermore, $|\vartheta|\le 1$, $|\underline{F}_j|\le F_j$, and $|\underline{G}_j|\le G_j$. As a result, $\underline{\boldsymbol{\Delta}}_j\ge\boldsymbol{\Delta}_j$. Hence, by Lemma 5.3, the perturbation remains of order $\mathcal{O}(n^{-1/2})$ after solving the system. Performing the same derivations as in Step 4 in Section 5, it can be established that $\underline{\boldsymbol{\Delta}}_n=\underline{\Delta}_n$. We finally end up with:

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\left(\rho^2\tilde t_{jj}^2\tilde d_j^2|\vartheta|^2\underline{\psi}_j+2\rho^2\tilde d_j\tilde t_{jj}^2\,\mathrm{Re}\left(\vartheta\underline{\theta}_{jj}\right)\right) &= \sum_{j=1}^{n}\frac{\underline{\boldsymbol{\Delta}}_{j-1}-\underline{\boldsymbol{\Delta}}_j}{\underline{\boldsymbol{\Delta}}_j}+\mathcal{O}(n^{-1/2})\ ,\\ &= -\log(\underline{\Delta}_n)+\mathcal{O}(n^{-1/2})\ ,\end{aligned}$$

which is the desired result.

6.2. **Estimates over $\Theta_n$.** In order to conclude the proof of Theorem 2.2, it remains to prove that $0<\liminf_n\Theta_n\le\limsup_n\Theta_n<\infty$.

Consider first the upper bound. By Lemma 3.5, $\sup_n(-\log\Delta_n)<\infty$. As $\underline{\Delta}_n\ge\Delta_n$, $\log\underline{\Delta}_n$ is defined and $\sup_n(-\log\underline{\Delta}_n)<\infty$. By Lemma 3.4, the cumulant term in the expression of $\Theta_n$ is bounded, hence $\limsup_n\Theta_n<\infty$.

We now prove that $\liminf\Theta_n>0$. To this end, write:

$$\begin{aligned}\Theta_n &= \sum_{j=1}^{n}\left(\frac{\boldsymbol{\Delta}_{j-1}-\boldsymbol{\Delta}_j}{\boldsymbol{\Delta}_j}+\frac{\underline{\boldsymbol{\Delta}}_{j-1}-\underline{\boldsymbol{\Delta}}_j}{\underline{\boldsymbol{\Delta}}_j}\right)+\kappa\frac{\rho^2}{n^2}\sum_{i=1}^{N}d_i^2t_{ii}^2\sum_{j=1}^{n}\tilde d_j^2\tilde t_{jj}^2+\mathcal{O}(n^{-1/2})\ ,\\ &= \sum_{j=1}^{n}\left(\frac{\gamma(G_j-G_{j-1})}{\boldsymbol{\Delta}_j}+\frac{|\vartheta|^2\underline{\gamma}(\underline{G}_j-\underline{G}_{j-1})}{\underline{\boldsymbol{\Delta}}_j}\right)\\ &\quad+\sum_{j=1}^{n}\left(\frac{2(F_j-F_{j-1})(1-F_j)}{\boldsymbol{\Delta}_j}+\frac{2\,\mathrm{Re}\left(\vartheta(\underline{F}_j-\underline{F}_{j-1})(1-\bar\vartheta\underline{\bar F}_j)\right)}{\underline{\boldsymbol{\Delta}}_j}\right)\\ &\quad+\frac{\rho^2}{n}\sum_{j=1}^{n}\tilde d_j^2\tilde t_{jj}^2\left(\frac{\gamma}{\boldsymbol{\Delta}_j}+|\vartheta|^2\frac{\underline{\gamma}}{\underline{\boldsymbol{\Delta}}_j}+\frac{\kappa}{n}\sum_{i=1}^{N}d_i^2t_{ii}^2\right)+\mathcal{O}(n^{-1/2})\ ,\\ &\triangleq Z_{1,n}+Z_{2,n}+Z_{3,n}+\mathcal{O}(n^{-1/2})\ .\end{aligned}$$

We prove in the sequel that $Z_{1,n} \geq 0$, $Z_{2,n} \geq 0$, and that $\liminf_n Z_{3,n} > 0$. It has already been noticed that $\underline{\boldsymbol{\Delta}}_j \geq \boldsymbol{\Delta}_j$; moreover, it can be proved by direct computation that $|\underline{G}_j - \underline{G}_{j-1}| \leq G_j - G_{j-1}$, hence $Z_{1,n} \geq 0$. As

$$(1-F_j) - \frac{\gamma(M_j+G_j)}{1-F_j} \leq \left|1-\vartheta\underline{F}_j\right| - \frac{|\vartheta|^2\underline{\gamma}\left(M_j+\underline{G}_j\right)}{\left|1-\vartheta\underline{F}_j\right|},$$

this implies that $\underline{\boldsymbol{\Delta}}_j^{-1}|1-\vartheta\underline{F}_j| \leq \boldsymbol{\Delta}_j^{-1}(1-F_j)$. Noticing in addition that $|\underline{F}_j - \underline{F}_{j-1}| \leq F_j - F_{j-1}$, we get $Z_{2,n} \geq 0$. The cumulant $\kappa = \mathbb{E}|X_{11}|^4 - 2 - |\vartheta|^2$ satisfies $\kappa \geq -1 - |\vartheta|^2$, hence

$$\begin{aligned}
Z_{3,n} &\geq \frac{\rho^2}{n^2}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2\left(\left(\frac{1}{\boldsymbol{\Delta}_j}-1\right)+|\vartheta|^2\left(\frac{1}{\underline{\boldsymbol{\Delta}}_j}-1\right)\right)\sum_{i=1}^N d_i^2 t_{ii}^2 \\
&\quad + \frac{\rho^2}{n}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2\Big(\frac{1}{n\boldsymbol{\Delta}_j}\sum_{\substack{k,\ell=1\\k\neq\ell}}^N d_k^2\left|t_{kl}\right|^2 + \frac{1}{n\underline{\boldsymbol{\Delta}}_j}\sum_{\substack{k,\ell=1\\k\neq\ell}}^N d_k^2\left(t_{kl}\right)^2\Big)\ , \\
&\geq \frac{\rho^2}{n^2}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2\left(\left(\frac{1}{\boldsymbol{\Delta}_j}-1\right)+|\vartheta|^2\left(\frac{1}{\underline{\boldsymbol{\Delta}}_j}-1\right)\right)\sum_{i=1}^N d_i^2 t_{ii}^2 = \frac{\rho^2}{n^2}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2 p_j \sum_{i=1}^N d_i^2 t_{ii}^2\ .
\end{aligned}$$

As the term $p_j$ is linear in $|\vartheta|^2 \in [0,1]$, $p_j \geq \min\left(\boldsymbol{\Delta}_j^{-1}(1-\boldsymbol{\Delta}_j), \boldsymbol{\Delta}_j^{-1} + \underline{\boldsymbol{\Delta}}_j^{-1} - 2\right)$. We have

$$\begin{aligned}
\boldsymbol{\Delta}_j\underline{\boldsymbol{\Delta}}_j &\leq \left((1-F_j)^2 - \gamma(M_j+G_j)\right)\left((1+F_j)^2 + \gamma(M_j+G_j)\right)\ , \\
&= (1-F_j^2)^2 - \gamma^2(M_j+G_j)^2 - 4\gamma F_j(M_j+G_j) \quad \leq \quad 1\ .
\end{aligned}$$

Hence $\boldsymbol{\Delta}_j^{-1} + \underline{\boldsymbol{\Delta}}_j^{-1} - 2 \geq \boldsymbol{\Delta}_j^{-1} + \boldsymbol{\Delta}_j - 2 = \boldsymbol{\Delta}_j^{-1}(1-\boldsymbol{\Delta}_j)^2$. As $1 - \boldsymbol{\Delta}_j \geq \gamma M_j$, we get $p_j \geq \gamma^2 M_j^2$, which implies that

$$\begin{aligned}
Z_{3,n} &\geq \frac{\rho^2\gamma^2}{n^2}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2 M_j^2 \sum_{i=1}^N d_i^2 t_{ii}^2\ , \\
&= \frac{\rho^2\gamma^2}{3}\frac{1}{n}\sum_{i=1}^N d_i^2 t_{ii}^2\left(\frac{1}{n}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2\right)^3 + \mathcal{O}(n^{-1})\ ,
\end{aligned}$$

whose liminf is positive by Lemma 3.4.

The estimates over the variance are therefore established. This completes the proof of Theorem 2.2.

## 7. Proof of Theorem 2.3 (bias)

The same arguments as in the companon article [21] allow to write the bias term as:

$$\mathcal{B}_n(\rho) = \int_\rho^\infty \mathrm{Tr}\left(T(-\omega) - \mathbb{E}Q(-\omega)\right)d\omega.$$

We shall prove that:

$$\mathrm{Tr}\left(T(-\omega) - \mathbb{E}Q(-\omega)\right) = \kappa\frac{A(\omega)}{B(\omega)}, \tag{7.1}$$

where,

$$\begin{aligned} A(\omega) &= \omega^2(1+\tilde{\delta})\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2\frac{1}{n}\operatorname{Tr}\,ST^2+\omega^2(1+\delta)\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}\tilde{T}^2-\omega\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2\ , \\ B(\omega) &= 1+\omega(1+\delta)\tilde{\gamma}+\omega(1+\tilde{\delta})\gamma\ . \end{aligned}$$

**Outline of the proof.** The proof of (7.1) will be carried out in two steps:

- We first introduce the following matrix: $R(-\omega)=\left(\omega\,(1+\tilde{\alpha})\,I_N+\frac{AA^*}{1+\alpha}\right)^{-1}$, where $\alpha=\frac{1}{n}\operatorname{Tr}\mathbb{E}Q$ and $\tilde{\alpha}=\frac{1}{n}\operatorname{Tr}\mathbb{E}\tilde{Q}$, and we prove that:
$$\operatorname{Tr}\,(T-\mathbb{E}Q)=\frac{\operatorname{Tr}(R-\mathbb{E}Q)}{1-\frac{1}{n}\frac{\operatorname{Tr}TAA^*R}{(1+\delta)(1+\alpha)}+\omega\frac{1}{n}\operatorname{Tr}TR}. \tag{7.2}$$
and then we prove that the denominator verify,
$$1-\frac{1}{n}\frac{\operatorname{Tr}TAA^*R}{(1+\delta)\,(1+\alpha)}+\omega\frac{1}{n}\operatorname{Tr}TR \;=\; \frac{1+\omega(1+\delta)\tilde{\gamma}+\omega(1+\tilde{\delta})\gamma}{1+\tilde{\delta}}+\varepsilon, \tag{7.3}$$
where, $\varepsilon$ converges to zero in probability.
- We deal with the term $\operatorname{Tr}(R-\mathbb{E}Q)$ and we show that it verify:
$$\begin{aligned}\operatorname{Tr}\,(R-\mathbb{E}Q) &= \kappa\omega^2(1+\tilde{\delta})\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2\frac{1}{n}\operatorname{Tr}\,ST^2 \\ &\quad+\omega^2(1+\delta)\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}\tilde{T}^2-\omega\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2+\varepsilon, \end{aligned}\tag{7.4}$$
where $\varepsilon$ is a random variable which converges to zero in probability.

7.1. **Proof of 7.2.** We have,

$$\operatorname{Tr}\,(T-\mathbb{E}Q)=\operatorname{Tr}\,(T-R)+\operatorname{Tr}\,(R-\mathbb{E}Q)\,.$$

On the other hand, with the help of the resolvent identity, we can prove easily that:

$$\operatorname{Tr}\,(T-R)=\omega\operatorname{Tr}\left(\mathbb{E}\tilde{Q}-\tilde{T}\right)\frac{1}{n}\operatorname{Tr}\,RT+\operatorname{Tr}\,(T-\mathbb{E}Q)\,\frac{1}{n}\frac{\operatorname{Tr}\,RAA^*T}{1+\delta}.$$

In appendix A, we prove the following identity:

$$\operatorname{Tr}\,(T-Q)=\operatorname{Tr}\left(\tilde{T}-\tilde{Q}\right). \tag{7.5}$$

We then have,

$$\operatorname{Tr}\,(T-\mathbb{E}Q)=\frac{\operatorname{Tr}(R-\mathbb{E}Q)}{1-\frac{1}{n}\frac{\operatorname{Tr}TAA^*R}{\left(1+\frac{1}{n}\operatorname{Tr}T\right)\left(1+\frac{1}{n}\operatorname{Tr}\mathbb{E}Q\right)}+\omega\frac{1}{n}\operatorname{Tr}TR}. \tag{7.6}$$

Results of [20] allow us to make the following approximations:

$$1-\frac{1}{n}\frac{\operatorname{Tr}TAA^*R}{\left(1+\frac{1}{n}\operatorname{Tr}T\right)\left(1+\frac{1}{n}\operatorname{Tr}\mathbb{E}Q\right)}+\omega\frac{1}{n}\operatorname{Tr}TR=1-\frac{1}{n}\frac{\operatorname{Tr}\,AA^*T^2}{(1+\delta)^2}+\omega\gamma+\varepsilon,$$

where $\varepsilon$ converges to zero in probability.

To prove equality (7.3) we use Woodbury's identity which gives:

$$\frac{A^*T^2A}{1+\delta}=\frac{A^*\left(\omega(1+\tilde{\delta})I_N+\frac{AA^*}{1+\delta}\right)^{-2}A}{(1+\delta)^2}=\frac{1}{1+\delta}\tilde{T}-\omega\frac{1+\delta}{1+\tilde{\delta}}\tilde{T}^2.$$

The proof of the first step is done.

7.2. **Proof of 7.4.** We shall develop $\mathrm{Tr}\,(\mathbb{E}Q - R)$ as

$$\mathrm{Tr}\,(\mathbb{E}Q - R) = \chi_1 + \chi_2 + \chi_3 + \chi_4 \tag{7.7}$$

where the expressions of $\chi_i$, for $i = 1 : 4$, will be given when requered.
In all this section, $\tilde{b}_j$ and $e_j$ will refer, respectively, to $\tilde{b}_j = \left(\omega\left(1 + \frac{1}{n}\,\mathrm{Tr}\,Q_j + a_j^* Q_j a_j\right)\right)^{-1}$ and $e_j = \eta_j^* Q_j \eta_j - \frac{1}{n}\,\mathrm{Tr}\,Q_j - a_j^* Q_j a_j$.
Let us begin by using the resolvent identity to develop the term in the left hand side of (7.7). We have,

$$\begin{aligned}
\mathrm{Tr}\,(\mathbb{E}Q - R) &= \mathbb{E}\,\mathrm{Tr}\,R\left(R^{-1} - Q^{-1}\right)Q \\
&= \mathbb{E}\,\mathrm{Tr}\,R\left(\omega(1+\tilde{\alpha})I_N + \frac{AA^*}{1+\alpha} - \omega I_N - \Sigma\Sigma^*\right)Q \\
&= \left(\omega\tilde{\alpha}\mathbb{E}\,\mathrm{Tr}\;RQ + \frac{1}{1+\alpha}\,\mathrm{Tr}\;RAA^*\mathbb{E}Q\right) - \mathrm{Tr}\;R\mathbb{E}\Sigma\Sigma^*Q = X_1 + X_2.
\end{aligned}$$

$X_2$ can be handled as,

$$\begin{aligned}
X_2 &= -\,\mathrm{Tr}\;R\mathbb{E}\Sigma\Sigma^*Q = -\mathbb{E}\sum_{j=1}^{n}\eta_j^* Q R\eta_j \\
&= -\mathbb{E}\sum_{j=1}^{n}\left(\eta_j^* Q_j R\eta_j - \omega\tilde{q}_{jj}\eta_j^* Q_j\eta_j\eta_j^* Q_j R\eta_j\right) \\
&= \omega\sum_{j=1}^{n}\mathbb{E}\left((\tilde{q}_{jj} - \tilde{b}_j)\eta_j^* Q_j\eta_j\eta_j^* Q_j R\eta_j\right) \\
&\quad + \sum_{j=1}^{n}\mathbb{E}\left(-\frac{1}{n}\,\mathrm{Tr}\;Q_j R - a_j^* Q_j R a_j + \omega\tilde{b}_j\eta_j^* Q_j\eta_j\eta_j^* Q_j R\eta_j\right) \\
&= \chi_1 + X_3.
\end{aligned}$$

and we have,

$$\begin{aligned}
X_3 &= \sum_{j=1}^{n}\mathbb{E}\left[-\frac{1}{n}\,\mathrm{Tr}\,Q_j R - a_j^* Q_j R a_j + \omega\tilde{b}_j\eta_j^* Q_j\eta_j\eta_j^* Q_j R\eta_j\right] \\
&= \sum_{j=1}^{n}\mathbb{E}\left[\omega\tilde{b}_j\left(\eta_j^* Q_j\eta_j - \frac{1}{n}\,\mathrm{Tr}\,Q_j - a_j^* Q_j a_j\right)\left(\eta_j^* Q_j R a_j - \frac{1}{n}\,\mathrm{Tr}\,Q_j R - a_j^* Q_j R a_j\right)\right] \\
&\quad + \sum_{j=1}^{n}\mathbb{E}\left[\omega\tilde{b}_j\left(\frac{1}{n}\,\mathrm{Tr}\,Q_j + a_j^* Q_j a_j\right)\left(\frac{1}{n}\,\mathrm{Tr}\,Q_j R + a_j^* Q_j R a_j\right) - \left(\frac{1}{n}\,\mathrm{Tr}\,Q_j R + a_j^* Q_j R a_j\right)\right] \\
&= \sum_{j=1}^{n}\mathbb{E}\left[\omega\tilde{b}_j\left(\eta_j^* Q_j\eta_j - \frac{1}{n}\,\mathrm{Tr}\,Q_j - a_j^* Q_j a_j\right)\left(\eta_j^* Q_j R a_j - \frac{1}{n}\,\mathrm{Tr}\,Q_j R - a_j^* Q_j R a_j\right)\right] \\
&\quad - \sum_{j=1}^{n}\mathbb{E}\left[\omega\tilde{b}_j\left(\frac{1}{n}\,\mathrm{Tr}\,Q_j R + a_j^* Q_j R a_j\right)\right] \\
&= \chi_2 + X_4.
\end{aligned}$$

$X_1 + X_4$ can be decomposed as:

$$\begin{aligned} X_1 + X_4 &= \left(\omega\tilde{\alpha}\mathbb{E}\,\mathrm{Tr}\; RQ + \frac{1}{1+\alpha}\,\mathrm{Tr}\; RAA^*\mathbb{E}Q\right) - \sum_{j=1}^{n}\mathbb{E}\left[\omega\tilde{b}_j\left(\frac{1}{n}\,\mathrm{Tr}\,Q_jR + a_j^*Q_jRa_j\right)\right] \\ &= \left(\omega\tilde{\alpha}\mathbb{E}\,\mathrm{Tr}\,RQ - \frac{\omega}{n}\sum_{j=1}^{n}\mathbb{E}\tilde{b}_j\,\mathrm{Tr}\,Q_jR\right) + \left(\frac{1}{1+\alpha}\,\mathrm{Tr}\,RAA^*EQ - \sum_{j=1}^{n}\omega a_j^*\mathbb{E}\tilde{b}_jQ_jRa_j\right) = \chi_3 + \chi_4. \end{aligned}$$

Hence, $\mathrm{Tr}\,(\mathbb{E}Q - R)$ is given by:

$$\mathrm{Tr}\,(\mathbb{E}Q - R) = \chi_1 + \chi_2 + \chi_3 + \chi_4. \tag{7.8}$$

In order to prove formula (7.4), we shall study separately terms in the right hand side of (7.8). Lemma **??**-2 will be of constant use in this study and, up to arguments in subsection (**??**), terms on $\zeta = \mathbb{E}|X_{11}|^2X_{11}$ will be considered as $\varepsilon \to 0$ in probability.
We begin by the treatment of $\chi_1$. We have,

$$\begin{aligned} \chi_1 &= \omega\sum_{j=1}^{n}\mathbb{E}\left((\tilde{q}_{jj} - \tilde{b}_j)\eta_j^*Q_j\eta_j\eta_j^*Q_jR\eta_j\right) \\ &= \omega\sum_{j=1}^{n}\mathbb{E}\left((\tilde{q}_{jj} - \tilde{b}_j)\left(\frac{1}{n}\,\mathrm{Tr}\,Q_j + a_j^*Q_ja_j\right)\left(\eta_j^*Q_jR\eta_j - \frac{1}{n}\,\mathrm{Tr}\,Q_jR - a_j^*Q_jRa_j\right)\right) \\ &\quad + \left[\omega\sum_{j=1}^{n}\mathbb{E}\left((\tilde{q}_{jj} - \tilde{b}_j)(\eta_j^*Q_j\eta_j - \frac{1}{n}\,\mathrm{Tr}\,Q_j - a_j^*Q_ja_j)\left(\frac{1}{n}\,\mathrm{Tr}\,Q_jR + a_j^*Q_jRa_j\right)\right)\right. \\ &\quad \left. + \omega\sum_{j=1}^{n}\mathbb{E}\left((\tilde{q}_{jj} - \tilde{b}_j)\left(\frac{1}{n}\,\mathrm{Tr}\,Q_j + a_j^*Q_ja_j\right)\left(\frac{1}{n}\,\mathrm{Tr}\,Q_jR + a_j^*Q_jRa_j\right)\right)\right] + \varepsilon \\ &= \chi_{11} + \chi_{12} + \varepsilon_1. \end{aligned}$$

where,

$$\mathbb{E}|\varepsilon_1| \le \sum_{j=1}^{n}\mathbb{E}\left|\omega(\tilde{q}_{jj} - \tilde{b}_j)\left(\eta_j^*Q_j\eta_j - \frac{1}{n}\,\mathrm{Tr}\,Q_j - a_j^*Q_ja_j\right)\left(\eta_j^*Q_jR\eta_j - \frac{1}{n}\,\mathrm{Tr}\,Q_jR - a_j^*Q_jRa_j\right)\right| \le \frac{K}{n}.$$

Treatment of $\chi_{11}$. Recall that: $\tilde{q}_{jj} - \tilde{b}_j = -\omega\tilde{b}_j^2e_j + \varepsilon_j$, where, $\mathbb{E}|\varepsilon_j| \le \frac{K}{n}$. Let $f_j = \frac{1}{n}\,\mathrm{Tr}\,Q_j + a_j^*Q_ja_j$ and $g_j = \frac{1}{n}\,\mathrm{Tr}\,Q_jR + a_j^*Q_jRa_j$, $\chi_{11}$ becomes:

$$\chi_{11} = -\sum_{j=1}^{n}\mathbb{E}\left(\omega^2\tilde{b}_j^2f_je_j\left(\eta_j^*Q_jR\eta_j - \mathbb{E}\eta_j^*Q_jR\eta_j\right)\right) + \varepsilon_{11}^1,$$

with $\mathbb{E}|\varepsilon_{11}| \le \frac{K}{\sqrt{n}}$ by lemma (**??**-(1)).
Using lemma (**??**-(2)), with $M = Q_j$ and $P = Q_jR$, we obtain,

$$\begin{aligned} \chi_{11} &= -\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\omega^2\tilde{b}^2f_j\left(\frac{1}{n}\,\mathrm{Tr}\,Q_j^2R + a_j^*Q_jRQ_ja_j + a_j^*Q_j^2Ra_j\right.\right. \\ &\quad \left.\left. + \frac{|\vartheta|^2}{n}\,\mathrm{Tr}\,Q_jR_j^tQ_j^t + \vartheta a_j^*Q_jRQ_j^ta_j + \bar{\vartheta}a_j^tQ_j^tQ_jRa_j + \frac{\kappa}{n}\sum_{i=1}^{N}[Q_j]_{ii}[Q_jR]_{ii}\right)\right) + \varepsilon_{11}, \end{aligned}$$

where, $\varepsilon_{11}$ converges to zero in probability.
We turn now to the treatment of $\chi_{12}$. We have,

$$\begin{aligned}\chi_{12} &= \omega\sum_{j=1}^n \mathbb{E}\left((\tilde{q}_{jj}-\tilde{b}_j)g_j(\eta_j^*Q_j\eta_j - f_j)\right) + \sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 e_j^2 f_j g_j\right)\\ &= -\sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 e_j^2 g_j\right) + \sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 e_j^2 f_j g_j\right) + \varepsilon_{12}^1\\ &\overset{(a)}{=} -\sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 e_j^2 g_j\right) + \varepsilon_{12},\end{aligned}$$

where (a) follows from the fact that $\omega^2\tilde{b}_j^2 f_j = \omega\tilde{b}_j(1-\omega\tilde{b}_j)$ and $\varepsilon_{12}^1$ converges to zero in probability. Lemma (**??**-(2)) imply,

$$\chi_{12} = -\frac{1}{n}\sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 g_j\left(\frac{1}{n}\operatorname{Tr} Q_j^2 + 2a_j^*Q_j^2a_j + \frac{|\vartheta|^2}{n}\operatorname{Tr} Q_jQ_j^t + 2\operatorname{Re}(\vartheta a_j^*Q_jQ_j^t\bar{a}_j) + \frac{\kappa}{n}\sum_{i=1}^N [Q_j]_{ii}^2\right)\right) + \varepsilon_{12}.$$

Then, $\chi_1$ becomes

$$\begin{aligned}\chi_1 &= -\frac{1}{n}\sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 f_j\left(\frac{1}{n}\operatorname{Tr} Q_j^2R + a_j^*Q_jRQ_ja_j + a_j^*Q_j^2Ra_j\right) + \omega^3\tilde{b}_j^3 g_j\left(\frac{1}{n}\operatorname{Tr} Q_j^2 + 2a_j^*Q_j^2a_j\right)\right)\\ &\quad -\frac{\kappa}{n}\sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 f_j\frac{1}{n}\sum_{i=1}^N [Q_j]_{ii}[Q_jR]_{ii} + \omega^3\tilde{b}_j^3 g_j\frac{1}{n}\sum_{i=1}^N [Q_j]_{ii}^2\right) + \chi_1(\vartheta) + \varepsilon_1,\end{aligned}$$

where,

$$\begin{aligned}\chi_1(\vartheta) &= -\frac{1}{n}\sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 f_j\left(\frac{|\vartheta|^2}{n}\operatorname{Tr} Q_jR_j^tQ_j^t + \vartheta a_j^*Q_jRQ_j^ta_j + \bar{\vartheta}a_j^tQ_j^tQ_jRa_j\right)\right)\\ &\quad -\frac{1}{n}\sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 g_j\left(\frac{|\vartheta|^2}{n}\operatorname{Tr} Q_jQ_j^t + 2\operatorname{Re}(\vartheta a_j^*Q_jQ_j^t\bar{a}_j)\right)\right).\end{aligned}$$

Treatment of $\chi_2$. We use again lemma (**??**-(2)),

$$\chi_2 = \frac{1}{n}\sum_{j=1}^n \mathbb{E}\left\{\omega\tilde{b}_j\left(\frac{1}{n}\operatorname{Tr} Q_j^2R + a_j^*Q_jRQ_ja_j + a_j^*Q_j^2Ra_j + \frac{\kappa}{n}\sum_{i=1}^N [Q_j]_{ii}[Q_jR]_{ii}\right)\right\} + \chi_2(\vartheta) + \varepsilon_2$$

where,

$$\chi_2(\vartheta) = \frac{1}{n}\sum_{j=1}^n \mathbb{E}\left(\omega\tilde{b}_j\left(\frac{|\vartheta|^2}{n}\operatorname{Tr} Q_jR^tQ_j^t + \vartheta a_j^*Q_jRQ_j^t\bar{a}_j + \bar{\vartheta}a_j^tQ_j^tQ_jRa_j\right)\right).$$

Treatment of $\chi_3$:

$$\begin{aligned}\chi_3 &= \frac{\omega}{n}\sum_{j=1}^{n}\mathbb{E}\left(\mathbb{E}\tilde{q}_{jj}\operatorname{Tr}RQ-\tilde{b}_j\operatorname{Tr}RQ_j\right)\\ &= \frac{\omega}{n}\sum_{j=1}^{n}\mathbb{E}\left(\mathbb{E}\tilde{q}_{jj}\left(\operatorname{Tr}RQ_j-\omega\tilde{q}_{jj}\operatorname{Tr}RQ_j\eta_j\eta_j^*Q_j\right)-\tilde{b}_j\operatorname{Tr}RQ_j\right)\\ &= \frac{\omega}{n}\sum_{j=1}^{n}\mathbb{E}\left((\mathbb{E}\tilde{q}_{jj}-\tilde{b}_j)\operatorname{Tr}RQ_j\right)-\frac{\omega^2}{n}\sum_{j=1}^{n}\mathbb{E}\tilde{q}_{jj}\mathbb{E}\left(\tilde{b}_j\eta_j^*Q_jRQ_j\eta_j\right)+\varepsilon_3\end{aligned}$$

We have,

$$\begin{aligned}\mathbb{E}\left(\tilde{q}_{jj}-\tilde{b}_j\right) &= \omega^2\mathbb{E}\left(\tilde{b}_j^3e_j^2\right)+\mathcal{O}(n^{-3/2})\\ &= \omega^2\mathbb{E}\left(\tilde{b}_j^3\left(\eta_j^*Q_j\eta_j-\frac{1}{n}\operatorname{Tr}Q_j-a_j^*Q_ja_j\right)^2\right)+\mathcal{O}(n^{-3/2})\\ &= \omega^2\mathbb{E}\left\{\tilde{b}_j^3\left(\frac{1}{n^2}\operatorname{Tr}Q_j^2+2\frac{1}{n}a_j^*Q_j^2a_j+\frac{|\vartheta|^2}{n^2}\operatorname{Tr}Q_jQ_j^t+2\operatorname{Re}(\frac{\vartheta}{n}a_j^*Q_jQ_j^ta_j)\right.\right.\\ &\quad\left.\left.+\frac{\kappa}{n^2}\sum_{i=1}^{N}[Q_j]_{ii}^2\right)\right\}+\varepsilon_j.\end{aligned}$$

we then obtain,

$$\begin{aligned}\chi_3 &= \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\omega^3\tilde{b}_j^3}{n}\operatorname{Tr}RQ_j\left\{\frac{1}{n}\operatorname{Tr}Q_j^2+2a_j^*Q_j^2a_j+\frac{\kappa}{n}\sum_{i=1}^{N}[Q_j]_{ii}^2\right\}\right]\\ &\quad-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\omega^2\tilde{b}_j^2\frac{1}{n}\operatorname{Tr}Q_jRQ_j+a_j^*Q_jRQ_ja_j\right)+\chi_3(\vartheta)+\varepsilon,\end{aligned}$$

where,

$$\chi_3(\vartheta) = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\omega^3\tilde{b}_j^3}{n}\operatorname{Tr}RQ_j\left(\frac{|\vartheta|^2}{n}\operatorname{Tr}Q_jQ_j^t+2\operatorname{Re}(\vartheta a_j^*Q_jQ_j^ta_j)\right)\right].$$

We turn now to $\chi_4$. We have,

$$\begin{aligned}\chi_4 &= \frac{1}{1+\alpha}\operatorname{Tr}RAA^*EQ-\sum_{j=1}^{n}\omega a_j^*\mathbb{E}(\tilde{b}_jQ_j)Ra_j\\ &= \sum_{j=1}^{n}\mathbb{E}\left(\frac{1}{1+\alpha}\left(a_j^*Q_jRa_j-\omega\tilde{q}_{jj}a_j^*Q_j\eta_j\eta_j^*Q_jRa_j\right)-\omega\tilde{b}_ja_j^*Q_jRa_j\right)\end{aligned}$$

using identity $\tilde{q}_{jj}=\tilde{b}_j-\omega\tilde{b}_j^2e_j+\omega^2\tilde{b}_j^3e_j^2+\mathcal{O}(n^{3/2})$, we obtain,

$$\begin{aligned}\chi_4 &= \sum_{j=1}^{n}\mathbb{E}\left[a_j^*Q_jRa_j\left(\frac{1-\omega\tilde{b}_ja_j^*Q_ja_j}{1+\alpha}-\omega\tilde{b}_j\right)\right]-\frac{1}{n(1+\alpha)}\sum_{j=1}^{n}\mathbb{E}\left(\omega\tilde{b}_ja_j^*Q_j^2Ra_j\right)\\ &\quad+\frac{1}{1+\alpha}\sum_{j=1}^{n}\omega^2\mathbb{E}\left(\tilde{b}_j^2e_ja_j^*Q_j\eta_j\eta_j^*Q_jRa_j\right)-\frac{1}{1+\alpha}\sum_{j=1}^{n}\mathbb{E}\left(\omega^3\tilde{b}_j^3e_j^2a_j^*Q_j\eta_j\eta_j^*Q_jRa_j\right)+\varepsilon_4\\ &= W_1+W_2+W_3+W_4+\varepsilon_4,\end{aligned}$$

where $\mathbb{E}|\varepsilon_4| \leq \frac{K}{\sqrt{n}}$.
Let us begin by the treatment of $W_1$. We have,

$$\frac{1-\omega\tilde{b}_j a_j^* Q_j a_j}{1+\alpha} - \omega\tilde{b}_j = \omega\tilde{b}_j \frac{\frac{1}{n}\operatorname{Tr}(Q_j - \mathbb{E}Q)}{1+\alpha} = \omega\tilde{b}_j \frac{\frac{1}{n}\operatorname{Tr}\mathbb{E}(Q - Q_j)}{1+\alpha} + \omega\tilde{b}_j \frac{\frac{1}{n}\operatorname{Tr}(Q_j - \mathbb{E}Q_j)}{1+\alpha},$$

then,

$$\begin{aligned} W_1 &= \frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega\tilde{b}_j a_j^* Q_j R a_j\right) \mathbb{E}\left(\omega\tilde{q}_{jj}\eta_j^* Q_j^2 \eta_j\right) + \frac{1}{1+\alpha} \sum_{j=1}^n \mathbb{E}\left(\omega\tilde{b}_j a_j^* Q_j R a_j \frac{1}{n}\operatorname{Tr}(Q_j - \mathbb{E}Q_j)\right) \\ &= \frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 a_j^* Q_j R a_j \left(a_j^* Q_j^2 a_j + \frac{1}{n}\operatorname{Tr} Q_j^2\right)\right) + \varepsilon, \end{aligned}$$

where $\varepsilon$ ?.
Let us now deal with $W_3$. We have,

$$\begin{aligned} W_3 &= \frac{1}{1+\alpha} \sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 e_j \left(\eta_j^* Q_j R a_j a_j^* Q_j \eta_j - \mathbb{E}\eta_j^* Q_j R a_j a_j^* Q_j \eta_j\right)\right) \\ &= \frac{1}{1+\alpha} \sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 \left(\frac{1}{n^2} a_j^* Q_j^3 R a_j + \frac{1}{n} a_j^* Q_j R a_j a_j^* Q_j^2 a_j + \frac{1}{n} a_j^* Q_j^2 R a_j a_j^* Q_j a_j \right.\right. \\ &\quad \left.\left. + \frac{|\vartheta|^2}{n^2} a_j^t R^t Q_j^t Q_j Q_j^t \bar{a}_j + \frac{\vartheta}{n} a_j^* Q_j R a_j a_j^* Q_j Q_j^t \bar{a}_j + \frac{\bar{\vartheta}}{n} a_j^t Q_j^t Q_j R a_j a_j^* Q_j a_j + \frac{\kappa}{n^2} \sum_{i=1}^N [Q_j]_{ii} \left[Q_j R a_j a_j^* Q_j\right]_{ii}\right)\right) \\ &= \frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left[\omega^2\tilde{b}_j^2 \left(a_j^* Q_j^2 R a_j a_j^* Q_j a_j + a_j^* Q_j R a_j a_j^* Q_j^2 a_j + \vartheta a_j^* Q_j R a_j a_j^* Q_j Q_j^t \bar{a}_j + \bar{\vartheta} a_j^t Q_j^t Q_j R a_j a_j^* Q_j a_j\right)\right] + \varepsilon. \end{aligned}$$

Treatment of $W_4$. We have,

$$\begin{aligned} W_4 &= -\frac{1}{1+\alpha} \sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 e_j^2 \left(a_j^* Q_j R a_j a_j^* Q_j a_j\right)\right) + \varepsilon \\ &= -\frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 a_j^* Q_j R a_j a_j^* Q_j a_j \left(\frac{1}{n}\operatorname{Tr} Q_j^2 + 2a_j^* Q_j^2 a_j + \frac{|\vartheta|^2}{n}\operatorname{Tr} Q_j Q_j^t + 2\operatorname{Re}(\vartheta a_j^* Q_j Q_j^t \bar{a}_j)\right)\right) \\ &\quad -\frac{\kappa}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 a_j^* Q_j R a_j a_j^* Q_j a_j \frac{1}{n} \sum_{i=1}^N [Q_j]_{ii}^2\right) + \varepsilon, \end{aligned}$$

We put terms toghether, we get,

$$\begin{aligned} \chi_4 &= \frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega^2\tilde{b}_j^2 a_j^* Q_j R a_j \left(a_j^* Q_j^2 a_j + \frac{1}{n}\operatorname{Tr} Q_j^2\right)\right) - \frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega\tilde{b}_j a_j^* Q_j^2 R a_j\right) \\ &\quad + \frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left[\omega^2\tilde{b}_j^2 \left(a_j^* Q_j^2 R a_j a_j^* Q_j a_j + a_j^* Q_j R a_j a_j^* Q_j^2 a_j + \vartheta a_j^* Q_j R a_j a_j^* Q_j Q_j^t \bar{a}_j + \bar{\vartheta} a_j^t Q_j^t Q_j R a_j a_j^* Q_j a_j\right)\right] \\ &\quad -\frac{1}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 a_j^* Q_j R a_j a_j^* Q_j a_j \left(\frac{1}{n}\operatorname{Tr} Q_j^2 + 2a_j^* Q_j^2 a_j + \frac{|\vartheta|^2}{n}\operatorname{Tr} Q_j Q_j^t + 2\operatorname{Re}(\vartheta a_j^* Q_j Q_j^t \bar{a}_j)\right)\right) \\ &\quad -\frac{\kappa}{n(1+\alpha)} \sum_{j=1}^n \mathbb{E}\left(\omega^3\tilde{b}_j^3 a_j^* Q_j R a_j a_j^* Q_j a_j \frac{1}{n} \sum_{i=1}^N [Q_j]_{ii}^2\right) + \varepsilon. \end{aligned}$$

Straightforward computations based on the identity: $\omega\tilde{b}_j\left(1+\frac{a_j^*Q_ja_j}{1+\alpha}\right)=(1+\alpha)^{-1}+\mathcal{O}(n^{-1/2})$ yield,

$$\begin{aligned}\chi_4 &= \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left[\omega^3\tilde{b}_j^3a_j^*Q_jRa_j\left(\frac{1}{n}\operatorname{Tr}Q_j^2+2a_j^*Q_j^2a_j\right)-\omega^2\tilde{b}_j^2a_j^*Q_j^2Ra_j\right]\\ &\quad-\frac{\kappa}{n(1+\alpha)}\sum_{j=1}^{n}\mathbb{E}\left(\omega^3\tilde{b}_j^3a_j^*Q_jRa_ja_j^*Q_ja_j\frac{1}{n}\sum_{i=1}^{N}[Q_j]_{ii}^2\right)+\chi_4(\vartheta)+\varepsilon,\end{aligned}$$

where,

$$\begin{aligned}\chi_4(\vartheta) &= |\vartheta|^2\left(-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\frac{\omega^3\tilde{b}_j^3}{1+\alpha}a_j^*Q_ja_j\frac{1}{n}\operatorname{Tr}Q_jQ_j^ta_j^*Q_jRa_j\right)\right)\\ &\quad+\vartheta\left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\frac{\omega^2\tilde{b}_j^2}{1+\alpha}a_j^*Q_jRa_ja_j^*Q_jQ_j^t\bar{a}_j\right)-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\frac{\omega^3\tilde{b}_j^3}{1+\alpha}a_j^*Q_ja_ja_j^*Q_jRa_ja_j^*Q_jQ_j^t\bar{a}_j\right)\right)\\ &\quad+\bar{\vartheta}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\frac{\omega^2\tilde{b}_j^2}{1+\alpha}a_j^*Q_ja_ja_j^tQ_j^tQ_jRa_j\right)-\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left(\frac{\omega^3\tilde{b}_j^3}{1+\alpha}a_j^*Q_ja_ja_j^*Q_jRa_ja_j^*Q_jQ_j^t\bar{a}_j\right)\right).\end{aligned}$$

**Final expression for** $\operatorname{Tr}(T-\mathbb{E}Q)$**.** The aim of this part is to prove that:

$$\operatorname{Tr}(\mathbb{E}Q-R)=\kappa\left(\omega^2(1+\tilde{\delta})\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2\frac{1}{n}\operatorname{Tr}\,ST^2+\omega^2(1+\delta)\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}\tilde{T}^2-\omega\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2\right)+\varepsilon, \tag{7.9}$$

where $\varepsilon$ converges to zero in probability.
For this end, two steps will be verified:

- The first step consists to prove that:

$$\operatorname{Tr}(\mathbb{E}Q-R)=\frac{\kappa}{n}\sum_{j=1}^{n}\omega^2\tilde{b}_j^2\left[\left(\frac{1}{n}\sum_{i=1}^{N}[Q_j]_{ii}[Q_jR]_{ii}\right)-\frac{a_j^*Q_jRa_j}{1+\alpha}\frac{1}{n}\sum_{i=1}^{N}[Q_j]_{ii}^2\right]+\varepsilon.$$

- By using standard approximations, especially the asymptotic behavior of the entries of the resolvent matrices $Q$, $Q_j$ and $R$, we get the desired formula for $\operatorname{Tr}(\mathbb{E}Q-R)$.

Let us begin by the first step. By using the identity: $\omega\tilde{b}_j f_j = 1 - \omega\tilde{b}_j$, it is easily to see that terms which not depend on $\vartheta$ nor on $\kappa$ vanish. Let us now deal with terms on $\vartheta$. We have,

$$\begin{aligned}
&\chi_1(\vartheta) + \chi_2(\vartheta) + \chi_3(\vartheta) + \chi_4(\vartheta) \\
= \;& \frac{|\vartheta|^2}{n}\sum_{j=1}^n \mathbb{E}\left( -\omega^2\tilde{b}_j^2 f_j \frac{1}{n}\operatorname{Tr} Q_j R_j^t Q_j^t - \omega^3\tilde{b}_j^3 g_j \frac{1}{n}\operatorname{Tr} Q_j Q_j^t + \omega\tilde{b}_j \frac{1}{n}\operatorname{Tr} Q_j R^t Q_j^t + \frac{\omega^3\tilde{b}_j^3}{n}\operatorname{Tr} R Q_j \frac{1}{n}\operatorname{Tr} Q_j Q_j^t \right. \\
& \left. - \frac{\omega^3\tilde{b}_j^3}{1+\alpha} a_j^* Q_j a_j \frac{1}{n}\operatorname{Tr} Q_j Q_j^t a_j^* Q_j R a_j \right) \\
& + \frac{\vartheta}{n}\sum_{j=1}^n \mathbb{E}\left( -\omega^2\tilde{b}_j^2 f_j a_j^* Q_j R Q_j^t a_j - \omega^3\tilde{b}_j^3 g_j a_j^* Q_j Q_j^t \bar{a}_j + \omega\tilde{b}_j a_j^* Q_j R Q_j^t \bar{a}_j + \frac{\omega^3\tilde{b}_j^3}{n}\operatorname{Tr} R Q_j a_j^* Q_j Q_j^t a_j \right. \\
& \left. + \frac{\omega^2\tilde{b}_j^2}{1+\alpha} a_j^* Q_j R a_j a_j^* Q_j Q_j^t \bar{a}_j - \frac{\omega^3\tilde{b}_j^3}{1+\alpha} a_j^* Q_j a_j a_j^* Q_j R a_j a_j^* Q_j Q_j^t \bar{a}_j \right) \\
& + \frac{\bar{\vartheta}}{n}\sum_{j=1}^n \mathbb{E}\left( -\omega^2\tilde{b}_j^2 f_j a_j^t Q_j^t Q_j R a_j - \omega^3\tilde{b}_j^3 g_j a_j^t Q_j^t Q_j a_j + \omega\tilde{b}_j a_j^t Q_j^t Q_j R a_j + \frac{\omega^3\tilde{b}_j^3}{n}\operatorname{Tr} R Q_j a_j^* Q_j Q_j^t a_j \right. \\
& \left. + \frac{\omega^2\tilde{b}_j^2}{1+\alpha} a_j^* Q_j a_j a_j^t Q_j^t Q_j R a_j - \frac{\omega^3\tilde{b}_j^3}{1+\alpha} a_j^* Q_j a_j a_j^* Q_j R a_j a_j^* Q_j Q_j^t \bar{a}_j \right) = \varepsilon..
\end{aligned}$$

We therfore obtain,

$$\operatorname{Tr}(\mathbb{E}Q - R) \;=\; \frac{\kappa}{n}\sum_{j=1}^n \omega^2\tilde{b}_j^2 \left[ \left( \frac{1}{n}\sum_{i=1}^N [Q_j]_{ii}[Q_j R]_{ii} \right) - \frac{a_j^* Q_j R a_j}{1+\alpha}\frac{1}{n}\sum_{i=1}^N [Q_j]_{ii}^2 \right] + \varepsilon.$$

where $\varepsilon$ converges to zero in probability. Standard approximations yield,

$$\operatorname{Tr}(\mathbb{E}Q - R) \;=\; \frac{\kappa}{n}\sum_{j=1}^n \omega^2\tilde{t}_{jj}^2 \left( \frac{1}{n}\sum_{i=1}^N t_{ii}[T^2]_{ii} \right) - \frac{\omega^2\tilde{t}_{jj}^2 a_j^* \mathcal{T}_j T a_j}{1+\delta}\frac{1}{n}\sum_{i=1}^N [T^2]_{ii} + \varepsilon.$$

Furthermore, we have the following identities,

$$a_j^* \mathcal{T}_j a_j = \frac{a_j^* T a_j}{\omega t_{jj}(1+\delta)} \quad \text{and} \quad \frac{A^* T^2 A}{(1+\delta)^2} = \frac{1}{1+\tilde{\delta}}\tilde{T} - \omega\frac{1+\delta}{1+\tilde{\delta}}\tilde{T}^2,$$

We then obtain,

$$\begin{aligned}
\operatorname{Tr}(\mathbb{E}Q - R) \;=\; & \frac{\kappa}{1+\tilde{\delta}}\left( \omega^2(1+\tilde{\delta})\frac{1}{n}\sum_{j=1}^n \tilde{t}_{jj}^2 \frac{1}{n}\sum_{i=1}^N t_{ii}[T]_{ii}^2 + \omega^2(1+\delta)\frac{1}{n}\sum_{j=1}^n \tilde{t}_{jj}[T]_{jj}^2 \sum_{i=1}^N t_{ii}^2 \right. \\
& \left. - \omega\frac{1}{n}\sum_{j=1}^n \tilde{t}_{jj}^2 \sum_{i=1}^N t_{ii}^2 \right) + \varepsilon.
\end{aligned}$$

Gathering this last identity into (7.6) we get,

$$\operatorname{Tr}(T - \mathbb{E}Q) \;=\; \kappa\frac{\omega^2(1+\tilde{\delta})\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2\frac{1}{n}\operatorname{Tr}\,ST^2 + \omega^2(1+\delta)\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}\tilde{T}^2 - \omega\frac{1}{n}\operatorname{Tr}\,S^2\frac{1}{n}\operatorname{Tr}\,\tilde{S}^2}{1+\omega(1+\delta)\tilde{\gamma} + \omega(1+\tilde{\delta})\gamma},$$

which ends the proof of the bias theorem.

## Appendix A. Proofs of some lemmas

A.1. **Proof of Lemma 3.4.** The two first upper bounds are easy to obtain, given that $\delta_n$ and $\tilde{\delta}_n$ are Stieltjes transforms of nonnegative measures with respective total mass $n^{-1}\operatorname{Tr} D$ and $n^{-1}\operatorname{Tr}\tilde{D}$. Now $\operatorname{Tr} DTDT \leq \boldsymbol{d}_{\max}^2 \operatorname{Tr} T^2$ by applying twice (3.19), which in turn is lower than $N\boldsymbol{d}_{\max}^2\rho^{-2}$, hence the third upper bound, and the fourth which can be proved similarly. Let us now prove the first lower bound.

$$\begin{aligned}
\operatorname{Tr} D &= \operatorname{Tr}(T^{\frac{1}{2}}DT^{\frac{1}{2}}T^{-1}) \overset{(a)}{\leq} \operatorname{Tr}(DT)\times\|T^{-1}\| \,, \\
&\leq \operatorname{Tr}(DT)\times\left(\rho(1+\tilde{\delta}_n\boldsymbol{d}_{\max})+\boldsymbol{a}_{\max}^2\|(I+\delta_n\tilde{D})^{-1}\|\right) \,, \\
&\overset{(b)}{\leq} \operatorname{Tr}(DT)\times\left(\rho+\boldsymbol{d}_{\max}\tilde{\boldsymbol{d}}_{\max}+\boldsymbol{a}_{\max}^2\right) \,,
\end{aligned}$$

where $(a)$ follows from (3.19) and $(b)$ from $\delta_n$'s upper bound. This readily yields $\delta_n$'s lower bound and $\tilde{\delta}_n$'s lower bound which can be proved similarly. The lower bound for $\gamma_n$ follows from the same ideas:

$$\begin{aligned}
\left(\frac{1}{N}\operatorname{Tr} D\right)^2 &\leq \frac{1}{N}\operatorname{Tr} D^2 = \frac{1}{N}\operatorname{Tr}(T^{\frac{1}{2}}D^2T^{\frac{1}{2}}T^{-1}) \,, \\
&\leq \frac{1}{N}\operatorname{Tr}(T^{\frac{1}{2}}D^2T^{\frac{1}{2}})\times\|T^{-1}\| = \frac{1}{N}\operatorname{Tr}(T^{\frac{1}{2}}DT^{\frac{1}{2}}T^{-1}T^{\frac{1}{2}}DT^{\frac{1}{2}})\times\|T^{-1}\| \,, \\
&\leq \frac{1}{N}\operatorname{Tr}(TDTD)\times\|T^{-1}\|^2 \,,
\end{aligned}$$

and one readily obtains $\gamma_n$'s lower bound (and similarly $\tilde{\gamma}_n$'s lower bound) using assumption **A-3** and the upper estimate previously obtained for $\|T^{-1}\|$.

The two last series of inequalities related to $n^{-1}\sum_{i=1}^N d_i^2 t_{ii}^2$ and $n^{-1}\sum_{j=1}^n \tilde{d}_j^2\tilde{t}_{jj}^2$ can be proved with similar arguments (lower bounds are in fact easier to obtain as one can directly get lower bounds for $t_{ii}$ and $\tilde{t}_{jj}$ - using (3.15) for instance).

A.2. **Proof of Lemma 3.5.** From (1.3), $TA(I+\delta\tilde{D})^{-1}A^* = I - \rho T(I+\tilde{\delta}D)$. Moreover, $(I+\delta\tilde{D})^{-1}\tilde{D} = \delta^{-1}I - \delta^{-1}(I+\delta\tilde{D})^{-1}$. Hence

$$\begin{aligned}
\frac{1}{n}\operatorname{Tr} D^{1/2}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}} &\leq \frac{1}{n\delta}\operatorname{Tr} DTA(I+\delta\tilde{D})^{-1}A^*T \\
&= 1-\frac{\rho}{n\delta}\operatorname{Tr} DT^2 - \rho\frac{\tilde{\delta}}{\delta}\gamma \qquad \text{(A.1)}
\end{aligned}$$

which proves the first assertion with the help of the results of Lemma 3.4. Similarly,

$$\frac{1}{n}\operatorname{Tr}\tilde{D}^{1/2}\tilde{T}A^*(I+\tilde{\delta}D)^{-2}DA\tilde{T}\tilde{D}^{\frac{1}{2}} \leq 1-\frac{\rho}{n\tilde{\delta}}\operatorname{Tr}\tilde{D}\tilde{T}^2 - \rho\frac{\delta}{\tilde{\delta}}\tilde{\gamma} \qquad \text{(A.2)}$$

We now show that the left hand sides of (A.1) and (A.2) are equal. Using the well known matrix identity $(I+UV)^{-1}U = U(I+VU)^{-1}$,

$$\begin{aligned}
TA(I+\delta\tilde{D})^{-1} &= \rho^{-1}(I+\tilde{\delta}D)^{-1}\left(I+\rho^{-1}A(I+\delta\tilde{D})^{-1}A^*(I+\tilde{\delta}D)^{-1}\right)^{-1}A(I+\delta\tilde{D})^{-1} \\
&= \rho^{-1}(I+\tilde{\delta}D)^{-1}A(I+\delta\tilde{D})^{-1}\left(I+\rho^{-1}A^*(I+\tilde{\delta}D)^{-1}A(I+\delta\tilde{D})^{-1}\right)^{-1} \\
&= (I+\tilde{\delta}D)^{-1}A\tilde{T},
\end{aligned}$$

and similarly, $(I+\delta\tilde{D})^{-1}A^*T = \tilde{T}A^*(I+\tilde{\delta}D)^{-1}$. Plugging these identities in the l.h.s. of (A.1), we obtain the result. As a consequence, we have

$$\left(1-\frac{1}{n}\operatorname{Tr} D^{1/2}TA(I+\delta\tilde{D})^{-2}\tilde{D}A^*TD^{\frac{1}{2}}\right)^2 \geq \left(\frac{\rho}{n\delta}\operatorname{Tr} DT^2+\rho\frac{\tilde{\delta}}{\delta}\gamma\right)\left(\frac{\rho}{n\tilde{\delta}}\operatorname{Tr}\tilde{D}\tilde{T}^2+\rho\frac{\delta}{\tilde{\delta}}\tilde{\gamma}\right)$$
$$\geq \rho^2\gamma\tilde{\gamma}+\frac{\rho}{n\delta}\operatorname{Tr} DT^2\frac{\rho}{n\tilde{\delta}}\operatorname{Tr}\tilde{D}\tilde{T}^2$$

hence, $\liminf \Delta_n > 0$ by Lemma 3.4. Lemma 3.5 is proved.

A.3. **Proof of Lemma 5.2.** Recalling the expression (3.15) of $\tilde{t}_{\ell\ell}$, we notice that $(1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell) = \rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)$ is bounded below. It results from theorem 3.3-(2) that

$$\sum_{\ell=1}^{j}\frac{\alpha_\ell\, u^*Ta_\ell\,\mathbb{E}\,[a_\ell^*Qu]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} = \sum_{\ell=1}^{j}\frac{\alpha_\ell\, u^*Ta_\ell\, a_\ell^*Tu}{\rho\tilde{t}_{\ell\ell}(1+\tilde{d}_\ell\delta)}+\mathcal{O}(n^{-1/2})$$

Moreover,

$$\sum_{\ell=1}^{j}\alpha_\ell\, u^*Ta_\ell\left(\frac{\mathbb{E}\,[a_\ell^*Qu]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell}-\mathbb{E}\,[a_\ell^*Q_\ell u]\right) = \sum_{\ell=1}^{j}\alpha_\ell\, u^*Ta_\ell\mathbb{E}\left[a_\ell^*Q_\ell u\left(\frac{1-\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell}-1\right)\right]$$
$$-\sum_{\ell=1}^{j}\alpha_\ell\, u^*Ta_\ell\frac{\mathbb{E}[\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell\, y_\ell^*Q_\ell u]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell} = \varepsilon_1+\varepsilon_2$$

We have $\varepsilon_1 = \sum_{\ell=1}^{j}\alpha_\ell\, u^*Ta_\ell\mathbb{E}\,[a_\ell^*Q_\ell u\,\xi_\ell]$ where $\mathbb{E}|\xi_\ell|^p \leq Kn^{-p/2}$ for $p\geq 2$. It results that $|\varepsilon_1| \leq \left(\sum_{\ell=1}^{j}\alpha_\ell^2|u^*Ta_\ell|^2\mathbb{E}\xi_\ell^2\right)^{1/2}\left(\sum_{\ell=1}^{j}\mathbb{E}|a_\ell^*Q_\ell u|^2\right)^{1/2} \leq K/\sqrt{n}$ by theorem 3.3-(1). By writing

$$\varepsilon_2 = -\sum_{\ell=1}^{j}\alpha_\ell\, u^*Ta_\ell\frac{\mathbb{E}\,[(\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell-\mathbb{E}[\rho\tilde{q}_{\ell\ell}a_\ell^*Q_\ell\eta_\ell])\; y_\ell^*Q_\ell u]}{1-\rho\tilde{t}_{\ell\ell}a_\ell^*\mathcal{T}_\ell a_\ell}$$

and proceeding similarly to $\varepsilon_1$, we obtain $|\varepsilon_2| \leq K/\sqrt{n}$, which completes the proof of lemma 5.2.

A.4. **Proof of Lemma 5.1.** Let us show that $\max_j \operatorname{var}(a^*\mathbb{E}_jQD\mathbb{E}_jQa) = \mathcal{O}(n^{-1})$. We have

$$\begin{aligned}
a^*\mathbb{E}_jQD\mathbb{E}_jQa &= \sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})(a^*\mathbb{E}_jQD\mathbb{E}_jQa)\\
&= \sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})\left\|D^{1/2}\mathbb{E}_j(Q_i-\rho\tilde{q}_{ii}Q_i\eta_i\eta_i^*Q_i)a\right\|^2\\
&= \sum_{i=1}^{j}(\mathbb{E}_i-\mathbb{E}_{i-1})\Big[-2\rho\operatorname{Re}\left(a^*(\mathbb{E}_jQ_i)D(\mathbb{E}_j\tilde{q}_{ii}Q_i\eta_i\eta_i^*Q_i)a\right)\\
&\qquad\qquad +\|\mathbb{E}_j(\rho\tilde{q}_{ii}\,\eta_i^*Q_ia\,D^{1/2}Q_i\eta_i)\|^2\Big]\\
&\triangleq 2\operatorname{Re}(X)+Z\;, \qquad\qquad (A.3)
\end{aligned}$$

and the variance of $a^*\mathbb{E}_j QD\mathbb{E}_j Qa$ is the sum of the variances of these martingale increments. Consider the term $X$. Recalling that $\tilde{q}_{ii} = \tilde{b}_i - \rho\tilde{q}_{ii}\tilde{b}_i e_i$,

$$X = -\rho\sum_{i=1}^{j}(\mathbb{E}_i - \mathbb{E}_{i-1})(\tilde{b}_i\,\eta_i^* Q_i aa^*(\mathbb{E}_j Q_i)DQ_i\eta_i) + \rho^2\sum_{i=1}^{j}(\mathbb{E}_i - \mathbb{E}_{i-1})(\tilde{b}_i\tilde{q}_{ii}e_i\eta_i^* Q_i aa^*(\mathbb{E}_j Q_i)DQ_i\eta_i)$$
$$\stackrel{\triangle}{=} X_1 + X_2\ .$$

Let $M_i = Q_i aa^*(\mathbb{E}_j Q_i)DQ_i$. The term $X_1$ satisfies

$$\mathbb{E}|X_1|^2 = \rho^2\sum_{i=1}^{j}\mathbb{E}\left|\mathbb{E}_i\tilde{b}_i\left(y_i^* M_i y_i - \frac{1}{n}\operatorname{Tr} M_i + y_i^* M_i a_i + a_i^* M_i y_i\right)\right|^2\ . \tag{A.4}$$

As $M_i$ is a rank one matrix, $\sum_{i=1}^{j}\mathbb{E}|y_i^* M_i y_i - \operatorname{Tr} M_i/n|^2 \le K/n$. Moreover,

$$\sum_{i=1}^{j}\mathbb{E}\,|y_i^* M_i a_i|^2 \;=\; \frac{1}{n}\sum_{i=1}^{j}\mathbb{E}\left(a^*Q_i^2 a\ \ |a^*(\mathbb{E}_j Q_i)DQ_i a_i|^2\right) \;\le\; \frac{K}{n}\sum_{i=1}^{j}\mathbb{E}\,|a^*(\mathbb{E}_j Q_i)DQ_i a_i|^2\,.$$

The summand at the r.h.s. of the inequality satisfies:

$$\begin{aligned}|a^*(\mathbb{E}_j Q_i)DQ_i a_i|^2 &\le 4\Big(|a^*(\mathbb{E}_j Q)DQa_i|^2 + |a^*(\mathbb{E}_j(Q_i - Q))D(Q_i - Q)a_i|^2 + \\ &\qquad |a^*(\mathbb{E}_j Q)D(Q_i - Q)a_i|^2 + |a^*(\mathbb{E}_j(Q_i - Q))DQa_i|^2\Big)\\ &\stackrel{\triangle}{=} 4(W_{i,1} + W_{i,2} + W_{i,3} + W_{i,4}).\end{aligned}$$

Recalling that $A_{1:j} = [a_1, \cdots, a_j]$, we have

$$\sum_{i=1}^{j} W_{i,1} = a^*(\mathbb{E}_j Q)DQA_{1:j}A_{1:j}^* QD(\mathbb{E}_j Q)a \le K\ .$$

Recalling (3.10) and (3.11), and writing $\xi_i = 1 + \eta_i^* Q_i\eta_i$, we have:

$$W_{i,2} = \frac{a_i^* Q\eta_i\eta_i^* Qa_i}{1 - \eta_i^* Q\eta_i} \times a^*\mathbb{E}_j\left(\xi_i\, Q\eta_i\eta_i^* Q\right)\ D\frac{Q\eta_i\eta_i^* Q}{1 - \eta_i^* Q\eta_i}D\ \mathbb{E}_j\left(\xi_i\, Q\eta_i\eta_i^* Q\right)a\ .$$

As $\|(1 - \eta_i^* Q\eta_i)^{-1}Q\eta_i\eta_i^* Q\| = \|Q - Q_i\| \le K$ and $\|Q\Sigma\| \le K$, we have $\sum_{i=1}^{j}\mathbb{E}W_{i,2} \le \sum_{i=1}^{j}\mathbb{E}\left[|a^*Q\eta_i|^2|\xi_i|^2\right]$. Writing $\xi_i = (\rho\tilde{b}_i)^{-1} + e_i$, and noticing that $(\rho\tilde{b}_i)^{-1}$ is bounded, we obtain:

$$\sum_{i=1}^{j}\mathbb{E}W_{i,2} \le 2\mathbb{E}a^*Q\Sigma_{1:j}\operatorname{diag}((\rho\tilde{b}_1)^{-1}, \ldots, (\rho\tilde{b}_j)^{-1})\Sigma_{1:j}^* Qa\ +\ K\sum_{i=1}^{j}\mathbb{E}|e_i|^2 \le K\ ,$$

where $\Sigma_{1:j} = [\eta_1, \cdots, \eta_j]$. The terms $W_{i,3}$ and $W_{i,4}$ can be handled by similar derivations. It results that $\sum_{i=1}^{j}\mathbb{E}|y_i^* M_i a_i|^2 \le K/n$. The terms $a_i^* M_i y_i$ at the right hand side of (A.4) satisfy $\sum_{i=1}^{j}\mathbb{E}\,|a_i^* M_i y_i|^2 \le Kn^{-1}\sum_{i=1}^{j}\mathbb{E}|a_i^* Q_i a|^2 \le K/n$, which proves that $\mathbb{E}|X_1|^2 \le K/n$.

We now consider $X_2$, which satisfies $\mathbb{E}|X_2|^2 \le 2\rho^4\sum_{i=1}^{j}\mathbb{E}|\tilde{b}_i\tilde{q}_{ii}e_i\eta_i^* M_i\eta_i|^2$. We have

$$\begin{aligned}\sum_{i=1}^{j}\mathbb{E}\left|\tilde{b}_i\tilde{q}_{ii}e_i a_i^* M_i a_i\right|^2 &\le K\sum_{i=1}^{j}\mathbb{E}\mathbb{E}^{(i)}\,|e_i a_i^* M_i a_i|^2\\ &\le \frac{K}{n}\sum_{i=1}^{j}\mathbb{E}|a_i^* M_i a_i|^2 \le \frac{K}{n}\sum_{i=1}^{j}\mathbb{E}|a_i^* Q_i a|^2 \le \frac{K}{n}\ ,\end{aligned}$$

where $\mathbb{E}^{(i)} = \mathbb{E}[\cdot|y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]$. Moreover,

$$\sum_{i=1}^{j} \mathbb{E}\left|\tilde{b}_i \tilde{q}_{ii} e_i y_i^* M_i a_i\right|^2 \leq K \sum_{i=1}^{j} (\mathbb{E}|e_i|^4)^{1/2} (\mathbb{E}|y_i^* M_i a_i|^4)^{1/2} \leq \frac{K}{n}$$

and similarly for the terms in $a_i^* M_i y_i$ and in $y_i^* M_i y_i$. It results that $\mathbb{E}|X_2|^2 = \mathcal{O}(n^{-1})$. We now turn to the term $Z$ of equation (A.3). To control the variance of $Z$, we only need to control the variances of the terms:

$$Z_1 = \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1})(\mathbb{E}_j \tilde{q}_{ii} a^* Q_i y_i \eta_i^* Q_i) D (\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i a_i^* Q_i a),$$

$$Z_2 = \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} y_i^* Q_i a\, D^{1/2} Q_i \eta_i)\|^2,$$

$$Z_3 = \sum_{i=1}^{j} (\mathbb{E}_i - \mathbb{E}_{i-1}) \|\mathbb{E}_j (\tilde{q}_{ii} a_i^* Q_i a\, D^{1/2} Q_i \eta_i)\|^2.$$

The first term satisfies

$$\begin{aligned}
\mathbb{E}|Z_1|^2 &\leq 2 \sum_{i=1}^{j} \mathbb{E}\left|(\mathbb{E}_j \tilde{q}_{ii} a^* Q_i y_i \eta_i^* Q_i) D (\mathbb{E}_j \tilde{q}_{ii} Q_i \eta_i a_i^* Q_i a)\right|^2 \\
&\leq 2 \sum_{i=1}^{j} \mathbb{E}\left|(\mathbb{E}_j a^* Q_i y_i \eta_i^* Q_i) D Q_i \eta_i a_i^* Q_i a\right|^2 \\
&= 2 \sum_{i=1}^{j} \mathbb{E}\left[|a_i^* Q_i a|^2\, \mathbb{E}^{(i)}\left|(\mathbb{E}_j a^* Q_i y_i \eta_i^* Q_i) D Q_i \eta_i\right|^2\right] \\
&\leq \frac{K}{n} \sum_{i=1}^{j} \mathbb{E}|a_i^* Q_i a|^2 \quad = \quad \mathcal{O}(n^{-1})\,,
\end{aligned}$$

where the second inequality comes from $\mathbb{E}|\mathbb{E}_j(X)\mathbb{E}_j(Y)|^2 = \mathbb{E}|\mathbb{E}_j(X\mathbb{E}_j(Y))|^2 \leq \mathbb{E}|X\mathbb{E}_j(Y)|^2$. The terms $Z_2$ and $Z_3$ can be handled similarly; details are omitted. The result is $\mathbb{E}|Z|^2 \leq K/n$.

Hence, $\operatorname{var}\left(a^*(\mathbb{E}_j Q)^2 a\right) = \mathcal{O}(n^{-1})$. The estimate $\operatorname{var}\left(\operatorname{Tr}(\mathbb{E}_j Q) D (\mathbb{E}_j Q)\right) = \mathcal{O}(1)$ can be established similarly.

A.5. **Proof of Lemma 5.3.** The $F_j$ increase to $F_n = n^{-1} \operatorname{Tr} D^{1/2} T A (1+\delta\tilde{D})^{-2} \tilde{D} A^* T D^{1/2} < 1$ by lemma 3.5. As $\gamma > 0$ and $M_j$ and $G_j$ are increasing, $\boldsymbol{\Delta}_j$ is decreasing. In order to show that $\boldsymbol{\Delta}_n = \Delta_n$, we only need to show that $M_n + G_n = \rho^2 \tilde{\gamma}$. We have

$$G_n = \frac{1}{n} \operatorname{Tr} \tilde{D}(I+\delta\tilde{D})^{-2} A^* T A \tilde{D} (I+\delta\tilde{D})^{-2} A^* T A - \frac{1}{n} \sum_{k=1}^{n} \left(\frac{\tilde{d}_k\, a_k^* T a_k}{(1+\delta\tilde{d}_k)^2}\right)^2$$

Recall from (3.15) and (3.16) that $(1+\delta\tilde{d}_k)^{-2} a_k^* T a_k = (1+\delta\tilde{d}_k)^{-1} - \rho\tilde{t}_{kk}$. Hence

$$\frac{1}{n} \sum_{k=1}^{n} \left(\frac{\tilde{d}_k\, a_k^* T a_k}{(1+\delta\tilde{d}_k)^2}\right)^2 = \frac{1}{n} \sum_{k=1}^{n} \rho^2 \tilde{d}_k^2 \tilde{t}_{kk}^2 - \frac{1}{n} \sum_{k=1}^{n} \frac{\rho \tilde{d}_k^2 \tilde{t}_{kk}}{(1+\delta\tilde{d}_k)} + \frac{1}{n} \sum_{k=1}^{n} \frac{\tilde{d}_k^2\, a_k^* T a_k}{(1+\delta\tilde{d}_k)^3}$$

which results in

$$M_n + G_n = \frac{\rho}{n} \operatorname{Tr} \tilde{D}\tilde{T}\tilde{D}(I+\delta\tilde{D})^{-1} - \frac{1}{n} \operatorname{Tr} \tilde{D}^2(I+\delta\tilde{D})^{-3}A^*TA + \frac{1}{n} \operatorname{Tr} \tilde{D}(I+\delta\tilde{D})^{-2}A^*TA\tilde{D}(I+\delta\tilde{D})^{-2}A^*TA.$$

Now, one can check with the help of (3.17) that $\rho^2\tilde{\gamma} = \rho^2 n^{-1} \operatorname{Tr} \tilde{D}\tilde{T}\tilde{D}\tilde{T}$ is equal to the r.h.s. of this equation. Lemma 5.3 is proved.

## Appendix B. Additional proofs (extended version)

In this section, we gather the proofs of many results mentionend without proofs in the short version of the paper.

B.1. **Proofs of Eq.** (3.15) **and Eq.** (3.16)**.** Proof of Eq. (3.15) mainly relies on matrix identities (3.6) and (3.7). To lighten the computations, let us introduce the following notations:

$$\mathcal{I} = \left(I_n + \tilde{\delta}D\right)^{-1}, \qquad \tilde{\mathcal{I}} = \left(I_{N-1} + \delta\tilde{D}_1\right)^{-1} .$$

In order to express a diagonal element of $\tilde{T}$, say $\tilde{t}_{11}$ (without loss of generality), let us first write:

$$\tilde{T} = \begin{bmatrix} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 & a_1^*\mathcal{I}A_1 \\ A_1^*\mathcal{I}a_1 & -z\tilde{\mathcal{I}}^{-1} + A_1^*\mathcal{I}A_1 \end{bmatrix}^{-1} .$$

Hence, according to (3.6):

$$\begin{aligned}
\frac{1}{\tilde{t}_{11}} &= -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 - a_1^*\mathcal{I}A_1\left[-z\tilde{\mathcal{I}}^{-1} + A_1^*\mathcal{I}A_1\right]^{-1}A_1^*\mathcal{I}a_1 \\
&\overset{(a)}{=} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 - a_1^*\mathcal{I}A_1\left[-\frac{1}{z}\tilde{\mathcal{I}} + \frac{1}{z}\tilde{\mathcal{I}}A_1^*\mathcal{T}_1A_1\tilde{\mathcal{I}}\right]A_1^*\mathcal{I}a_1 \\
&\overset{(b)}{=} -z(1+\delta\tilde{d}_1) + a_1^*\mathcal{I}a_1 + \frac{1}{z}a_1^*\mathcal{I}\left(\mathcal{T}_1^{-1} + z\mathcal{I}^{-1}\right)\mathcal{I}a_1 \\
&\qquad -\frac{1}{z}a_1^*\mathcal{I}A_1\tilde{\mathcal{I}}A_1^*\left[I_N + z\mathcal{T}_1\mathcal{I}^{-1}\right]\mathcal{I}a_1 \\
&\overset{(b)}{=} -z(1+\delta\tilde{d}_1) + 2a_1^*\mathcal{I}a_1 + \frac{1}{z}a_1^*\mathcal{I}\mathcal{T}_1^{-1}\mathcal{I}a_1 \\
&\qquad -\frac{1}{z}a_1^*\mathcal{I}\left[\mathcal{T}_1^{-1} + z\mathcal{I}^{-1}\right]\left[I_N + z\mathcal{T}_1\mathcal{I}^{-1}\right]\mathcal{I}a_1 \\
&= -z(1+\delta\tilde{d}_1) - a_1^*\mathcal{T}_1a_1 ,
\end{aligned}$$

where $(a)$ follows from (**??**), $(b)$ from equalities

$$\mathcal{T}_1A_1\tilde{\mathcal{I}}A_1^* = I_N + z\mathcal{T}_1\mathcal{I}^{-1} \quad \text{and} \quad A_1\tilde{\mathcal{I}}A_1^* = \mathcal{T}_1^{-1} + z\mathcal{I}^{-1}$$

which follow from the mere definition of $\mathcal{T}_1$. Finally, (3.15) is established.

Let us now turn to the proof of (3.16). Notice first that $T$ writes:

$$T = \left[-z(I_N + \tilde{\delta}D) + A_1(I_{n-1} + \delta\tilde{D}_1)^{-1}A_1^* + \frac{a_1a_1^*}{1+\delta\tilde{d}_1}\right]^{-1} .$$

Applying (**??**) readily yields:

$$T = \mathcal{T}_1 - \mathcal{T}_1 a_1 \left(1 + \delta \tilde{d}_1 + a_1^* \mathcal{T}_1 a_1\right) a_1^* \mathcal{T}_1 \ .$$

It remains to multiply by $a_1^*$ (left), $b$ (right) and to use (3.15) to establish (3.16).

B.2. **Proof of Inequality** (3.20)**.** We provide here some elements to establish that $\mathbb{E}|e_j|^p = \mathcal{O}(n^{-p/2})$. Recall the definition (3.3) of $e_j$ and write:

$$\mathbb{E}|e_j|^p \leq 2^{p-1} \left\{ \mathbb{E} \left| y_j^* Q_j y_j - \frac{\tilde{d}_j}{n} \mathrm{Tr} D Q_j \right|^p + \mathbb{E} \left| a_j^* Q_j y_j \right|^p + \mathbb{E} \left| y_j^* Q_j a_j \right|^p \right\} \ .$$

The first term of the r.h.s. can be directly estimated with the help of Lemma 3.2. The two remaining terms are similar and can be estimated in the following way:

$$\begin{aligned} \mathbb{E} \left| a_j^* Q_j y_j \right|^p &= \mathbb{E} \left( y_j^* Q_j^* a_j a_j^* Q_j y_j \right)^{p/2} \\ &\leq 2^{p-1} \left( \mathbb{E} \left| y_j^* Q_j^* a_j a_j^* Q_j y_j - \frac{d_j^2}{n} \mathrm{Tr} Q_j^* a_j a_j^* Q_j \right|^{p/2} + \mathbb{E} \left| \frac{1}{n} \mathrm{Tr} Q_j^* a_j a_j^* Q_j \right|^{p/2} \right) \ . \end{aligned}$$

The first term of the r.h.s. can be handled with the help of Lemma 3.2 (notice that $\mathrm{Tr}(Q_j^* a_j a_j^* Q_j)^{p/2}$ is of order 1), and the second term is directly of the right order.

B.3. **Proofs of Theorem 3.3-**(3)**.** The estimate established in [22] is:

$$\left| \frac{1}{n} \mathrm{Tr}\, U (T(-\rho) - \mathbb{E}Q(-\rho)) \right| \leq \frac{K}{\sqrt{n}} \ .$$

The speed $\mathcal{O}(n^{-1})$ stated in the theorem is higher and therefore does not directly follow from [22]. We provide here the needed extra arguments.

B.4. **Proof of Theorem 3.3-**(4)**.** s

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