On the capacity of Lagrangians in the cotangent disc bundle of the torus

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Abstract

We prove that a Lagrangian torus in T^*T^n Hamiltonianly isotopic to the zero section and contained in the unit disc bundle has bounded γ-capacity, where $\gamma(L)$ is the norm on Lagrangian submanifolds defined in [Vit1]. On one hand this gives new obstructions to Lagrangian embeddings or isotopies, of a quantitative kind. On the other hand, it gives a certain control on the γ topology in terms of the Hausdorff topology. Finally this result is a crucial ingredient in establishing symplectic homogenization theory in [Vit5].

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Contents

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1 Introduction

The goal of this paper is to gain a better understanding of a symplectically invariant metric on the set $\mathcal{L}(T^*N)$ of Lagrange submanifolds hamiltonianly isotopic to the zero section in a cotangent bundle T^*N.

The metric is the metric γ defined using generating functions (see [Vit1], and also [C-V]) on T^*N. We shall show that if g is some Riemannian metric on N, and

$$D_gT^*N = \{(x,p) \in T^*N|\ |p|_g \leq 1\}$$

there $\gamma(L) \leq C(g)$ for any L contained in D_gT^*N.

The case where N is simply connected will be dealt with in [Vit4] and we shall here study the case where N is a torus.

Our main result is:

Theorem 1.1. *Let g be a metric on T^n. There is a constant $C(g)$ such that if L is any element of $\mathcal{L}(T^*T^n)$ contained in $D_gT^*T^n$, then*

$$\gamma(L) \leq C(g)\ .$$

In other words, let L be a Lagrangian $\mathcal{L}(T^*T^n)$.
Then

$$\gamma(L) \leq C(g)\|L\|$$

where $\|L\|$ is defined as

$$\|L\| = \sup\{|p|_g \mid (q,p) \in L\}$$

Corollary 1.2. *Let f be a function with oscillation larger than $C(g)$. Then there is no Hamiltonian isotopy preserving the zero section and sending the graph of df, $\Gamma_f = \{(x, df(x)) \mid x \in T^n\}$ in the unit disc bundle of T^*T^n*

Proof. Indeed, according to [Vit1], $\gamma(\Gamma_f) = \text{osc(f)}$. □

Another consequence is as follows. Let $d_H(L_0, L_1)$ be the Hausdorff distance between L_0 and L_1, that is:

$$d_H(L_0, L_1) = \min\{\varepsilon \mid \forall z_0 \in L_0, z_1 \in L_1, d(z_0, L_1) \leq \varepsilon, d(z_1, L_0) \leq \varepsilon\}$$

Similarly, we could consider for two symplectomorphisms ϕ_0, ϕ_1 the distance $d_H(\Gamma(\phi_0), \Gamma(\phi_1)$. It turns out that this is equal to the C^0 distance

$$d_{C^0}(\phi_0, \phi_1) = \sup\{\max(d(\phi_0(x), \phi_1(x)), d(\phi_0^{-1}(x), \phi_1^{-1}(x))), \mid x \in M\}$$

Corollary 1.3. *The distance γ is bounded by a multiple of the Hausdorff distance. Therefore for all $L_0, L_1 \in \mathcal{L}(T^*T^n)$, we have*

$$\gamma(L_0, L_1) \leq C(L_0) \cdot d_H(L_0, L_1)$$

Thus if L_ν is a sequence in $\mathcal{L}(T^*T^n)$, converging for the Hausdorff distance to L, then it converges for γ. Similarly, if ϕ_ν converges C^0 to ϕ in $\mathcal{H}(T^*T^n)$, the set of time one maps of a Hamiltonian flow, then ϕ_ν converges to ϕ for γ.

Proof. Indeed, let ψ be such that $\psi(L_0) = 0_{T^n}$. We have $\gamma(L_0, L_1) = \gamma(0_{T^n})$ and $d_H(\psi(L_0), \psi(L_1)) \leq C(L_0) d_H(L_0, L_1)$. Hence we can restrict ourselves to the case $L_0 = 0_{T^n}$ in which case it is just our main theorem.

□

Remark 1.4.

1. The main theorem has some important applications to the so-called *symplectic homogenization* that can be found in [Vit5].

2. It would be important to better understand the metric γ. A long term goal should be to obtain a characterization of compact sets for γ. Examples of compact sets are already very interesting. Our main theorem gives an example of bounded subsets. We conjecture that a bounded subset of $\mathcal{L}(T^*N)$ are of the form

$$\Omega_C = \{\mathcal{H}(T^*N, N) \cdot L \mid \|L\| \leq C\}$$

where $\mathcal{H}(T^*N, N)$ is the set of Hamiltonian isotopies preserving the zero section, L is a Lagrangian submanifold, so that

$$\mathcal{H}(T^*N, N) \cdot L = \{\varphi(L) \mid \varphi \in \mathcal{H}(T^*N)\}$$

Of course, one could make a bolder conjecture, and replace L by Γ_f the graph of df with osc(f) ≤ C.

3. We defined in [C-V] a metric on $\mathscr{DH}(T^*N)$, the set of time one maps of Hamiltonian isotopies in T^*N, by

$$\widetilde{\gamma}(\varphi) = \sup\{\gamma(\varphi(L), L) | L \in \mathcal{L}(T^*N)\}$$

It does not follow from Theorem 1.1 that if the support of the isotopy $(\varphi^t)_{t\in[0,1]}$ is contained in $D_g T^*T^n$ we have $\widetilde{\gamma}(\varphi) \leq \widetilde{C}(g)$. Indeed, this is false as would follow from Polterovich's book [Pol]proposition 7.1.A (even though this is stated for the Hofer norm, the same result holds for γ), or from the main theorem in [Vit5].

4. One may define the Hofer metric on Lagrangians in $\mathcal{L}(T^*N)$ as

$$d_H(L) = \inf\left\{\int_0^1 [\max_x H(t,x) - \min_x H(t,x)]dt \mid \phi_H^1(0_N) = L\right\}$$

An interesting open question (even in the case of $N = T^n$) is whether for Lagrangians satisfying the assumption of our theorem, the quantity $d_H(L)$ is bounded.

Acknowledgements

We are very grateful to Yuri Chekanov for pointing out, while our proof was not yet complete, that the case $n = 1$ was already non-trivial. I am very glad to acknowledge Denis Auroux's contribution, who pointed out a mistake in the original proof, and later made a crucial suggestion to the present proof. We also thank the University of Beijing, for hospitality during the preparation of this work and to the participants of the RIMS conference for enduring a first version of these results.

The results and methods of this paper bear some relation to previous work of Fukaya, Seidel, and Smith on the Fukaya category of T^*N and in particular to the fact that this category is generated by "vertical Lagrangians". This point of view, which seems to have been initiated by Fukaya and Smith, materializes most beautifully in [F-S-S1, F-S-S2]. In spite of attending several talks on this subject, and of illuminating comments by Kenji Fukaya and Paul Seidel ([Seidel]), I do not feel competent to usefully discuss such relationship. Nonetheless our paper's debt to the above mentioned works, even though it only became conscious at a later stage, remains fundamental.

2 Reformulation of the problem

First of all it is enough to prove our theorem for some given metric g_0 since there exists $\lambda(g, g_0)$ such that

$$D_g T^*T^n \subset \lambda(g, g_0) D^*_{g_0} T^*T^n$$

We shall take for g_0 the Finsler metric given by

$$|p|_{g_0} = \max\{|p_j| \mid j = 1, \cdots, n\}$$

Thus $D_{g_0} T^*T^n = T_1^*T^n = (S^1 \times [-1, 1])^n$.
Let Λ_1 be the exact Lagrangian in T^*S^1 given by figure 1.

Since $\gamma(\Lambda_1)$ equals some integral $\int_c pdq$ where $c : S^1 \longrightarrow T^*S^1$ is a path in Λ_1 connecting two intersection points of $\Lambda_1 \cap 0_{T^n}$ (cf. [Vit1]), we may conclude that $\gamma(\Lambda_1)$ is of the order of $2\pi + 2\delta$. We can always arrange the data on figure 1 so that $\gamma(\Lambda_1) \leq 2\pi + 4\delta$. Let us now consider $\Lambda = \Lambda_1^n \subset T^*T^n$. Since $\gamma(L_1 \times L_2) = \gamma(L_1) + \gamma(L_2)$, we get

Lemma 2.1.

$$\gamma(\Lambda) \leq (2\pi + 4\delta)n$$

Let us now denote by V_0 the fiber of T^*T^n over $(0, \cdots, 0)$ and for $\varepsilon \in \{-1, 1\}^n$, we denote by $V_\varepsilon(\delta)$ the fiber over $(\varepsilon_1\delta, \cdots, \varepsilon_n\delta)$.

The main property of Λ is that

$$\Lambda \cap T_1^*T^n = \left(\bigcup_{\varepsilon \in \{-1,1\}^n} V_\varepsilon(\delta) \right) \cap T_1^*T^n.$$

This will enable us to compute $\gamma(\Lambda, L)$ where L is in $\mathcal{L}$ and contained in $T_1^*T^n$. We must then prove

Proposition 2.2. *For any L in $\mathcal{L}$ contained in $T_1^*T^n$ we have*

$$\gamma(L, \Lambda) \leq (2\pi + 4\delta)n\ .$$

Since $\gamma(L) \leq \gamma(L, \Lambda) + \gamma(\Lambda)$ (see [Vit3]) proposition 2.2 implies theorem 1.1 Moreover this yields that the constant $C(g_0)$ is bounded by $4\pi n$.

We are now going to sketch the proof of proposition 2.2

Let $V(\delta) = \bigcup_{\varepsilon \in \{-1,1\}^n} V_\varepsilon(\delta)$

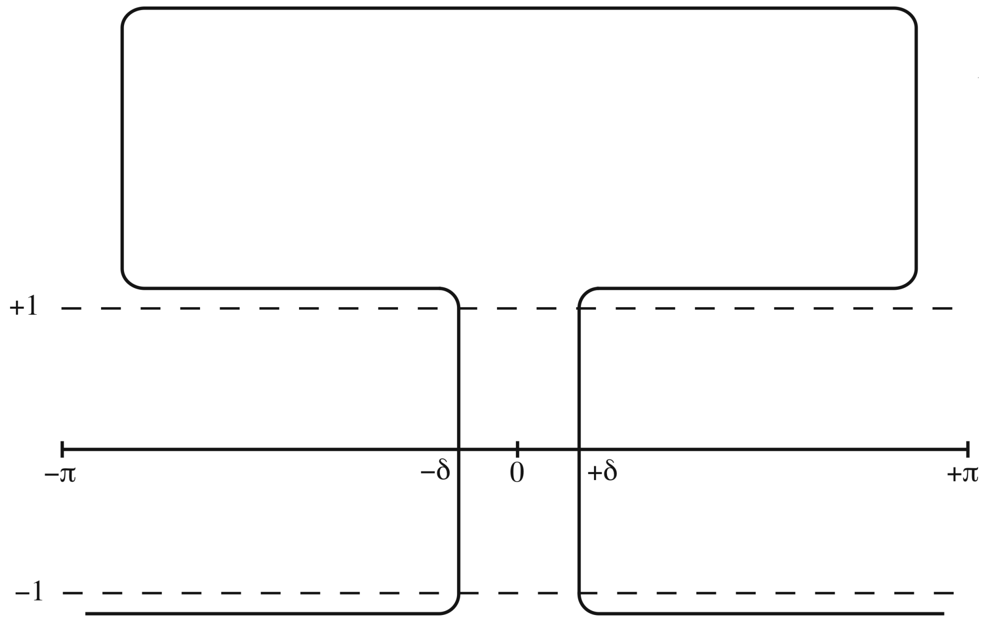

Figure 1: The curve Λ_1

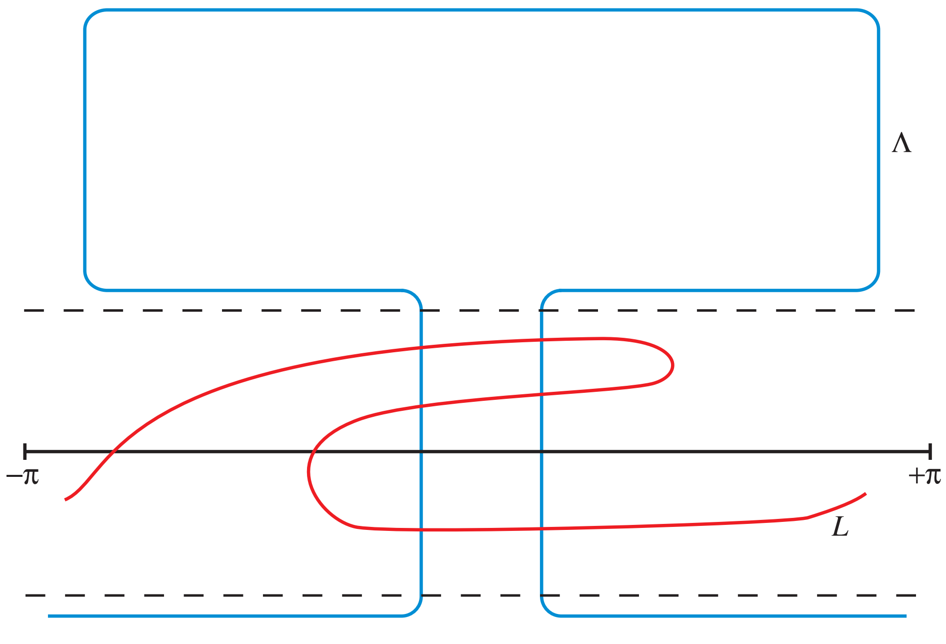

Figure 2: L and Λ

We shall first remind the reader about the identification of generating function cohomology and Floer cohomology as follows.

The notation f^b will designate the sublevel set

$$f^b = \{x | f(x) \leq b\} \ .$$

Let S_L (resp. S_Λ) be a generating function quadratic at infinity for L (resp. Λ). Lisa Traynor defined in [Tr], the groups $GH^*(L, \Lambda; a, b)$ as follows:

Definition 2.3. *Let* $S_L \ominus S_\Lambda(x, \xi, \eta) = S_L(x, \xi) - S_\Lambda(x, \eta)$. *We define* $GH^*(L, \Lambda; a, b)$ *as* $H^*((S_L \ominus S_\Lambda)^b, (S_L \ominus S_\Lambda)^a)$.

Remarks 2.4.

1. A similar group $FH^*(L, \Lambda; a, b)$ is defined using Floer cohomology. We refer to [Viterbo - FCFH2] or section 3 for the precise definition.
2. In defining $FH^*(L, \Lambda; a, b)$ or $GH^*(L, \Lambda; a, b)$ we need a normalizing constant for the action on the generating function. One way to define this normalization, is to specify the value of the action (or the g.f.q.i.) at an intersection point of $L \cap \Lambda$.

According to [Vit2],

Proposition 2.5. *We have an isomorphism*

$$GH^*(L, \Lambda; a, b) \simeq FH^*(L, \Lambda; a, b)$$

As a result, if we define $GH^*(L, \Lambda)$ and $FH^*(L, \Lambda)$ as $GH^*(L, \Lambda; -\infty, +\infty)$ and $(FH^*(L, \Lambda; -\infty, +\infty)$, both are isomorphic, according to Floer, to $H^*(T^n)$. Indeed for any Hamiltonian diffeomorphism φ, we have

$$FH^*(L, \Lambda) = FH^*(\varphi(L), \varphi(\Lambda)) \ .$$

If $\varphi(\Lambda) = O_{T^n}$, we get

$$\begin{aligned} FH^*(\varphi(L), O_N) &\simeq FH^*(\varphi(L)) \\ &\simeq H^*(T^n) \end{aligned}$$

according to Floer's main theorems of [Fl1, Fl2].

Of course we have maps

$$\sigma_G^c : GH^*(L, \Lambda; -\infty, c) \longrightarrow GH^*(L, \Lambda; -\infty, +\infty)$$

and

$$\sigma_F^c : FH^*(L, \Lambda; -\infty, c) \longrightarrow FH^*(L, \Lambda; -\infty, +\infty)$$

We may thus define for $\alpha \in H^*(T^n)$ the number $c(\alpha, L, \Lambda)$ as

Definition 2.6.

$$c(\alpha; L, \Lambda) = \inf\{c \mid \alpha \quad \textit{is in the image of} \quad \sigma_G^c\} = \inf\{c \mid \alpha \quad \textit{is in the image of} \quad \sigma_F^c\}$$

Now μ and 1 being the respective generators of $H^n(T^n)$ and $H^0(T^n)$,

Definition 2.7. *We set* $\gamma(L, \Lambda) = c(\mu, L, \Lambda) - c(1, L, \Lambda)$.

Once we know that γ can be defined using Floer cohomology, we may notice that Λ and $V(\delta)$ have the same intersection with $T_1^*T^n$, and we may hope that there is an isomorphism between $FH^*(L, \Lambda)$ and

$$(\star) \qquad FH^*(L, V(\delta)) \simeq \bigoplus_{\varepsilon \in \{-1,1\}^n} FH^*(L, V_\varepsilon(\delta)) \ .$$

This isomorphism obviously holds at the chain level, and we have to show that the coboundary maps are isomorphic. Of course, $V_\varepsilon(\delta)$ is not isotopic to the zero section, so that the above Floer cohomology, although well defined, is not given by the cohomology of one of the manifolds. It can nonetheless be computed, by isotoping L to the zero section. Since $0_{T^n} \cap V_0 = \{(0, ..., 0)\}$, $FC^*(0_{T^n}, V) \simeq \mathbb{Z}$, hence $FH^*(L, V) \simeq FH^*(0_{T^n}, V) \simeq \mathbb{Z}$.

However the isomorphism $(\star)$ does not respect the action filtration, as we already see for $n = 1$: on figure 1, the two points of $\Lambda \cup O_N$, that is $(-\delta, 0)$ and $(0, \delta)$ have action difference approximately given by $(2\pi + 2\delta)$. Considered as intersection points of $\Lambda \cap V_{-1}(\delta)$ and $\Lambda \cap V_{+1}(\delta)$, their action would depending on the choice of the normalizing constants on $\Lambda \cap V_{-1}(\delta)$ and $\Lambda \cap V_{+1}(\delta)$ and since these are independent, nothing can be said at this point. Indeed, we see that for generic L, the intersection points of L with Λ appear in families of cardinal 2^n. Indeed each point x in $L \cap V_0$ yields a point in $L \cap V_\varepsilon(\delta)$, that we denote by x_ε. Thus to a normalization of the action for $L \cap \Lambda$ corresponds a normalization of each of the terms $L \cap V_\varepsilon(\delta)$.

We will denote by $|\varepsilon|$ the sum $\sum_{j=1}^{n} \varepsilon_j$.

Proposition 2.8. *We have an isomorphism*

$$FH^*(L, \Lambda; -\infty, c) \simeq \bigoplus_{\varepsilon \in \{-1,1\}^n} FH^{*-\frac{(n+|\varepsilon|)}{2}}(L, V_\varepsilon(\delta); -\infty, c + c(\varepsilon))$$

where $c(\varepsilon)$ *is approximately given by*

$$c(\varepsilon) \simeq (n + |\varepsilon|)(\pi + \delta)$$

As explained earlier, the isomorphism of proposition 2.8 assumes we associate to a normalization of the action of the points in $L \cap \Lambda$ a normalization of the action of the points in $L \cap V_\varepsilon(\delta)$. Moreover for δ small enough, there is a natural bijection between $L \cap V_\varepsilon(\delta)$ and $L \cap V_0$.

We thus choose a point $x_0(-1, \cdots, -1)$ in $\Lambda \cap L$ contained in $V_{(-1,\cdots,-1)}(\delta)$ and decide its action has value 0. We then decide that the corresponding point $x_0(-1, \cdots, -1)$ in $V_{(-1,\cdots,-1)}(\delta)$ has also value 0 as a point in $L \cap V_{(-1,\cdots,-1)}(\delta)$.

It is however obvious that as a point in $L \cap \Lambda$, the difference in action between $x_0(\varepsilon)$ and $x_0(-1, \cdots, -1)$ is given by $c(\varepsilon) - c \simeq (n + \sum_{j=1}^{n} \varepsilon_j)(\pi + \delta)$. A similar argument holds for the grading of the cohomology via the Maslov index.

Since for $c = +\infty$ each term of the right hand side corresponds to a generator of $H^*(T^n)$, and $FH^*(L, V_\varepsilon(\delta)) \simeq \mathbb{Z}$ we have the following result:

Corollary 2.9. *If $\alpha \in H^*(T^n)$ corresponds to an element in $FH^*(L, V_\varepsilon(\delta))$ then*

$$c(\alpha, L, \Lambda) = c(1, L, V_\varepsilon(\delta)) - c(\varepsilon)$$

In particular, since μ corresponds to $\varepsilon = (+1, \cdots, +1)$ and 1 to $\varepsilon = (-1, \cdots, -1)$, we have:

$$\gamma(L, \Lambda) = c(\mu, L, \Lambda) - c(1, L, \Lambda) = c(1, L, V_{(1,\cdots,1)}(\delta)) - c(1, L, V_{(-1,\cdots,-1)}(\delta)) = c(1, \cdots, 1) - c(-1, \cdots, -1) \simeq 2n(\pi + \delta) \ .$$

Proof of corollary assuming proposition 2.8: Since degree $(\mu) = n$ and degree $(1) = 0$, the class μ is associated to some $V_\varepsilon(\delta)$ such that $n + |\varepsilon| = 2n$ i.e. $\varepsilon = (+1, \cdots, +1)$ while 1 is associated to ε such that $n + |\varepsilon| = 0$ i.e. $\varepsilon = (-1, \cdots, -1)$.

The computation of the critical values is then obvious. □

Clearly corollary 2.9 implies proposition 2.2, thus concluding our proof of the theorem 1.1

3 Sketch of the proof of Proposition 2.8. First steps

Let us now sketch the proof of proposition 2.8. First of all we denote from now on by $FC^*(L, V(\delta))$ the normalized chain complex associated to $(L, V(\delta))$, i.e. the complex with grading shifted by $\frac{(n+|\varepsilon|)}{2}$ and filtration shifted by $c(\varepsilon)$:

$$\widetilde{FC}^*(L, V(\delta); a, b) = \bigoplus_{\varepsilon\in\{-1,1\}^n} FC^{*-\frac{(n+|\varepsilon|)}{2}}(L, V_\varepsilon(\delta); a + c(\varepsilon), b + c(\varepsilon))$$

since $\Lambda \cap L = L \cap V(\delta)$, the chain complexes $FC^*(L, \Lambda)$ and $\widetilde{FC}^*(L, V(\delta))$ are obviously isomorphic. Moreover $\widetilde{FC}^*$ is defined so that the isomorphism preserves grading and filtration. We are thus left to consider the coboundary maps, δ_Λ and δ_V. As is well known since Floer's work ([Fl1, Fl2]), these maps "count" the number of holomorphic strips, i.e. holomorphic maps $u : [-1, 1] \times \mathbb{R} \to (T^*T^n, J)$ where J is some almost complex structure on T^*T^n, and $u(-1, t) \in L$, $u(1, t) \in \Lambda$ (resp. V). In fact given x, y in $L\cup\Lambda$ the set of such holomorphic maps $\mathcal{M}_\Lambda(x, y)$ (resp. $\mathcal{M}_V(x, y)$) has the obvious $\mathbb{R}$-action $\tau^*u(s, t) = u(s, t + \tau)$. The quotient space $\widehat{\mathcal{M}}_\Lambda(x, y) = \mathcal{M}(x, y)/\mathbb{R}$ (resp. $\widehat{\mathcal{M}}_V(x, y) = \mathcal{M}_V(x, y)/\mathbb{R}$) is discrete if x and y have (Maslov) index difference one. We shall content ourselves with counting the number of points mod 2 (we could get an integer at some extra cost, but this will not be necessary). We thus have

$$\sharp_2\widehat{\mathcal{M}}_\Lambda(x, y) = \langle\delta_\Lambda x, y\rangle \ .$$

$$\sharp_2\widehat{\mathcal{M}}_V(x, y) = \langle\delta_V x, y\rangle \ .$$

Our proposition follows from

Proposition 3.1. *For a suitable choice of J, we have*

$$\sharp_2\widehat{\mathcal{M}}_\Lambda(x, y) = \sharp_2\widehat{\mathcal{M}}_V(x, y) \ .$$

The idea of the proof is to choose J so that on one hand $T_1^*T^n$ is pseudo convex and on the other hand $V_\varepsilon(\delta)$ is totally real i.e. it is locally the fixed point set of a anti-holomorphic map.

With such a choice of J, the proposition being obvious for 0_N, we show in section 4 that as we isotope 0_N to $\varphi(0_N)$ through $\varphi^t(0_N)$, the holomorphic strips contained in $T_1^*T^n$ do not interact with those not contained in this pseudo convex set.

Obviously those contained in $T_1^*T^n$ contribute in the same way to $\sharp_2\widehat{\mathcal{M}}_\Lambda(x, y)$ and $\sharp_2\widehat{\mathcal{M}}_V(x, y)$, since a holomorphic strip contained in $T_1^*T^n$ bounding L and Λ is the same as a holomorphic strip bounding L and V.

Our last step is to prove that the strips exiting from $T_1^*T^n$ do not contribute to $\sharp_2\widehat{\mathcal{M}}_\Lambda(x, y)$ or $\sharp_2\widehat{\mathcal{M}}_V(x, y)$. This is completed in section 5.

4 Holomorphic strips in $T_1^*T^n$

Let J be an almost complex structure on T^*T^n satisfying the following properties

(a) $$T_\nu^*T^n = \{(q,p) \in T^*T^n | \ \ |p|_1 \leq \nu\}$$

is J-pseudo convex for $\nu \geq 1$

(b) the verticals $V_\varepsilon(\delta)$ are fixed points of a locally defined anti-holomorphic involution.

Any almost complex structure will be implicitly assumed to satisfy the above assumptions. Our first goal is to prove

Proposition 4.1. *Let $u : [-1,1] \times \mathbb{R} \to T^*T^n$ be a J-pseudo holomorphic with boundary in $L \cup V_\varepsilon(\delta)$ i.e. $u(-1,t) \in L$, $u(1,t) \in V_\varepsilon(\delta)$.*

Assume J satisfies (a) and (b), and L is contained in $T_1^*T^n$. Then $u([-1,1] \times \mathbb{R})$ is contained in $T_1^*T^n$.

Proof : Consider the function $f = |p \circ u|_1$ on $[-1,1] \times \mathbb{R}$. Arguing by contradiction, we assume the supremum of f is > 1. Since

$$\lim_{t \to \pm\infty} u(s,t) \in L \cap V \subset T_1^*T^n$$

the supremum is not achieved as t goes to $\pm\infty$, hence it is a maximum achieved either at an interior point (i.e. in $]-1,1[\times\mathbb{R})$ but this is impossible due to the maximum principle, or at boundary point.

Since $u(\{-1\} \times \mathbb{R}) \subset L \subset T_1^*T^n$, the boundary point must be $(1,t_0)$ and by translation we may assume it is $(1,0)$.

Now let τ be the anti holomorphic involution fixing $V_\varepsilon(\delta)$, and consider the map w defined in a neighbourhood of $(-1,0)$ in $\mathbb{R}^2$ by

- $$w(s,t) = u(s,t) \text{ for } s \geq -1$$
- $$w(s,t) = \tau \cdot u(-2-s,t) \text{ for } s \leq -1$$

Clearly w is continuous, J-holomorphic and $|p \circ w|_1$ has a local maximum at $(-1,0)$. This is a contradiction, hence our proposition follows. □

As a consequence of proposition 4.1, we see that a holomorphic strip with boundary in $L \cup \Lambda$ is

- either contained in $T_1^*T^n$ and therefore has boundary in $L \cup V_\varepsilon(\delta)$ for some $\varepsilon \in \{-1,1\}^n$

- is not contained in $T_{(1+\alpha)}^*T^n$ for some positive α depending only on Λ. For example

$$1 + \alpha = \sup\{|p| \mid (x,p) \in \Lambda \Longleftrightarrow (x,p) \in V_\varepsilon(\delta)\}$$

5 The complex $(FC^*(L,\Lambda),\delta_\Lambda)$

Our goal in this section is to prove that if x in $FC^*(L,\Lambda) \simeq FC^*(L,V(\delta))$ satisfies $\delta_\Lambda(x)=0$ then $\delta_V(x)=0$. In fact for a suitable filtration of $FC^*(L,\Lambda)$, denoted $FC_t^*(L,\Lambda)$ we have

Proposition 5.1. *There is a filtration $FC_t^*(L,\Lambda)$ such that δ_Λ increases the filtration (i.e. $\delta_\Lambda : FC_t^*(L,\Lambda) \to FC_t^*(L,\Lambda)$) and the induced map*

$$\bar{\delta}_\Lambda : FC_t^*/FC_{t-1}^* \to FC_t^*/FC_{t-1}^*$$

coincides with the map $\bar{\delta}_V$ induced by δ_V.

Moreover $FC_t^*/FC_{t-1}^* \simeq FH^*(L,V_\varepsilon(\delta))$ for some $\varepsilon=\varepsilon(t)$.

Remark 5.2. This result is in fact a vanishing theorem for a spectral sequence (essentially the one in [F-S-S1]). According to [Seidel] this is essentially included in the wonderful paper [F-S-S1] (see also [F-S-S2], and for an alternative approach to [N, N-Z]) since this is a case where the spectral sequence discussed there degenerates at E_1. As pointed out by Paul Seidel this is a case where the identification of the E_2 term is still conjectural.

Proof. The idea of the proof is based on a modification of the symplectic form, so that holomorphic strips inside $T_1^*T^n$ will have much smaller area than those exiting from $T_1^*T^n$. This new area yields a filtration by levels of the action on $FC^*(L,\Lambda)$ which has the required properties.

Indeed let r_ν be the map defined on $T^*\mathcal{S}^1$ by :

$$\begin{aligned} r_\nu(x,p) &= (x,P_\nu(p)) \text{ where} \\ P_\nu(p) &= \frac{1}{\nu}\cdot p \text{ for } |p|\leq 1 \\ &= p \text{ for } |p| \geq 1+\alpha \end{aligned}$$

and moreover $p \to P_\nu(p)$ is monotone

Note that $r_\nu(\Lambda)=\Lambda, r_\nu(V_\varepsilon)=V_\varepsilon$. Denoting by R_ν the product map

$$R_\nu(x_1,p_1,\cdots,x_n,p_n) = (r_\nu(x_1,p_1),\cdots,r_\nu(x_n,p_n))$$

we have again $R_\nu(\Lambda)=\Lambda$, $R_\nu(V_\varepsilon)=V_\varepsilon$ and $R_\nu(L)\subset T_{1/c}^*T^n$.

Let $\sigma_n=\sigma_1\oplus\ldots\oplus\sigma_1$ be the standard symplectic form on $T^*T^n, J_0 = J_0\oplus\ldots\oplus J_0$ the standard complex structure.

The J-holomorphic strips are mapped by R_ν to (R_ν^*J)-holomorphic strips.

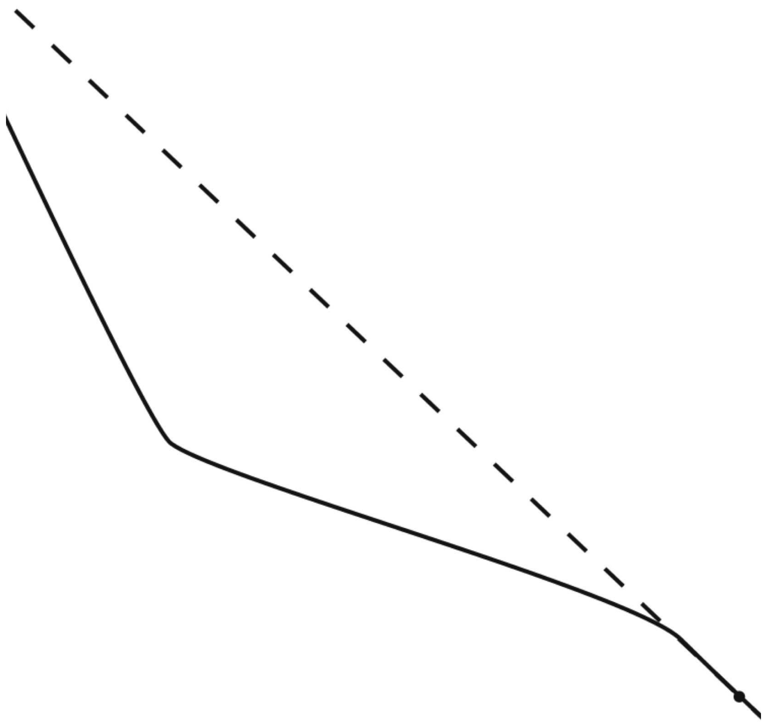

Figure 3: The function P_ν

If such a strip is contained in $T_1^*T^n$, and has area A, its image by R_ν has area $\frac{1}{\nu}A$.

On the other hand if such a strip exits from $T_1^*T^n$, we claim its area is bounded from below independently from ν. □

Lemma 5.3. *There is a constant $\beta_0 > 0$ such that any $R_\nu^*(J_0)$-holomorphic strip with boundary in $R_\nu(L) \cup \Lambda$ exiting $T_1^*T^n$ has area greater than β_0.*

Proof. Even though the lemma holds for any J and Λ (see Appendix of [Vit2]) we shall take advantage of the special structure.

Indeed, a holomorphic strip for $(R_\nu^* J_0)$ is given by n maps $(u_1, \cdots, u_n)$ $u_j :]-1,1[\times\mathbb{R} \to T^*S^1$ is $r_\nu^*(J_0)$-holomorphic $u_j(1,t) \in \Lambda_1$ $u_j(-1,t) \in T_1^*S^1$ (since $(u_1, \cdots, u_n)(-1,t) \in R_\nu(L) \subset T_1^*T^n$).

If the holomorphic strip is not contained in $T_1^*T^n$, one of the u_j, for example u_1, will exit from $T_1^*S^1$. If is then clear that $\int_{[-1,1]\times\mathbb{R}} u_1^*\sigma_1$ must be greater than the area of the upper or lower loop of figure 2, i.e. greater than 2π.

Since $\int_{[-1,1]\times\mathbb{R}} u_j^*\sigma_1 > 0$ and

$$\int_{[-1,1]\times\mathbb{R}} u^*\sigma = \sum_{j=1}^n \int_{[-1,1]\times\mathbb{R}} u_j^*\sigma_1$$

we get that this integral is greater than 2π. □

On the other hand, $(R_\nu^* J_0)$-holomorphic strips contained in $T_1^*T^n$ have area $0(\frac{1}{\nu})$ and for ν large enough (since there are only finitely many strips) they all have area less than $\frac{\pi}{2}$. □

Proof of proposition 2.8. Let us summarize our findings. Consider δ_Λ^ν the coboundary map associated to $(R_\nu^* J_0)$ for c large enough. Then δ_Λ^ν is the sum of the contributions corresponding to holomorphic strips inside $T_1^*T^n$, denoted by δ_Λ^{int}, and holomorphic strips exiting T_1^*, T^n, denoted by δ_Λ^{ext}.

Of course $\delta_\Lambda = \delta_\Lambda^{int} + \delta_\Lambda^{ext}$, and if $A(x)$ is the value of the action at x, we have

$$-\langle \delta_\Lambda^{int} x, y\rangle \neq 0 \Longrightarrow A(y) - A(x) \leq 0\left(\frac{1}{\nu}\right)$$

$$-\langle \delta_\Lambda^{ext} x, y\rangle \neq 0 \Longrightarrow A(y) - A(x) \geq 2\pi$$

Moreover $\delta_\Lambda^{int} = \delta_V$.

Therefore let us choose $x_0(\varepsilon)$ in $V_\varepsilon(\delta)$ so that for c large enough,

$$A(x_0(\varepsilon)) - A(x_0(\varepsilon')) \simeq \pi \sum_{j=1}^{n} (\varepsilon_j - \varepsilon'_j)$$

so that $A(x_0(\varepsilon)) \simeq \pi(\sum_{j=1}^{n} \varepsilon_j)$.

We then set $y \in FC_t \backslash FC_{t-1}$ if and only if $A(y) \simeq A(x_0(\varepsilon))$ with $t = \sum_{j=1}^{n} \varepsilon_j$.

It is now clear that δ_Λ^{int} sends FC_t to FC_t, and δ_Λ^{ext} sends FC_t to $\sum_{\delta \geq t+1} FC_s$. As a result, the cohomology of (FC^*, δ_Λ) can be computed by a spectral sequence associated to the level filtration. The computation of this spectral sequence is done by first taking the cohomology of $(FC^*, \delta_\Lambda^{int})$, that is $\bigoplus_{\varepsilon \in \{-1,1\}^n} FH^*(L, V_\varepsilon(\delta))$ and then computing the cohomology induced by δ_Λ^{ext}. But $FH^*(L, \Lambda)$ has rank 2^k and so has $\bigoplus_{\varepsilon \in \{-1,1\}^n} FH^*(L, V_\varepsilon(\delta))$ therefore the map induced by δ_Λ^{ext} must vanish.

This proves proposition 2.8. □

6 Appendix: extension to more general cotangent bundles

We shall here assume M is a compact manifold admitting a perfect Morse function, i.e. a Morse function for which Morse inequalities are in fact equalities.

We claim that the above proof works exactly in the same way. The crucial steps are

1. Construct the analogue of Λ. To obtain this, we start from a perfect Morse function f, and consider the graph of df. We then dilate this to $c \cdot df$ for c going to infinity. Then the intersection of the unit tube and this graph will be close to the vertical fibres over critical points of f. We may then isotope this graph inside the tube of radius 2 to make it equal to the vertical over such points. We denote by Λ this manifold. The minimal possible value of $\gamma(\Lambda)$ is a constant depending only on (M, g).

2. The inflation of the symplectic form so that the integral of ω over a J-holomorphic curve outside the unit tube is arbitrarily large. Now we may assume that Λ is essentially the union of two copies of a

vertical Lagrangian and a closed Lagrangian outside the tube. Let h be a plurisubharmonic function such that $\omega = dJ^*dh$. We may take $h(q,p) = |p|^2$. We are going to replace h by $\phi \circ h$ where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing convex function. It is then well-known that $\phi \circ h$ is again plurisubharmonic. Then the Liouville form $J^*df = \lambda = pdq$ is replaced by $\lambda_\phi = \phi'(f)\lambda$. Then any vertical Lagrangian (i.e. $q = q_0$) is again Lagrangian. Assume that $\phi(t) = a \cdot t$ for t greater than 2 and for a large enough. Then a Lagrangian outside the tube of radius 2 is still Lagrangian (since $\omega_\phi = a\omega$) and a vertical Lagrangian remains Lagrangian since $\lambda_\phi = \phi'(f)\lambda$ and $\lambda = pdq$ vanishes on vertical Lagrangians. From this it follows that Λ is Lagrangian for ω_ϕ, but the area of the portion of a holomorphic curve contained outside the tube of radius 2 is multiplied by a.

3. For a large enough we may then proceed as in the proof of proposition 5.1. The vanishing of the spectral sequence is derived in the same way as in the torus case, since the number of vertical parts is twice the number of critical points of h.

We thus proved

Theorem 6.1. *Let M be a manifold such that M carries a perfect Morse function. Then here exists a constant $C_M(g)$ such that any Lagrangian L in D_gT^*M satisfies*

$$\gamma(L) \leq C_M(g)$$

Remark 6.2. Note that we do not need M to carry a perfect Morse function. If a fiber bundle $F \to P \to M$ carries a perfect Morse function, then the lift $\tilde{L}$ of L to T^*P is also contained in the unit tube and thus satisfies

$$\gamma(L) \leq \gamma(\tilde{L}) \leq C_P(\tilde{g})$$

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