# $L^p-$solution of Backward Stochastic Differential Equation with Barrier

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**Abstract**

In this paper, we are interested in solving generalized backward stochastic differential equation with one barrier (RGBSDE for short). We deal with the case of fixed terminal and also the random terminal case. The study use some new technical aspects of the stochastic calculus related to the reflected Generalized BSDE, (see [5] for the case of BSDE) to derive a priori estimates and prove existence and uniqueness of solution in $L^p, p > 1$. The result extended the one of Aman et al [1]. The need for this type of extension comes from the desire to prove a probabilistic representation of $L^p-$ solution of semi-linear partial differential equation with nonlinear Neumann boundary condition when $p \in (1, 2)$.

**MSC Subject Classification:** 60F25; 60H20.
**Key Words**: Backward stochastic differential equation; Monotone generator; $p-$integrable data.

## 1 Introduction

A linear version of backward stochastic differential equations (BSDE's in short) was first considered by Bismut ([2], [3]) in the framework of optimal stochastic control. In 1990, Pardoux and Peng [12] have introduced a nonlinear version of BSDE's. Since, this kind of equations found several fields of applications. For instance, in mathematical finance (El Karoui et al [7], Cvitanic and Ma [6]), in stochastic control and stochastic game (Hamadène and Lepeltier [9],[10]). It also provided probabilistic interpretation of semi-linear PDE (Pardoux and Peng [11]). However in most of the previous works, solutions are taken in $L^2$ space or in $L^p, p > 2$. The first work where they were in $L^p, p \in (1, 2)$ is the one done by El Karoui et al [7] when the generator is Lipschitz continuous. Pardoux et al [5] generalized this result. They provided the existence and uniqueness of solution of BSDE when the data $\xi$ and $f(t, 0, 0)_{t\in[0,T]}$ are in

$L^p, p \in (1,2)$ with a monotone generator, both for equation on a fixed and random time interval. On other hand the so-called generalized BSDE have been considered by Pardoux and Zhang [13]. This equation involves the integral with respect to an increasing process and provide a probabilistic representation for solution of parabolic or elliptic semi-linear PDE with Neumann boundary condition. The reflected case in the domain of a convex function or above a given stochastic process have been investigate respectively by Essaky et al [8] and Aman et al [1] in the case that solution is in $L^2$. In this paper we generalize the above result dealing with a class of reflected generalized BSDE and searching for solution in $L^p, p \in (1,2)$. The paper is organized as follows: the next section contains all the notations and assumptions, while section 3 give essential estimates. Section 4 is devoted to existence and uniqueness result in the case where the data were in $L^p, p \in (1,2)$ on the fixed time interval. Finally in section 5 we deal with the same problem but on random time interval.

## 2 Preliminaries

Let $B = \{B_t\}_{t>0}$ be a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ augmented natural filtration of $B$ which satisfies the usual conditions.
For any real $p > 0$, let us define the following spaces:
$\mathcal{S}^p(\mathbb{R}^k)$, denote set of $\mathbb{R}^k-$ valued, adapted càdlàg processes $\{X_t\}_{t\in[0,T]}$ such that

$$||X||_{\mathcal{S}^p} = \mathbb{E}\left(\sup_{0\leq t\leq T} |X_t|^p\right)^{1\wedge\frac{1}{p}} < +\infty;$$

and $\mathcal{M}^p(\mathbb{R}^k))$ is the set of predictable processes $\{X_t\}_{t\in[0,T]}$ such that

$$||X||_{\mathcal{M}^p} = \mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{\frac{p}{2}}\right]^{1\wedge\frac{1}{p}} < +\infty.$$

If $p \geq 1$, then $||.||_{\mathcal{S}^p}$ (resp $||X||_{\mathcal{M}^p}$) is a norm on $\mathcal{S}^p(\mathbb{R}^k)$ (resp. $\mathcal{M}^p(\mathbb{R}^k)$) and these spaces are banach spaces. But if p$\in (0,1)$, $(X, X') \longmapsto \|X - X'\|_{\mathcal{S}^p(\mathbb{R}^k)}$ (resp $\|X - X'\|_{\mathcal{M}^p}$) define a distance on $\mathcal{S}^p(\mathbb{R}^k)$, (resp. $\mathcal{M}^p(\mathbb{R}^k)$)ands under this metric, $\mathcal{S}^p(\mathbb{R}^k)$ (resp. $\mathcal{M}^p(\mathbb{R}^k)$ is complete.

Now let us give the following assumptions:

**(A1)** $(G_t)_{t\geq 0}$ is a continuous real valued increasing $\mathcal{F}_t-$progressively measurable process

**(A2)** $f$ and $g$ are $\mathbb{R}-$values measurable functions defined respectively on $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$ and $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ such that there are constants $\mu \in \mathbb{R}$, $\beta < 0$, $\lambda > 0$ and $[1, +\infty)-$

valued process $\{\varphi_t,\ \psi_t\}_{t\leq 0}$ verifying

$(i)$ $\forall t, \forall z, y \longmapsto (f(t,y,z), g(t,y))$ is continuous

$(ii) \forall y, z, (\omega,t) \longmapsto (f(\omega,t,y,z), g(\omega,t,y))$ is $\mathcal{F}_t -$ progressively measurable

$(iii)$ $\forall t, \forall y, \forall (z,z'),\quad |f(t,y,z) - f(t,y,z')| \leq \lambda |z-z'|$

$(iv)$ $\forall t, \forall z, \forall (y,y'),\ (y-y')(f(t,y,z) - f(t,y',z)) \leq \mu |y-y'|^2$

$(v)$ $\forall t, \forall z, \forall (y,y'),\ (y-y')(g(t,y) - g(t,y')) \leq \beta |y-y'|^2$

$(vi)$ $\forall t, \forall y, \forall z,\ |f(t,y,z)| \leq \varphi_t + K(|y| + |z|),\quad |g(t,y)| \leq \psi_t + K|y|$

$(vii)$ $\mathbb{E}\left[\left(\int_0^T \varphi(s)ds\right)^p + \left(\int_0^T \psi(s)dG_s\right)^p\right] < \infty.$

(**A3**) For any $r > 0$, we define the processes $\pi_r$ and $\phi_r$ in $L^1([0,T]\times\Omega, m\otimes\mathbb{P})$ such that

$(i)$ $\pi_r(t) = \sup_{|y|\leq r} |f(t,y,0) - f(t,0,0)|,$

$(ii)$ $\phi_r(t) = \sup_{|y|\leq r} |g(t,y) - g(t,0)|.$

(**A4**) $\xi$ is a $\mathcal{F}_T-$measurable variable such that $\mathbb{E}(|\xi|^p) < +\infty$

(**A5**) $(S_t)_{t\geq 0}$ is a continuous progressively measurable real-valued process satisfying:

$(i)$ $\quad \mathbb{E}\left(\sup_{0\leq t\leq T}(S_t^+)^p\right) < +\infty$

$(ii)$ $\quad S_T \leq \xi$ $\mathbb{P}$ a.s.

Before of all let us give meanning of a $L^p-$ solution of reflected BSDE.

**Definition 2.1** *A $L^p-$ solution of reflected generalized BSDE associated to the data $(\xi, f, g, S)$ is a triplet $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ of progressively measurable processes taking values in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ and satisfying:*

*$(i)$ $Y$ is a continuous process*

*$(ii)$ $Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T f(s,Y_s,Z_s)dG_s - \int_t^T Z_s dW_s + K_T - K_t$*

*$(iii)$ $Y_t \geq S_t$*

*$(iv)$ $\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^p + \left(\int_0^T |Z_s|^2 ds\right)^{p/2}\right) < +\infty,$*

*$(v)$ $K$ is an increasing process such that $K_0 = 0$ and $\int_0^T (Y_s - S_s)^{p-1} dK_s = 0$ a.s.*

Let us now introduced the notation $\widehat{x} = |x|^{-1}x\mathbf{1}_{\{x\neq 0\}}$ in order to give basic inequality which is analogue of corollary 2.3 in [5].

# 3 A priori estimates

**Lemma 3.2** *If $(Y, Z, K)$ is a solution of reflected generalized BSDE associated to $(\xi, f, g, S)$, $p \geq 1$, $c(p) = p\left[(p-1)\wedge 1\right]/2$. Then*

$$\begin{aligned} & |Y_t|^p + c(p)\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z|^2 ds \\ \leq\ & |\xi| + p\int_t^T |Y_s|^{p-1}\widehat{Y}_s\ f(s, Y_s, Z_s)\,ds + p\int_t^T |Y_s|^{p-1}\widehat{Y}_s\ g(s, Y_s)\,dG_s \\ & +p\int_t^T |Y_s|^{p-1}\widehat{Y}_s\ dK_s - p\int_t^T |Y_s|^{p-1}\widehat{Y}_s\ Z_s dW_s. \end{aligned}$$

Now we state some estimate for solution of the reflected generalized BSDE associated to $(\xi, f, g, S)$ in $L^p$. Indeed, let $p > 1$, and $\xi, f, g$ given above. In view of $(\mathbf{A2})$, we assume the following: $\forall\ (t, y, z) \in [0, T]\times\mathbb{R}\times\mathbb{R}^d$

$$\begin{aligned} \widehat{y}\ f(t, y, z) &\leq \varphi_t + \mu|y| + \lambda|z|, \\ \widehat{y}\ g(t, y) &\leq \psi_t + \beta|y|,\ \mathbb{P}-\text{a.s.} \end{aligned} \tag{3.1}$$

First of all we give an estimate which permit to control the process $Z$ with the data and the process $Y$.

**Lemma 3.3** *Assume $(\boldsymbol{A1}) - (\boldsymbol{A5})$ and $(3.1)$. Moreover let $(Y, Z, K)$ be the solution of generalized reflected BSDE associated to $(\xi, f, g, S)$. If $Y \in \mathcal{S}^p$ then $Z$ belong to $\mathcal{M}^p$ and there exist a>0, constant $C_p$ depending only on $p$ such that for any $a \geq \mu + \lambda^2$,*

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T e^{2a(r+G_r)}|Z_r|^2 dr\right)^{p/2}\right] \leq\ & C_p\mathbb{E}\left\{\sup_{0\leq t\leq T} e^{ap(t+G(t))}|Y_t|^p + \left(\int_0^T e^{a(r+G_r)}\varphi(r)dr\right)^p\right. \\ & \left. + \left(\int_0^T e^{a(r+G_r)}\psi(r)dG_r\right)^p\right\}. \end{aligned}$$

**Proof.** Let us fix $a \geq \mu + \lambda^2$ and define $\tilde{Y}_t = e^{a(t+G(t))}Y_t$, $\tilde{Z}_t = e^{a(t+G(t))}Z_t$ and $\tilde{K}_t = e^{a(t+G(t))}K_t$. Then $\left(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t\right)$ solves the reflected generalized BSDE

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{f}(s, Y_s, Z_s)ds + \int_t^T \tilde{g}(s, Y_s)dG_s - \int_t^T \tilde{Z}_s dW_s + \tilde{K}_T - \tilde{K}_t, 0 \leq t \leq T,$$

where $\tilde{\xi} = e^{a(T+G(T))}\xi$, $\tilde{f}(s,y,z) = f(t, e^{-a(t+G(t))}y, e^{-a(t+G(t))}z) - ay$, $\tilde{g}(s,y) = g(t, e^{-a(t+G(t))}y) - ay$ which satisfied assumption (3.1) with $\tilde{\varphi}_t = e^{-a(t+G(t))}\varphi_t$, $\tilde{\psi}_t = e^{-a(t+G(t))}\psi_t$, $\tilde{\lambda} = \lambda$, $\tilde{\mu} = \mu - a$ and $\tilde{\beta} = \beta - a$. Since we are working on the compact time interval, integrability conditions are equivalent with or without the superscript $\tilde{}$. Thus, with this change of variable we reduce to the case $a = 0$ and $\mu + \lambda^2 \leq 0$ and we forget superscript $\tilde{}$ for notational convenience. For each integer $n \geq 1$ let us introduce the stopping time

$$\tau_n = inf\left\{t \in [0,T], \int_0^t |Z_r|dr \geq n\right\} \wedge T.$$

Itô's formula yields

$$\begin{aligned}
& |Y_0| + \int_0^{\tau_n} |Z_r|^2 dr \\
= \; & |Y_{\tau_n}|^2 + 2\int_0^{\tau_n} Y_r \; f(r, Y_r, Z_r)dr + 2\int_0^{\tau_n} Y_r \; g(r, Y_r)dG_r \\
& +2\int_0^{\tau_n} Y_r \; dK_r - 2\int_0^{\tau_n} Y_r \; Z_r dW_r. \qquad (3.2)
\end{aligned}$$

In view of assumption (3.1),we have

$$\begin{aligned}
y \; f\,(t,y,z) \; & \leq \; 2|y|\varphi_t + 2(\mu + \lambda^2)|y|^2 + \frac{1}{2}|z|^2, \\
y \; g\,(t,y) \; & \leq \; 2|y|\psi_t + 2\beta|y|^2.
\end{aligned}$$

So inequality 3.2 leads to

$$\begin{aligned}
& |Y_0|^2 + \int_0^{\tau_n} |Z_r|^2 dr \\
\leq \; & |Y_{\tau_n}|^2 + 2\int_0^T |Y_r| \; |\varphi_r|dr + 2\int_0^T |Y_r| \; |\psi_r|dG_r \\
& +2(\mu+\lambda^2)\int_0^T |Y_r|^2 dr + 2\beta\int_0^T |Y_r|^2 dG_r \\
& +\frac{1}{2}\int_0^{\tau_n} |Z_r|^2 dr + 2\int_0^T Y_r dK_r \\
& -2\int_0^T Y_r Z_r dW_r. \qquad (3.3)
\end{aligned}$$

Since $\mu + \lambda^2 \leq 0$, $\beta < 0$, $\tau_n \leq T$, we deduce from (3.3) that

$$\begin{aligned}
\frac{1}{2}\int_0^{\tau_n} |Z_r|^2 dr \; & \leq \; Y_*^2 + 2Y_*^2\int_0^T |\varphi_r|dr \\
& +2Y_*^2\int_0^T |\psi_r|dG_r \\
& +2Y_*^2.K_T + 2\sup\left|\int_0^{\tau_n} \langle Y_r, Z_r dW_r\rangle\right|,
\end{aligned}$$

with $Y_*^p = \sup_{0\le t\le T}|Y_t|^p$ for $p \ge 1$.
Therefore

$$\begin{aligned}\int_0^{\tau_n}|Z_r|^2dr &\le C\left\{Y_*^2 + \left(\int_0^T\varphi_r dr\right)^2 + \left(\int_0^T\psi_r dG_r\right)^2\right. \\ &\quad \left. + \sup\left|\int_0^{\tau_n}\langle Y_r, Z_r dW_r\rangle\right|\right\} + \gamma|K_T|^2.\end{aligned}$$

We have

$$|K_T|^2 \le c\left\{Y_*^2 + \left(\int_0^T\varphi_r dr\right)^2 + \left(\int_0^T\psi_r dG_r\right)^2 + \int_0^{\tau_n}|Z_r|^2dr\right\}.$$

By choosing $\gamma = \frac{1}{2c}$ we obtain

$$\left(\int_0^{\tau_n}|Z_r|^2dr\right)^{p/2} \le C_p\left\{Y_*^2 + \left(\int_0^T\varphi_r dr\right)^p + \left(\int_0^T\psi_r dG_r\right)^p \right. \tag{3.4}$$

$$\left. + \left|\int_0^{\tau_n}\langle Y_r, Z_r dW_r\rangle\right|^{p/2}\right\}. \tag{3.5}$$

On the other hand it follows by using Burlkölder-Davis-Gundy and Young inequalities that

$$\begin{aligned}C_p\mathbb{E}\left(\left|\int_0^{\tau_n}\langle Y_r, Z_r dW_r\rangle\right|^{p/2}\right) &\le d_p\mathbb{E}\left(\int_0^{\tau_n}|Y_r|^2|Z_r|^2dr\right)^{p/4} \\ &\le d_p\mathbb{E}\left[Y_*^{p/2}\left(\int_0^{\tau_n}|Z_r|^2dr\right)^{p/4}\right] \\ &\le \frac{d_p^2}{2}\mathbb{E}\left(Y_*^p\right) + \frac{1}{2}\mathbb{E}\left(\int_0^{\tau_n}|Z_r|^2dr\right)^{p/2}. \end{aligned} \tag{3.6}$$

Now putting (3.6) in (3.5) we have for each $n \ge 1$

$$\mathbb{E}\left(\int_0^{\tau_n}|Z_r|^2dr\right)^{p/2} \le C_p\left\{Y_*^2 + \left(\int_0^T\varphi_r dr\right)^p + \left(\int_0^T\psi_r dG_r\right)^p\right\}.$$

Finally Fatou's Lemma implies that

$$\mathbb{E}\left(\int_0^{T}|Z_r|^2dr\right)^{p/2} \le C_p\left\{Y_*^p + \left(\int_0^T\varphi_r dr\right)^p + \left(\int_0^T\psi_r dG_r\right)^p\right\}.$$

■

The second estimation gives a necessary condition for existence and uniqueness of solution of the reflected generalized BSDE associated to the data $(\xi, f, g, S)$.

**Lemma 3.4** *Suppose* (3.1) *hold and assume* $(\boldsymbol{A1})-(\boldsymbol{A5})$. *Let* $(Y,Z,K))$ *be a solution of the reflected BDSE associated to the data* $(\xi,f,g,S)$ *where* $Y$ *belong to* $\mathcal{S}^p$. *Then there exist a constant* $C_p$ *depending only on* $p$ *such that for any* $a\geq\mu+\lambda^2/[1\wedge(p-1)]$

$$\begin{aligned}&\mathbb{E}\left\{\sup_{0\leq t\leq T}e^{ap(t+G(t))}|Y_t|^p+\left(\int_0^T e^{2a(s+G(s))}|Z_s|^2ds\right)^{p/2}\right\}\\ \leq\ &\mathbb{E}\left\{e^{ap(T+G(T))}|\xi|^p+\left(\int_0^T e^{a(s+G(s))}\varphi_s ds\right)^p\right.\\ &\left.+\left(\int_0^T e^{a(s+G(s))}\psi_s dG_s\right)^p+\sup_{0\leq t\leq T}e^{ap(t+G(t))}(S_t^+)^p\right\}.\end{aligned}$$

**Proof.** Let us fix $a\geq\mu+\lambda^2/[1\wedge(p-1)]$. The same argument as the previous proof reduce to the case $a=0$ and $\mu+\lambda^2/[1\wedge(p-1)]\leq 0$. So we have to prove without the superscript $\tilde{}$

$$\begin{aligned}&\mathbb{E}\left\{\sup_{0\leq t\leq T}|Y_t|^p+\left(\int_0^T|Z_s|^2ds\right)^{p/2}\right\}\\ \leq\ &\mathbb{E}\left\{|\xi|^p+\left(\int_0^T\varphi_s ds\right)^p\right.\\ &\left.+\left(\int_0^T\psi_s dG_s\right)^p+\sup_{0\leq t\leq T}(S_t^+)^p\right\}.\end{aligned}$$

By virtue of Lemma 3.2 and (3.1) we get

$$\begin{aligned}&|Y_t|^p+c(p)\int_t^T|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq0\}}|Z_s|^2ds\\ \leq\ &|\xi|^p+p\int_0^T(|Y_s|^{p-1}\varphi_s+\mu|Y_s|^p)ds+p\int_0^T(|Y_s|^{p-1}\psi_s+\beta|Y_s|^p)dG_s\\ &+p\lambda\int_0^T|Y_s|^{p-1}|Z_s|ds+p\int_t^T(Y_s-S_s)^{p-1}dK_s\\ &+p\int_t^T(S_s^+)^{p-1}dK_s-p\int_0^T|Y_s|^{p-1}\tilde{Y}_sZ_sdW_s\\ \leq\ &|\xi|^p+p\int_0^T(|Y_s|^{p-1}\varphi_s+\mu|Y_s|^p)ds+p\int_0^T(|Y_s|^{p-1}\psi_s+\beta|Y_s|^p)dG_s\\ &+p\lambda\int_0^T|Y_s|^{p-1}|Z_s|ds+p\int_t^T(S_s^+)^{p-1}dK_s-p\int_0^T|Y_s|^{p-1}\tilde{Y}_sZ_sdW_s.\end{aligned}\qquad(3.7)$$

From the previous inequality it not difficult to check that, $\mathbb{P}-a.s.$,

$$\int_t^T|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq0\}}|Z_s|^2ds<\infty.$$

Moreover, by the Young inequality we show easily that

$$p\lambda\int_0^T |Y_s|^{p-1}|Z_s| \leq \frac{p\lambda^2}{1\wedge(p-1)}\int_0^T |Y_s|^p ds + \frac{c(p)}{2}\int_0^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2,$$

and since $\mu + \lambda^2/[1\wedge(p-1)] \leq 0$ and $\beta < 0$ it follows from (3.7) that

$$\begin{aligned} & |Y_t|^p + \frac{c(p)}{2}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds \\ \leq\ & |\xi|^p + p\int_0^T |Y_s|^{p-1}|f_s^0|ds + p\int_0^T |Y_s|^{p-1}|g_s^0|dG_s \\ & +p\int_t^T (S_s^+)^{p-1}dK_s - p\int_0^T |Y_s|^{p-1}\tilde{Y}_s Z_s dW_s. \end{aligned} \tag{3.8}$$

Let us denote

$$X = |\xi|^p + p\int_0^T |Y_s|^{p-1}\varphi_s ds + p\int_0^T |Y_s|^{p-1}\psi_s dG_s + p\int_t^T (S_s^+)^{p-1}dK_s;$$

So inequality (3.8) become

$$|Y_t|^p + \frac{c(p)}{2}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds \leq X - p\int_0^T |Y_s|^{p-1}\tilde{Y}_s Z_s dW_s. \tag{3.9}$$

Let us define $M_t = \int_t^T |Y_s|^{p-1}\tilde{Y}_s Z_s dW_s$. Then $\{M_t\}_{0\leq t\leq T}$ is a uniformly integrable martingale. Indeed, by young's inequality we have

$$\mathbb{E}\left[\langle M,M\rangle_T^{1/2}\right] \leq \frac{p-1}{p}\mathbb{E}(Y_*^p) + \frac{1}{p}\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^{p/2};$$

the last term being finite since $Y$ belongs to $\mathcal{S}^p$ and then by Lemma 3.2 $Z$ is in $\mathcal{M}^p$. Coming back to inequality (3.9) and taking the expectation for $t = 0$, we get both

$$\frac{c(p)}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds \leq \mathbb{E}(X) \tag{3.10}$$

and

$$\mathbb{E}(Y_*^p) \leq \mathbb{E}(X) + k_p\mathbb{E}\left[\langle M,M\rangle_T^{1/2}\right]. \tag{3.11}$$

But

$$k_p\mathbb{E}\left[\langle M,M\rangle_T^{1/2}\right] \leq \frac{1}{2}\mathbb{E}(Y_*^p) + \frac{k_p^2}{2}\mathbb{E}\int_t^T |Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2 ds. \tag{3.12}$$

Combining the inequalities (3.10), (3.11) and (3.12) we obtain

$$\mathbb{E}(Y_*^p) \leq d_p \mathbb{E}(X).$$

Using once again Young's inequality, we have

$$\begin{aligned} & pd_p \mathbb{E}\left\{\int_0^T |Y_s|^{p-1}\varphi_s ds + \int_0^T |Y_s|^{p-1}\psi_s dG_s\right\} \\ \leq\ & \frac{1}{2}\mathbb{E}(Y_*^p) + d'_p \mathbb{E}\left\{\left(\int_0^T \varphi_s ds\right)^p + \left(\int_0^T \psi_s dG_s\right)^p\right\}. \end{aligned}$$

According to definition of $X$ we have

$$\begin{aligned} \mathbb{E}(Y_*^p) &\leq C'_p \mathbb{E}\left\{|\xi|^p + \left(\int_0^T \varphi_s ds\right)^p + \left(\int_0^T \psi_s dG_s\right)^p + \int_0^T (S_s^+)^{p-1} dK_s\right\} \\ &\leq C'_p \mathbb{E}\left\{|\xi|^p + \left(\int_0^T \varphi_s ds\right)^p + \left(\int_0^T \psi_s dG_s\right)^p + \sup_{t\leq s\leq T}(S_s^+)^p\right\} + \frac{1}{\gamma}|K_T|^p. \quad (3.13) \end{aligned}$$

We now give an estimation of $\mathbb{E}\left(|K_T|^p\right)$ of the form

$$\begin{aligned} \mathbb{E}(|K_T|^p) \leq\ & C_p \mathbb{E}\left\{|\xi|^p + \left(\int_0^T \varphi_s ds\right)^p + \left(\int_0^T \psi_s dG_s\right)^p \right. \\ & \left. + \sup_{0\leq t\leq T}(S_t^+)^p\right\}. \end{aligned} \quad (3.14)$$

Recalling inequality (3.13) and in view of (3.14) we get

$$\begin{aligned} \mathbb{E}(Y_*^p) \leq\ & C_p \mathbb{E}\left\{|\xi|^p + \left(\int_0^T \varphi_s ds\right)^p + \left(\int_0^T \psi_s dG_s\right)^p \right. \\ & \left. + \sup_{0\leq t\leq T}(S_t^+)^p\right\}. \end{aligned}$$

The result follows from Lemma 3.2 ■

# 4 Existence and uniqueness of a solution

In this section we prove existence and uniqueness result for the reflected generalized BSDE associated to data $(\xi, f, g, S)$ in $L^p$, with the help of priori estimates given above.

**Theorem 4.5** *Assume $(A1)-(A5)$, the reflected generalized BSDE with data $(\xi, f, g, S)$ has a unique solution $(Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}^p \times \mathcal{S}^p$.*

**Proof. Uniqueness**
Let us consider $(Y, Z, K)$ and $(Y', Z', K')$ two solutions of our reflected generalized BSDE in the appropriate space. We denote by $(\bar{Y}, \bar{Z}, \bar{K}) = (Y - Y', Z - Z', K - K')$ solution to the following reflected generalized BSDE

$$\bar{Y}_t = \int_t^t f_1(s, \bar{Y}, \bar{Z})ds + \int_t^t g_1(s, \bar{Y})dG_s - \int_t^t \bar{Z}dW_s + \bar{K}_T - \bar{K}_t, 0 \leq t \leq T;$$

where $f_1$ and $g_1$ are defined by

$$f_1(t, \bar{y}, \bar{z}) = f(t, y + Y'_t, z + Z'_t) - f(t, Y', Z')$$

$$g_1(t, \bar{y}) = g(t, y + Y'_t) - g(t, Y').$$

In virtue of assumption $(A2)$ we note that $f_1$ and $g_1$ satisfied (3.3) with $f_1 \equiv 0$ and $g_1 \equiv 0$; and by Lemma 3.4 it follows that $(\bar{Y}, \bar{Z}, \bar{K}) \equiv 0$.
**Existence** It will be split into two steps
**Step 1.** In this part $\xi, \sup\varphi_t, \sup\psi_t, \sup S_t^+$ are supposed bounded random variables and $r$ a positive real such that

$$\|\xi\|_\infty + T\|\varphi\|_\infty + \|G_T\|_\infty\|\psi\|_\infty + \|S^+\|_\infty < r.$$

We consider $\theta_r$ a smooth function such that $0 \leq \theta_r \leq 1$, and define by

$$\theta_r(y) = \begin{cases} 1 \text{ for } |y| \leq r \\ 0 \text{ for } |y| \geq r + 1; \end{cases}$$

denote for each $n \in \mathbb{N}^*$, $q_n(z) = z\frac{n}{|z| \vee n}$. Let us set

$$h_n(t, y, z) = \theta_r(y)(f(t, y, q_n(z)) - f_t^0)\frac{n}{\pi_{r+1}(t) \vee n} + f_t^0$$

$$l_n(t, y) = \theta_r(y)(g(t, y) - g_t^0)\frac{n}{\phi_{r+1}(t) \vee n} + g_t^0,$$

where $f_t^0 = f(t, 0, 0)$; $g_t^0 = g(t, 0)$ .

It not difficult to prove that there exist two constants $\kappa$ and $\eta$ depending to $n$ such that

$$(y - y')(h_n(t, y, z) - h_n(t, y', z)) \leq \kappa|y - y'|^2$$

$$(y - y')(l_n(t, y) - h_n(t, y')) \leq \eta|y - y'|^2.$$

Then data $(\xi, h_n, l_n, S)$ satisfied condition in Aman et al [1]; hence equation associated has a unique solution $(Y^n, Z^n, K^n)$ belong in space $\mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{S}^2$.
In the other hand since

$$\begin{aligned} y\ h_n(t,y,z) &\leq |y|\ \|f^0\|_\infty + \lambda |y|\ |z| \\ y\ l_n(t,y) &\leq |y|\ \|g^0\|_\infty \end{aligned}$$

and $\xi$ bounded, the same computation of Lemma 2.2 of [4] adapted to reflected generalized BSDE show that the process $Y^n$ satisfies the inequality $\|Y^n\|_\infty \leq r$ and in view of Lemma 3.3, $\|Z^n\|_{\mathcal{M}^2} \leq r'$ where $r'$ is another constant. As a byproduct $(Y^n, Z^n, K^n)$ is a solution to the reflected generalized BSDE associated to $(\xi, f_n, g_n, S)$ where

$$\begin{aligned} f_n(t,y,z) &= (f(t,y,q_n(z)) - f_t^0)\frac{n}{\pi_{r+1}(t) \vee n} + f_t^0 \\ g_n(t,y) &= (g(t,y) - g_t^0)\frac{n}{\phi_{r+1}(t) \vee n} + g_t^0 \end{aligned}$$

that satisfied assumption $(\mathbf{A2}iv)$ and $(\mathbf{A2}v)$.
Now let denote $(\bar{Y}^{n,i}, \bar{Z}^{n,i}, \bar{K}^{n,i})$ by

$$\bar{Y}^{n,i} = Y^{n+i} - Y^n,\ \ \bar{Z}^{n,i} = Z^{n+i} - Z^n,\ \ \bar{K}^{n,i} = K^{n+i} - K^n,\ \ i, n \in \mathbb{N}^*$$

and $\Phi(t) = \exp[2(\lambda^2 + \mu)t]$.
Applying Itô's formula to the function $\Phi(t)|\bar{Y}^n|^2$, we have

$$\begin{aligned} & \Phi(t)|\bar{Y}_t^{n,i}|^2 + 2(\lambda^2+\mu)\int_t^T \Phi(s)|\bar{Y}_s^{n,i}|^2 ds + \int_t^T \Phi(s)|\bar{Z}_s^{n,i}|^2 ds \\ \leq\ & 2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(f_{n+i}(s, Y_s^{n+i}, Z_s^{n+i}) - f_n(s, Y_s^n, Z_s^n))ds \\ & +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(g_{n+i}(s, Y_s^{n+i}) - g_n(s, Y_s^n))dG_s \\ & +2\int_t^T \Phi(s)\bar{Y}_s^{n,i} d\bar{K}_s^{n,i} - 2\int_t^T \Phi(s)\bar{Y}_s^{n,i} dW_s \\ =\ & A_1 + A_2 + 2\int_t^T \Phi(s)\bar{Y}_s^{n,i} d\bar{K}_s^{n,i} - 2\int_t^T \Phi(s)\bar{Y}_s^{n,i} dW_s. \end{aligned} \tag{4.15}$$

It follows by virtue of (**A2***iv*) and (**A2***v*) that

$$\begin{aligned}
A_1 &\leq 2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(f_{n+i}(s,Y_s^{n+i},Z_s^{n+i}) - f_{n+i}(s,Y_s^n,Z_s^{n+i}))ds \\
&\quad +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(f_{n+i}(s,Y_s^n,Z_s^{n+i}) - f_{n+i}(s,Y_s^n,Z_s^n))ds \\
&\quad +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n))ds \\
&\leq 2(\lambda^2+\mu)\int_t^T \Phi(s)|\bar{Y}_s^{n,i}|^2 ds + \frac{1}{2}\int_t^T \Phi(s)|\bar{Z}_s^{n,i}|^2 ds \\
&\quad +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n))ds, \qquad (4.16)
\end{aligned}$$

$$\begin{aligned}
A_2 &\leq 2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(g_{n+i}(s,Y_s^{n+i}) - g_{n+i}(s,Y_s^n))dG_s \\
&\quad +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(g_{n+i}(s,Y_s^n) - g_n(s,Y_s^n))dG_s \qquad (4.17) \\
&\leq 2\beta\int_t^T \Phi(s)|\bar{Y}_s^{n,i}|^2 dG_s + 2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(g_{n+i}(s,Y_s^n) - g_n(s,Y_s^n))dGs.
\end{aligned}$$

Pluging (4.16) and (4.17) in (4.15) and since $\beta < 0$, we obtain

$$\begin{aligned}
&\Phi(t)|\bar{Y}_t^{n,i}|^2 + \frac{1}{2}\int_t^T \Phi(s)|\bar{Z}_s^{n,i}|^2 ds \\
&\leq 2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n))ds \\
&\quad +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}(g_{n+i}(s,Y_s^n) - g_n(s,Y_s^n))dGs \\
&\quad +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}d\bar{K}_s^n - 2\int_t^T \Phi(s)\bar{Y}_s^{n,i}dW_s.
\end{aligned}$$

But since $\|\bar{Y}^{n,i}\|_\infty \leq 2r$, it follows

$$\begin{aligned}
&\Phi(t)|\bar{Y}_t^{n,i}|^2 + \frac{1}{2}\int_t^T \Phi(s)|\bar{Z}_s^{n,i}|^2 ds \\
&\leq 2r\int_t^T \Phi(s)|f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)|ds \\
&\quad +\int_t^T \Phi(s)|g_{n+i}(s,Y_s^n) - g_n(s,Y_s^n)|dGs \\
&\quad +2\int_t^T \Phi(s)\bar{Y}_s^{n,i}d\bar{K}_s^{n,i} - 2\int_t^T \Phi(s)\bar{Y}_s^{n,i}dW_s.
\end{aligned}$$

Showing $\mathbb{E}\int_0^T \bar{Y}^{n,i} d\bar{K}^{n,i} \leq 0 \quad \mathbb{P}-a.s.$, Burkölder-Davis-Gundy inequality yields that there exist $C$ (depending only on $\lambda,\ \mu$ and $T$) such that

$$\begin{aligned} & \mathbb{E}\left(\sup_{0\leq t\leq T} |\bar{Y}_t^{n,i}|^2 + \frac{1}{2}\int_0^T |\bar{Z}_s^{n,i}|^2 ds\right) \\ \leq \; & Cr\mathbb{E}\left\{\int_0^T |f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)|ds \right. \\ & \left. + \int_0^T |g_{n+i}(s,Y_s^n) - g_n(s,Y_s^n)|dGs\right\}. \end{aligned} \tag{4.18}$$

Moreover recalling $\|Y^n\|_\infty \leq r$, we prove that

$$|f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)| \;\leq\; 2\lambda|Z^n|\mathbf{1}_{\{|Z^n|\ >n\}} + 2\lambda|Z^n|\mathbf{1}_{\{\pi_{r+1}>n\}} + 2\pi_{r+1}\mathbf{1}_{\{\pi_{r+1}>n\}}$$

$$|g_{n+i}(s,Y_s^n) - g_n(s,Y_s^n)| \;\leq\; 2\phi_{r+1}\mathbf{1}_{\{\phi_{r+1}>n\}}$$

from which we deduce according assumption (**A3**) and inequality (4.18) that $(Y^n, Z^n)$ is a cauchy sequence in $\mathcal{S}^2\times\mathcal{M}^2\times$, then it converge to the limit $(Y,Z)$ in $\mathcal{S}^2\times\mathcal{M}^2$ satisfied $Y_t \geq S_t;\ \forall\ 0\leq t\leq T$.
Let us define

$$K_t^n = Y_0 - Y_t - \int_0^t f(s,Y_s^n,Z_s^n)ds - \int_0^t g(s,Y_s^n)dG_s + \int_0^t Z_s^n dW_s.$$

Then it follows from Aman and al [1] that:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |K_t^n - K_t^p|^2\right) \longrightarrow 0, as\ n,p\longrightarrow\infty,$$

hence there exist a non decreasing process $K(K_0=0)$ such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |K_t^n - K_t|^2\right) \longrightarrow 0, as\ n\longrightarrow\infty$$

and

$$\int_0^T (Y_s - S_s)^{p-1} dK_s = 0, \text{ for every } T\geq 0.$$

Passing to the limit in the reflected generalized BSDE with data $(\xi, f_n, g_n, S)$ it follows that $(Y,Z,K)$ is solution of reflected generalized BSDE associated to data $(\xi,f,g,S)$.
**Step 2.** we now treat the general case. For each $n\in\mathbb{N}^*$, let us define

$$\begin{aligned} \xi_n &= q_n(\xi) \\ f_n(t,y,z) &= \frac{n}{|\varphi_t|\vee n}\left[f\left(t, \frac{|\varphi_t|\vee n}{n}y, \frac{|\varphi_t|\vee n}{n}z\right) - f_t^0\right] + \frac{n}{|\varphi_t|\vee n}f_t^0 \\ g_n(t,y) &= \frac{n}{|\psi_t|\vee n}\left[g\left(t, \frac{|\psi_t|\vee n}{2}y\right) - g_t^0\right] + \frac{n}{|\varphi_t|\vee n}g_t^0 \\ S_t^n &= q_n(S_t). \end{aligned}$$

The data $(\xi_n, f_n, g_n, S^n)$ satisfies the condition of **step1** and reflected generalized BSDE associated to it has a unique solution $(Y_n, Z_n, K_n) \in L^2$ thanks to the first step of this proof; but according to Lemma 3.3, $(Y_n, Z_n, K_n)$ is also in $L^p,\ p > 1$. Further applying Lemma 3.4, for $(i, n) \in \mathbb{N} \times \mathbb{N}^*$, there exist constant depending on $T,\ \lambda$ such that

$$\begin{aligned} & \mathbb{E}\left\{\sup_{0\le t\le T}|Y_t^{n+i} - Y_t^n|^p + \left(\int_0^T |Z_t^{n+i} - Z_t^n|^2 ds\right)^{p/2}\right\} \\ \le\ & C\mathbb{E}\left\{|\xi_{n+i} - \xi_n|^p + \int_0^T |q_{n+i}(\varphi_s) - q_n(\varphi_s)|^p ds \right. \\ & \left. + \int_0^T |q_{n+i}(\psi_s) - q_n(\psi_s)|^p dG_s + \sup_{0\le t\le T}|q_{n+i}(S_t^+) - q_n(S_t^+)|^p\right\} \end{aligned}$$

We show that the right-hand side of the last inequality converge uniformly on $i$ to 0 as $n \longrightarrow \infty$, so we conclude that $(Y^n, Z^n)$ is a cauchy sequence in $\mathcal{S}^p \times \mathcal{M}^p$ and

$$\mathbb{E}\left(\sup_{0\le t\le T}|K_t^n - K_t|^p\right) \longrightarrow 0, as\ n \longrightarrow \infty.$$

Passing to the limit in equation associated to $(\xi_n, f_n, g_n, S^n)$, the triplet $(Y, Z, K)$ is a $L^p-$solution to the reflected generalized BSDE with determinist time associated to $(\xi, f, g, S)$. ∎

# 5 $L^p-$solution of reflected generalized BSDE with random terminal time

Now let us assume that $T$ is a $\{\mathcal{F}_t\}_{t\ge 0}-$stopping time. For instance hypothesis $(A2vii), (A4)$ will be replaced respectively by $(A6), (A7)$ and in $(A3)$ processes $\pi_r, \phi_r$ are in $L^1((0,n) \times \Omega, m \otimes \mathbb{P})$.

$(A6)$ For some $\rho > \alpha = |\mu| + \frac{\lambda^2}{2(p-1)}$ and $\nu > \theta = |\beta|$ ($\mu, \beta$ and $\lambda$ are constants appearing in assumption $(A2)$) $\mathbb{E}\left(\int_0^T e^{p(\rho s+\nu G(s))}\left(|\varphi_s|^p ds + |psi_s|^p dG_s\right)\right) < \infty$.

$(A7)$ $\xi$ is $\mathcal{F}_T-$measurable and

$$\begin{aligned} \mathbb{E}\left\{\int_0^T e^{p(\rho s+\nu G(s))}|f(s, e^{-(\alpha s+\theta G(s))}\bar{\xi}_s, e^{-(\alpha s+\theta G(s))}\bar{\eta}_s)|^p ds \right. \\ \left. + \int_0^T e^{p(\rho s+\nu G(s))}|g(s, e^{-(\alpha s+\theta G(s))}\bar{\xi}_s)|^p dG_s\right\} < \infty. \end{aligned}$$

where $\bar{\xi} = e^{(\alpha T+\theta G(T))}\xi,\ \bar{\xi}_t = \mathbb{E}(e^{(\alpha T+\theta G(T))}\xi/\mathcal{F}_t)$ and $\bar{\eta}$ a predictable process such that

$$\bar{\xi} = \mathbb{E}(\bar{\xi}) + \int_0^\infty \bar{\eta}_s dW_s,\ \mathbb{E}\left(\int_0^\infty |\bar{\eta}_s|^2 ds\right)^{p/2}.$$

**Definition 5.6** *A triplet* $(Y,Z,K)$ *of progressively measurable processes with values in* $I\!R\times I\!R^d\times I\!R$ *is a solution of the reflected generalized BSDE with random terminal time* $T$ *and data* $(\xi,f,g,S)$ *if on the set* $\{t\leq T\}$ $Y_t=\xi,\ Z_t=0,\ K_t=K_T$ $I\!P-a.s$; $t\longmapsto \mathbf{1}_{\{t\leq T\}}f(t,Y_t,Z_t),\ t\longmapsto \mathbf{1}_{\{t\leq T\}}g(t,Y_t)$ *belong to* $L^1_{loc}(0,\infty),\ t\longmapsto Z_t$ *belong to* $L^2_{loc}(0,\infty)$ *and* $I\!P-a.s,$ *for all* $0\leq t\leq u$,

*(i)* $Y_t=Y_{T\wedge u}+\int_t^{T\wedge u}f(s,Y_s,Z_s)ds+\int_{t\wedge u}^{T\wedge u}g(s,Y_s)dG_s-\int_{t\wedge u}^{T\wedge u}Z_sdW_s+K_{T\wedge u}-K_{t\wedge T}$

*(ii)* $Y_t\geq S_t$

*(iii)* $K$ *is an non-decreasing process such that* $K_0=0$ *and* $\int_0^{T\wedge u}(Y_t-S_t)^{p-1}dK_t=0$

*A solution is said to be in* $L^p$ *if we have moreover*

$$I\!E\left(\sup_{0\leq t\leq T}e^{p(\rho t+\nu G(t))}|Y_t|^p+\int_0^T e^{p(\rho s+\nu G(s))}\left[|Y_s|^{p-2}(|Y_s|^2+|Z_s|^2)ds+|Y_s|^pdG_s\right]\right)<\infty.$$

Now let us give essential result of this section.

**Theorem 5.7** *Assume* $(A1)-(A2)$ *and* $(A5)-(A6)$. *Then the reflected BSDE with random terminal time associated to data* $(\xi,f,g,S)$ *has a unique solution satisfying*

$$\begin{aligned}
&I\!E\left\{\sup_{0\leq t\leq T}e^{p(\rho t+\nu G(t))}|Y_t|^p+\int_0^T e^{p(\rho s+\nu G(s))}|Y_s|^{p-2}|Z_s|^2ds+\int_0^T e^{p(\rho s+\nu G(s))}|Y_s|^pdG_s\right\}\\
\leq\ & C I\!E\left\{e^{p(\rho T+\nu G(T))}|\xi|^p+\int_0^T e^{p(\rho s+\nu G(s))}|\varphi_s|^pds+\int_0^T e^{p(\rho s+\nu G(s))}|\psi_s|^pdG_s\right.\\
&\left.+\sup_{0\leq t\leq T}e^{p(\rho t+\nu G(t))}(S_t^+)^p\right\}
\end{aligned}$$

*with* $C$ *some constant depending upon* $p,\lambda,\rho$ *and* $\mu$.

**Proof.** The proof follows the steps of the proof of the deterministic case. But to reduce the terminal condition $\bar{\xi}$ which belong to $L^p$, we firstly make the change of variables $\tilde{X}_t=e^{\alpha s+\theta G(s)}X_t$. Now let us derive a priori estimate in $L^p$ with $p\in(1,2)$, which is the only difference with the above case. By virtue of Lemma 3.1, we get

$$\begin{aligned}
& e^{p(\rho(T\wedge t)+\nu G(T\wedge t))}|Y_{T\wedge t}|^p + c(p)\int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^{p-2}|Z_s|^2 ds \\
& + \int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^p(\rho ds + \nu dG_s) \\
\leq\ & e^{p(\rho(T\wedge u)+\nu G(T\wedge u))}|Y_{T\wedge u}|^p + p\int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^{p-1}\widehat{Y}_s f(s,Y_s,Z_s)ds \\
& + p\int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^{p-1}\widehat{Y}_s g(s,Y_s)dG_s + p\int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^{p-1}\widehat{Y}_s dK_s \\
& - p\int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^{p-1}\widehat{Y}_s Z_s dW_s
\end{aligned}$$

Then since $\mathbb{E}\left(\int_0^{T\wedge u}(Y_s - S_s)^{p-1}dK_s\right) = 0$, assumptions on $f$ and $g$ with Young's inequality together and choosing $0 < \delta < (p-1)/2$ small enough so that $0 \leq \rho - (|\mu| + \delta + \lambda^2/(2(p-1-2\delta))) = \bar{\rho}$ and $0 \leq \nu - (|\beta| - \delta) = \bar{\nu}$ we have the following

$$\begin{aligned}
& e^{p(\rho(T\wedge t)+\nu G(T\wedge t))}|Y_{T\wedge t}|^p + p\delta\int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^{p-2}|Z_s|^2 ds \\
& + \int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^p(\rho ds + \nu dG_s) \\
\leq\ & C(p,\delta)\Big(e^{p(\rho(T\wedge u)+\nu G(T\wedge u))}|Y_{T\wedge u}|^p + \int_0^{T\wedge u} e^{p(\rho s+\nu G(s))}|\varphi_s|^p ds \\
& + \int_0^{T\wedge u} e^{p(\rho s+\nu G(s))}|\psi_s|^p dG_s + \int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|S_s|^{p-1}dK_s\Big) \\
& - p\int_{T\wedge t}^{T\wedge u} e^{p(\rho s+\nu G(s))}|Y_s|^{p-1}\widehat{Y}_s Z_s dW_s. \qquad (5.19)
\end{aligned}$$

Using the same estimation as in the proof of Lemma 3.3 we get

$$\begin{aligned}
& \mathbb{E}\left(\sup_{0\leq t\leq T} e^{p(\rho t+\nu G(t))}|Y_t|^p\right) + p\delta\mathbb{E}\int_0^T e^{p(\rho s+\nu G(s))}|Y_s|^{p-2}|Z_s|^2 ds \\
& + p\mathbb{E}\int_0^T e^{p(\rho s+\nu G(s))}|Y_s|^p(\bar{\rho} ds + \bar{\nu} dG_s) \\
\leq\ & C(p,\delta)\mathbb{E}\Big\{e^{p(\rho T+\nu G(T))}|\xi|^p + \int_0^T e^{p(\rho s+\nu G(s))}|\varphi_s|^p ds \\
& + \int_0^T e^{p(\rho s+\nu G(s))}|\psi_s|^p dG_s + \sup_{0\leq t\leq T} e^{p(\rho s+\nu G(s))}(S_t^+)^p\Big\}
\end{aligned}$$

which end the proof. ■

**Remark 5.8** *In the case $p = 2$, the condition $\rho > \mu + \frac{\lambda^2}{2(p-1)}$ reduce to $\rho > \mu + \frac{\lambda^2}{2}$ which is the condition in Aman et al [1]*
*No result for the case $p = 1$ can be deduce from the above.*

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