POLARIZATIONS OF PRYM VARIETIES FOR WEYL GROUPS VIA ABELIANIZATION

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Abstract. Let $\pi : Z \to X$ be a Galois covering of smooth projective curves with Galois group the Weyl group of a simple and simply connected Lie group G. For any dominant weight λ consider the curve $Y = Z/\mathrm{Stab}(\lambda)$. The Kanev correspondence defines an abelian subvariety P_λ of the Jacobian of Y. We compute the type of the polarization of the restriction of the canonical principal polarization of $\mathrm{Jac}(Y)$ to P_λ in some cases. In particular, in the case of the group E_8 we obtain families of Prym-Tyurin varieties. The main idea is the use of an abelianization map of the Donagi-Prym variety to the moduli stack of principal G-bundles on the curve X.

1. Introduction

1.1. **Verlinde spaces.** Let X be a smooth complex projective curve of genus g and let G be a simple, simply-connected complex Lie group. We denote by $\mathcal{M}_X(G)$ the moduli stack of principal G-bundles and by $\mathcal{L}$ the ample generator of its Picard group. The celebrated Verlinde formula ([Fa1], [So1], [So2]) computes the dimension $N_{g,l}(G)$ of the space of global sections $H^0(\mathcal{M}_X(G), \mathcal{L}^{\otimes l})$ for any level l. The Verlinde numbers at level $l = 1$ for the groups of type ADE are given in the following table.

G	$\mathrm{SL}(m)$	$\mathrm{Spin}(2m)$	E_6	E_7	E_8
$N_{g,1}(G)$	m^g	4^g	3^g	2^g	1

The number m^g for $\mathrm{SL}(m)$ coincides with the number of level-m theta functions on the Jacobian of X (see [BNR]). For the even Spin group the Verlinde number equals the number of theta characteristics of X (see [O]). The striking simplicity of the Verlinde numbers for E_6, E_7 and E_8 was the main motivation for us to try to relate these Verlinde spaces to spaces of theta functions on polarized abelian varieties (the Prym varieties) and compute the induced polarizations — see Main Theorem.

1.2. **Abelianization of principal G-bundles.** The abelianization program of principal G-bundles, or more precisely G-Higgs bundles, takes its origin in Hitchin's papers [Hi1] and [Hi2]. For the case $G = \mathrm{SL}(m)$ it is shown in [BNR] that for a sufficiently ramified spectral cover $\psi : Y \to X$ the direct image map

$$\mathrm{Prym}(Y/X) \longrightarrow \mathcal{M}_X(\mathrm{SL}(m))$$

induces by pull-back an isomorphism between the $\mathrm{SL}(m)$-Verlinde space at level 1 and the space of abelian theta functions $H^0(\mathrm{Prym}(Y/X), L_Y)$. For general structure groups G the abelianization theory has been worked out by Faltings [Fa2], by Donagi [Don1], [Don2] and Donagi-Gaitsgory [DG].

1.3. **Correspondences on spectral and cameral covers.** For general structure groups G Prym varieties can be constructed via correspondences on covers of the curve X:

In [K1] Kanev constructs from the data of a rational map $f : \mathbb{C} \to \mathfrak{g} = \mathrm{Lie}(G)$ and an irreducible representation $\rho_\lambda : G \to \mathrm{GL}(V)$ a spectral cover $\psi : Y \to \mathbb{P}^1$ equipped with a correspondence. He shows that if G is of type ADE, the weight λ minuscule and f sufficiently general, then the Prym variety $P_\lambda \subset \mathrm{Jac}(Y)$ associated with Kanev's correspondence is a Prym-Tyurin variety (i.e. the polarization on P_λ induced from the principal polarization of $\mathrm{Jac}(Y)$ is a multiple of a principal polarization). Note that Kanev's construction is carried out in the case $X = \mathbb{P}^1$.

A different but related construction of Prym varieties is given by Donagi in [Don1], [Don2]: let $T \subset G$ be a maximal torus, W the Weyl group of G and $\mathbb{S}_\omega = \mathrm{Hom}(T, \mathbb{C}^*)$ the weight lattice. For any cameral cover, i.e. a Galois cover with Galois group W

$$\pi : Z \longrightarrow X$$

satisfying some conditions on the ramification, Donagi introduces the Prym variety $\mathrm{Prym}(\pi, \mathbb{S}_\omega) := \mathrm{Hom}_W(\mathbb{S}_\omega, \mathrm{Jac}(Z))$ parametrizing W-equivariant homomorphisms from $\mathbb{S}_\omega$ to $\mathrm{Jac}(Z)$.

In this paper we generalize Kanev's construction to an arbitrary base curve X. Given a Galois cover $\pi : Z \to X$ and a dominant weight $\lambda \in \mathbb{S}_\omega$ we consider the cover of curves

$$\psi : Y \longrightarrow X, \qquad \text{with} \qquad Y = Z/\mathrm{Stab}(\lambda).$$

Kanev's construction generalizes (see section 3) to give a correspondence $\overline{K}_\lambda$ on the curve Y defining an abelian subvariety

$$P_\lambda \subset \mathrm{Jac}(Y),$$

which is isogenous to the Donagi-Prym variety $\mathrm{Prym}(\pi, \mathbb{S}_\omega)$ (Proposition 6.13).

1.4. **The polarization on the Prym variety P_λ.** Let L_Y denote a line bundle defining the canonical principal polarization on the Jacobian $\mathrm{Jac}(Y)$. The aim of this paper is to compute the induced polarization $L_Y|P_\lambda$ under certain assumptions. In fact, if q_λ denotes the exponent of the correspondence $\overline{K}_\lambda$ and d_λ the Dynkin index of λ, our main result is the following theorem (we prove a slighty more precise version, see Theorem 8.1) . We use the notation of [Bo] for the weights.

Main Theorem. *We suppose that the W-Galois cover $\pi : Z \to X$ is étale. Then in the cases given in the table below the induced polarization $L_Y|P_\lambda$ is divisible by q_λ,i.e. $L_Y|P_\lambda = M^{\otimes q_\lambda}$ and the polarization M on P_λ is of type $K(M) = (\mathbb{Z}/m\mathbb{Z})^{2g}$:*

Weyl group of type	weight λ	$q_\lambda = d_\lambda$	$K(M)$
A_n	$\varpi_i;\ (i, n+1) = 1$	$\binom{n-1}{i-1}$	$(\mathbb{Z}/(n+1)\mathbb{Z})^{2g}$
$D_n;\ n$ odd	ϖ_{n-1}, ϖ_n	2^{n-3}	$(\mathbb{Z}/4\mathbb{Z})^{2g}$
E_6	ϖ_1, ϖ_6	6	$(\mathbb{Z}/3\mathbb{Z})^{2g}$
E_7	ϖ_7	12	$(\mathbb{Z}/2\mathbb{Z})^{2g}$
E_8	ϖ_8	60	0

We observe that in all the cases of this table we have an equality (see table in section 1.1)

$$\dim H^0(P_\lambda, M) = N_{g,1}(G).$$

A priori it is not clear (e.g. in the case $G = E_8$) whether the linear map between the Verlinde space and $H^0(P_\lambda, M)$ induced by the abelianization map Δ_θ — see below — is nonzero.

It is well known that the Weyl group of type E_k is closely related to the del Pezzo surface of degree $9-k$ for $5 \leq k \leq 8$. In fact, a slightly modified lattice of the weight lattice is isomorphic to the Picard lattice of the corresponding del Pezzo surface (see [K1, section 8.7]). Moreover, for $4 \leq k \leq 7$ the Kanev correspondence is given essentially by the incidence correspondence of lines of the corresponding del Pezzo surface. For $k = 8$ there are multiplicities due to the fact that the weight ϖ_8 is only quasi-minuscule (see [K1]). Notice that in these cases the polarization M on P_λ is of type $(\mathbb{Z}/d\mathbb{Z})^{2g}$ where d is the degree of the correponding del Pezzo surface.

In particular in the case of $W = W(E_8)$ we obtain a family of Prym-Tyurin varieties, i.e. the pairs (P_λ, M) are principally polarized abelian varieties. It is easy to see that any curve X of genus $g \geq 4$ admits an étale Galois covering with Galois group $W(E_8)$ and we get a family of Prym-Tyurin varieties of dimension $8(g-1)$ of exponent 60. We plan to study this family in a subsequent paper.

Probably there is an analogous result in the case that the Galois covering $\pi : Z \to X$ admits simple ramification. Certainly the paper [DG] will be essential for this. We plan to come back to it subsequently.

Note that our results are disjoint from the results in [K1]. First of all, Kanev considers only Galois coverings over $\mathbb{P}^1$ which are necessarily ramified. Moreover, his correspondence satisfies a quadratic equation and he uses his criterion ([K2]) to show that the associated abelian subvarieties are principally polarized. In our case the corresponding correspondence satisfies a cubic equation (see Theorem 3.9) and a quadratic equation only on the Prym variety $\mathrm{Prym}(Y/X)$ of the covering $\psi : Y \to X$. However, the abelian variety $\mathrm{Prym}(Y/X)$ is not principally polarized and thus we cannot apply Kanev's criterion [K2] in order to compute the polarization of P_λ.

Instead we proceed as follows: We work out a general result on restrictions of polarizations to abelian subvarieties (Proposition 2.10) roughly saying that, if the restricted polarization equals the q-fold of a polarization, where q is the exponent of the abelian subvariety, then its type can be computed.

The main idea of the proof is then the use of an abelianization map

$$\Delta_\theta : \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n \to \mathcal{M}_X(G),$$

(see Section 7). Here $\mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$ denotes a certain component of the Donagi-Prym variety $\mathrm{Prym}(\pi, \mathbb{S}_\omega)$. The fact, that the restricted polarization is the q_λ-fold of a polarization is a consequence of the existence of a commutative diagram

$$\begin{tikzcd} \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n \arrow[r, "\widetilde{ev}_\lambda"] \arrow[d, "\Delta_\theta"] & T_\alpha(P_\lambda) \arrow[d, "\psi_*"] \\ \mathcal{M}_X(G) \arrow[r, "\tilde{\rho}_\lambda"] & \mathcal{M}_X(\mathrm{SL}(m)) \end{tikzcd}$$

(here $\widetilde{ev}_\lambda$ denotes evaluation at λ, the map $\tilde{\rho}_\lambda$ is induced by the representation with dominant weight λ, and T_α is a certain translation) together with a theorem of Laszlo-Sorger ([LS], [So2]) saying that the pull-back of the determinant bundle on $\mathcal{M}_X(\mathrm{SL}(m))$ by $\tilde{\rho}_\lambda$ is the d_λ-fold of the ample generator of $\mathrm{Pic}(\mathcal{M}_X(G))$ and the assumption $q_\lambda = d_\lambda$.

The contents of the paper is as follows: In Section 2 we prove the result just mentioned concerning restriction of a polarization to an abelian subvariety. In Section 3 we prove that the Schur- and the Kanev correspondence belong to the same abelian subvariety. This was shown in [LRo] for the special case $X = \mathbb{P}^1$ and for a left action of the group. In Section 4 we compute the invariants of our main examples mentioned above. In Section 5 we introduce the Donagi-Prym variety and derive the properties we need. Section 6 contains some results on Mumford groups. In Section 7 we introduce the abelianization map and prove the commutativity of the above diagram. Finally, Section 8 contains the proof of our main theorem.

Finally one word to the group actions: The group W acts on the curve Z as well as on the weight lattice $\mathbb{S}_\omega$ and these actions have to be consistently either both left actions or both right actions. Since traditionally principal bundles are defined via right actions, we are forced to use right actions in both cases.

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2. Restriction of polarizations to abelian subvarieties

Let S be an abelian variety with polarization L and associated isogeny $\varphi_L : S \longrightarrow \hat{S}$. Denote as usual $K(L) = \ker \varphi_L$. We consider an endomorphism $u \in \mathrm{End}(S)$ satisfying the following properties

- u is symmetric with respect to the Rosati involution, i.e.,

(1) $$\varphi_L\, u = \hat{u}\, \varphi_L,$$

  where $\hat{u} \in \mathrm{End}(\hat{S})$ is the dual endomorphism.
- there exists a positive integer q such that

(2) $$u^2 = qu.$$

We introduce the abelian subvarieties of S

(3) $$A = \mathrm{im}(u) \subset \ker(q-u) \qquad \text{and} \qquad P = \mathrm{im}(q-u) \subset \ker(u)$$

and we wish to study the induced polarizations $L_A = L|A$ on A and $L_P = L|P$ on P, i.e. determine the subgroups $K(L_P)$ and $K(L_A)$.

In the sequel we use the following notation

$$\iota_A : A \hookrightarrow S, \qquad \iota_P : P \hookrightarrow S, \qquad \pi_A : S \longrightarrow S/A, \qquad \pi_P : S \longrightarrow S/P.$$

where the ι's are inclusions and π's are projections.

Remark 2.1. (i): For any abelian subvariety A of S there is an endomorphism u satisfying (1) and (2). In fact, this is valid for the composition

$$u = \iota_A\, \psi_{L_A}\, \hat{\iota}_A\, \varphi_L \in \mathrm{End}(S)$$

(see [BL, Lemma 5.3.1]). Here $\psi_{L_A} = q\, \varphi_L^{-1} : \hat{A} \to A$ is an isogeny.

(ii): Let $\psi : Y \longrightarrow X$ be a degree d cover of smooth projective curves. If we identify the Jacobians $\mathrm{Jac}(X)$ and $\mathrm{Jac}(Y)$ with their duals via the canonical polarizations, then the dual $\hat{\psi}^*$ of the map $\psi^* : \mathrm{Jac}(X) \longrightarrow \mathrm{Jac}(Y)$ coincides with the norm map $\mathrm{Nm} : \mathrm{Jac}(Y) \longrightarrow \mathrm{Jac}(X)$. Moreover the endomorphism

$$t = \psi^* \,\mathrm{Nm} \in \mathrm{End}(\mathrm{Jac}(Y))$$

satisfies (1) and (2) with $q = d$. The abelian subvariety $P = \mathrm{im}(d - t)$ defined in (3) coincides with the usual Prym variety (not necessarily principally polarized) of the morphism ψ, which we denote by $\mathrm{Prym}(Y/X)$, i.e.,

$$P = \mathrm{Prym}(Y/X) = (\ker \mathrm{Nm})_0.$$

The endomorphism t is induced by the trace correspondence $\overline{T}$ on Y — see section 3. We denote the kernel $\ker \psi^* : \mathrm{Jac}(X) \longrightarrow \mathrm{Jac}(Y)$ by K. We also have the equality

$$\mathrm{im}(\psi^*) = \mathrm{Jac}(X)/K = \mathrm{im}(t).$$

Since the norm map Nm is the dual of ψ we easily obtain that the group of connected components of the fibre $\mathrm{Nm}^{-1}(0)$ is isomorphic to K.

(iii): With u also $u' = q - u$ satisfies (1) and (2). Replacing u by u' interchanges the abelian subvarieties A and P. Hence for the following Lemmas it suffices to prove one of the two statements concerning A and P and we will do so without further mentioning.

Recall that the addition map μ of S satisfies $\mu = \iota_A + \iota_P$ and that the following sequence is exact

$$0 \longrightarrow A \cap P \longrightarrow A \times P \xrightarrow{\mu} S \longrightarrow 0. \tag{4}$$

For any abelian variety B and any positive integer n let B_n denote the subgroup of n-torsion points of B. Then we have, denoting by $|M|$ the cardinality of a set M,

Lemma 2.2. *The finite subgroup $A \cap P$ is contained in A_q and P_q.*

Proof. Let $x \in A \cap P$. Then $u(x) = qx$, since $x \in A$ and $u(x) = 0$, since $x \in P$. This implies the assertion. □

Lemma 2.3. $|K(L_A)| \cdot |K(L_P)| = |A \cap P|^2 \cdot |K(L)|.$

Proof. According to [BL, Corollary 5.3.6] the polarization $\mu^*(L)$ splits, i.e. $\varphi_{\mu^*(L)} = \varphi_{L_A} \times \varphi_{L_P}$, which gives

$$|K(\mu^* L)| = |K(L_A)| \cdot |K(L_P)|.$$

On the other hand, (4) implies $\deg(\mu) = |A \cap P|$ and thus

$$|K(\mu^* L)| = |\ker(\mu \varphi_L \hat{\mu})| = \deg(\mu)^2 \cdot |\ker(\varphi_L)| = |A \cap P|^2 \cdot |K(L)|.$$

Combining both equations gives the assertion. □

Lemma 2.4. *We have the following equalities of abelian subvarieties of $\hat{S}$,*

$$\varphi_L(P) = \widehat{S/A} \subset \hat{S} \qquad \textit{and} \qquad \varphi_L(A) = \widehat{S/P} \subset \hat{S}.$$

Proof. The endomorphism u factorizes as $i_A v$ with $v : S \to A$. Taking the dual gives the factorization
$$\hat{u} : \hat{S} \xrightarrow{\hat{\iota}_A} \hat{A} \xrightarrow{\hat{v}} \hat{S},$$
from which we deduce that
$$\ker \hat{\iota}_A = \widehat{S/A} \subset \ker \hat{u}.$$
On the other hand we have $P \subset \ker u$ and so (1) implies $\varphi_L(P) \subset \ker \hat{u}$. This gives the first equality, since both abelian subvarieties are the connected component of the origin of $\ker \hat{u}$, as they are of the same dimension. □

We will give two descriptions of the subgroups $K(L_P)$ $(= \ker(\varphi_{L_P} : P \xrightarrow{\iota_P} S \xrightarrow{\varphi_L} \hat{S} \xrightarrow{\hat{\iota}_P} \hat{P}))$ and $K(L_A)$. We observe that $\ker \hat{\iota}_P = \widehat{S/P}$ and, according to Lemma 2.4, $\widehat{S/P} = \varphi_L(A)$. Hence we obtain
$$K(L_P) = \bigcup_{x \in K(L)} (A + x) \cap P,$$
where $A + x$ denotes the image of A under translation by x. Similarly we have
$$K(L_A) = \bigcup_{x \in K(L)} (P + x) \cap A.$$
In particular,
$$A \cap P \subset K(L_P) \quad \text{and} \quad A \cap P \subset K(L_A).$$

For the second description consider the isogenies
$$\alpha = \pi_A \, \iota_P : P \xrightarrow{\iota_P} S \xrightarrow{\pi_A} S/A \qquad \text{with} \qquad \ker \alpha = A \cap P$$
and
$$\beta = \pi_P \, \iota_A : A \xrightarrow{\iota_A} S \xrightarrow{\pi_P} S/P \qquad \text{with} \qquad \ker \beta = A \cap P.$$
Moreover, define the isogenies
$$\varphi_P : P \longrightarrow \widehat{S/A} \quad \text{and} \quad \varphi_A : A \longrightarrow \widehat{S/P}$$
by the factorizations $\varphi_L|P : P \xrightarrow{\varphi_P} \widehat{S/A} \hookrightarrow \hat{S}$ and $\varphi_L|A : A \xrightarrow{\varphi_A} \widehat{S/P} \hookrightarrow \hat{S}$ of Lemma 2.4. With this notation we have

Lemma 2.5. *The following sequences are exact*

(5) $\quad 0 \longrightarrow A \cap P \hookrightarrow K(L_P) \xrightarrow{\alpha} \ker(\hat{\varphi}_P) \longrightarrow 0 \quad$ *with* $\quad K(L) \cap P \cong \ker(\hat{\varphi}_P) \subset S/A,$

(6) $\quad 0 \longrightarrow A \cap P \hookrightarrow K(L_A) \xrightarrow{\beta} \ker(\hat{\varphi}_A) \longrightarrow 0 \quad$ *with* $\quad K(L) \cap A \cong \ker(\hat{\varphi}_A) \subset S/P.$

Proof. The dual isogeny of α factorizes as
$$\hat{\alpha} : \widehat{S/A} \xrightarrow{\hat{\pi}_A} \hat{S} \xrightarrow{\hat{\iota}_P} \hat{P}.$$
By the previous lemma $\varphi_L(P) = \widehat{S/A}$, which implies that
$$\varphi_{L_P} = \hat{\iota}_P \, \varphi_L \, \iota_P = \hat{\alpha} \, \varphi_L \, \iota_P.$$
Hence φ_{L_P} factorizes as $\varphi_{L_P} : P \xrightarrow{\varphi_P} \widehat{S/A} \xrightarrow{\hat{\alpha}} \hat{P}$. Taking the dual and using $\hat{\varphi}_{L_P} = \varphi_{L_P}$ we obtain the following factorization
$$\varphi_{L_P} : P \xrightarrow{\alpha} S/A \xrightarrow{\hat{\varphi}_P} \hat{P}$$

which gives the exact sequence (5). The assertion on the kernel follows from $\ker(\hat{\varphi}_P) \cong \ker(\varphi_P) = K(L) \cap P$. $\square$

In particular we obtain

$$|K(L_P)| = |A \cap P| \cdot |K(L) \cap P| \quad \text{and} \quad |K(L_A)| = |A \cap P| \cdot |K(L) \cap A|.$$

Inserting this into Lemma 2.3 we conclude

Corollary 2.6. $|K(L) \cap A| \cdot |K(L) \cap P| = |K(L)|.$

Consider the finite groups

$$G_P = \ker u/P \qquad \text{and} \qquad G_A = \ker(q-u)/A.$$

They are the groups of connected components of $\ker u$ and $\ker(q-u)$ respectively.

Lemma 2.7. *There are canonical exact sequences*

$$0 \longrightarrow A \cap P \longrightarrow A_q \longrightarrow G_P \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow A \cap P \longrightarrow P_q \longrightarrow G_A \longrightarrow 0.$$

Proof. Since $P \subset \ker u$ the endomorphism u descends to an isogeny

$$\bar{u} : S/P \longrightarrow A, \qquad \text{with} \qquad \ker \bar{u} = G_P.$$

Moreover we know that $u|A = q$, which implies that the composite isogeny

$$A \xrightarrow{\pi_P \iota_A} S/P \xrightarrow{\bar{u}} A$$

equals q. Taking kernels leads to the first exact sequence of the Lemma. $\square$

We denote by

$$u_q : S_q \longrightarrow A_q \quad \text{and} \quad (q-u)_q : S_q \longrightarrow P_q$$

the restrictions of u and $q-u$ to the q-torsion points S_q. Note that u_q and $(q-u)_q$ are nilpotent: $u_q^2 = (q-u)_q^2 = 0$. Note moreover that $G_P \subset (S/P)_q$ and $G_A \subset (S/A)_q$.

Lemma 2.8. *We have the following equalities of finite groups*

$$\operatorname{im} u_q = A \cap P, \quad \operatorname{coker} u_q = G_P \quad \textit{and} \quad \operatorname{im}(q-u)_q = A \cap P, \quad \operatorname{coker}(q-u)_q = G_A.$$

Proof. First we claim that $\operatorname{im} u_q \subset A \cap P$. For the proof let $a \in \operatorname{im} u_q$. Write $a = u(s)$ with $s \in S_q$. Then $qs = 0$ and therefore $a = (u-q)(s) \in P$.

In order to show equality of these subgroups of A_q we compute their indices. According to Lemma 2.7 the index of $A \cap P$ in A_q is $|G_P|$. The index of $\operatorname{im} u_q$ in A_q equals

$$\frac{|A_q|}{|\operatorname{im} u_q|} = \frac{|A_q|}{|S_q|} \cdot |\ker u_q| = \frac{|\ker u_q|}{|P_q|}.$$

Moreover since $G_P \subset (S/P)_q$ the exact sequence

$$0 \longrightarrow P \longrightarrow \ker u \longrightarrow G_P \longrightarrow 0$$

remains exact after taking q-torsion points. This implies the first equality. The second equality is obvious. $\square$

Consider again the isogenies $\hat{\varphi}_P : S/A \longrightarrow \hat{P}$ and $\hat{\varphi}_A : S/P \longrightarrow \hat{A}$.

Lemma 2.9. *We have the following equalities of subgroups of S/A and S/P respectively:*

$$\ker \hat{\varphi}_P = \pi_A(K(L)) \quad \textit{and} \quad \ker \hat{\varphi}_A = \pi_P(K(L)).$$

Proof. The inclusion $\pi_A(K(L)) \subset \ker \hat{\varphi}_P$ follows from the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi_L} & \hat{S} \\ \big\downarrow{\scriptstyle \pi_A} & & \big\downarrow{\scriptstyle \hat{\imath}_P} \\ S/A & \xrightarrow{\hat{\varphi}_P} & \hat{P} \end{array}$$

So it will be enough to show that these two groups have the same order. We have, using (5),

$$|\ker \hat{\varphi}_P| = |K(L) \cap P| \qquad \text{and} \qquad |\pi_A(K(L))| = \frac{|K(L)|}{|K(L) \cap A|}.$$

Equality follows from Corollary 2.6. □

The following proposition will be applied in the proof of the main theorem (section 8).

Proposition 2.10. (a)*: If $L_A = M^q$ for a polarization M on A, we have the following equality of subgroups of A,*

$$K(M) = u(K(L)),$$

(b)*: If $L_P = N^q$ for a polarization N on P, we have the following equality of subgroups of P,*

$$K(N) = (q - u)(K(L)).$$

Note that $L_A = M^q$ for a polarization M on A if and only if the polarization L_A is divisible by q, meaning that $A_q \subset K(L_A)$. In this case

$$K(M) = K(L_A)/A_q.$$

Proof. We take the quotient by $A \cap P$ of the inclusion $A_q \subset K(L_A)$. Since by Lemma 2.7, $A_q/A \cap P \cong G_P$ and by Lemma 2.5, $K(L_A)/A \cap P \cong \ker \hat{\varphi}_A$, this gives an inclusion

$$G_P \subset \ker \hat{\varphi}_A.$$

Recall that $G_P = \ker u/P$ and by Lemma 2.9, $\ker \hat{\varphi}_A = \pi_P(K(L))$. Therefore

$$\begin{aligned} K(M) = K(L_A)/A_q = \ker \hat{\varphi}_A/G_P &= ((K(L) + P)/P)/(\ker u/P) \\ &= (K(L) + P)/\ker u = u(K(L)). \end{aligned}$$

□

3. The Schur and Kanev correspondences

3.1. **Representations of the Weyl group W.** Let W be a Weyl group and let $\widehat{W}$ denote the set of its irreducible characters. It is known (see e.g [Sp] Corollary 1.15) that all irreducible representations of W are defined over $\mathbb{Q}$. Therefore any irreducible representation of W is also absolutely irreducible. Given $\omega \in \widehat{W}$ we choose an irreducible $\mathbb{Z}[W]$-module $\mathbb{S}_\omega$ such that

$$V_\omega := \mathbb{S}_\omega \otimes_{\mathbb{Z}} \mathbb{Q}$$

is the irreducible representation of W corresponding to ω. As outlined in the introduction, we consider every representation of W as a right-representation. In particular $\mathbb{S}_\omega$ is a right-$\mathbb{Z}[W]$-module with action $(\lambda, g) \mapsto \lambda g$ for $\lambda \in \mathbb{S}_\omega$ and $g \in W$.

Fix a *weight* λ of the lattice $\mathbb{S}_\omega$, which is by definition (see [K1] page 158) a vector

$$\lambda \in V_\omega \ \text{ such that } \ \lambda g - \lambda \in \mathbb{S}_\omega \ \ \forall g \in W.$$

Let $\pi : Z \to X$ denote a Galois covering of smooth projective curves with Galois group W. We consider the action of W on Z as a right-action $(z, g) \mapsto z^g$ for $z \in Z$ and $g \in W$. Let $H := \mathrm{Stab}(\lambda) \subset W$ denote the stabilizer subgroup of the weight $\lambda \in V_\omega$. Then π factorizes as follows

$$\begin{tikzcd} Z \arrow[dd, "\pi"'] \arrow[dr, "\varphi"] & \\ & Y \arrow[dl, "\psi"] \\ X & \end{tikzcd}$$

where Y denotes the quotient Z/H. Schur's orthogonality relations induce a correspondence on Z which we denote by S_λ. On the other hand, Kanev [K1] defined in the case $X = \mathbb{P}^1$ a correspondence on the curve Y, which we denote by $\overline{K}_\lambda$. It is the aim of this section to generalize Kanev's construction to an arbitrary base curve X and to work out the relation between the two correspondences S_λ and $\overline{K}_\lambda$.

3.2. **Schur correspondence.** Since V_ω is an absolutely irreducible W-representation, there is a unique negative definite W-invariant symmetric form $(\ ,\ )$ on V_ω with

(1) $(\lambda, \mu) \in \mathbb{Z}$ for all $\mu \in \mathbb{S}_\omega$,
(2) any W-invariant form on V_ω satisfying (1) is an integer multiple of $(\ ,\ )$.

The *Schur correspondence* associated to the pair $(\mathbb{S}_\omega, \lambda)$ is by definition the rational correspondence on Z over X defined by

$$S_\lambda = \sum_{g \in W} (\lambda g, \lambda) \Gamma_g.$$

Here $\Gamma_g \in Z \times Z$ denotes the graph of the automorphism g of Z. Note that $(\lambda g, \lambda)$ need not be an integer. Considered as a map of Z into the group $\mathrm{Div}_{\mathbb{Q}}(Z)$ of rational divisors on Z, S_λ is given by

$$S_\lambda(z) = \sum_{g \in W} (\lambda g, \lambda) z^g.$$

The correspondence S_λ descends to a correspondence $\overline{S}_\lambda$ on Y in the usual way, namely

$$\overline{S}_\lambda = (\varphi \times \varphi)_* S_\lambda \subset Y \times Y.$$

In order to express this correspondence as a map $Y \longrightarrow \mathrm{Div}_{\mathbb{Q}}(Y)$, denote $d = [W : H]$ and let $\{g_1 = 1, g_2, \dots, g_d\}$ denote a set of representatives for the right cosets of H in W:

$$W = \cup_{i=1}^d H g_i.$$

Proposition 3.1. *For any $y = \varphi(z) \in Y$ we have*

$$\overline{S}_\lambda(y) = |H|^2 \sum_{i=1}^d (\lambda g_i, \lambda) \varphi(z^{g_i^{-1}}).$$

Note that the left-hand side of this equality does not depend neither on the choice of the point z in the fibre over y nor on the set of representatives $\{g_i\}$.

Proof. Since H is the stabilizer of λ, we have by definition of $\overline{S}_\lambda$,

$$\overline{S}_\lambda(y) = \sum_{i=1}^{d} \sum_{h \in H} (\lambda h g_i, \lambda) \sum_{h' \in H} \varphi(z^{h' h g_i}) = |H| \sum_{i=1}^{d} \sum_{k \in H} (\lambda g_i, \lambda) \varphi(z^{k g_i}) = |H| \sum_{g \in W} (\lambda g, \lambda) \varphi(z^g).$$

Now, if $\{g_i\}$ is a set of representatives for the right cosets of H, then $\{g_i^{-1}\}$ is a set of representatives for the left cosets of H. Hence for any pair $(k, i) \in H \times \{1, \ldots, d\}$ there is a unique pair $(h, j) \in H \times \{1, \ldots, d\}$ such that $kg_i = g_j^{-1} h$. Moreover, if kg_i runs exactly once through W, so do the elements $g_j^{-1} h$. This implies, since $(\ ,\ )$ is W-invariant and since H stabilizes λ,

$$\overline{S}_\lambda(y) = |H| \sum_{i=1}^{d} \sum_{h \in H} (\lambda g_i, \lambda) \varphi(z^{g_i^{-1} h}) = |H| \sum_{i=1}^{d} \sum_{h \in H} (\lambda g_i, \lambda) \varphi(z^{g_i^{-1}}).$$

This implies the assertion. $\square$

3.3. **Kanev correspondence.** In order to define the Kanev correspondence on Y, let $U \subset X$ denote the complement of the branch locus of π and fix a point $\xi_0 \in U$. Since H is the stabilizer of λ, the group W acts on the set $\{\lambda = \lambda g_1, \lambda g_2, \ldots, \lambda g_d\}$.

The group W also acts on the fibre $\psi^{-1}(\xi_0)$. Note that the group W does not act on the curve Y, since the subgroup H is not necessarily a normal subgroup of W. However we can define the action of W on the set $\psi^{-1}(\xi_0)$ via the monodromy at the point ξ_0 as follows: consider the fundamental group $\pi_1(U, \xi_0)$ and let $U_Z = \pi^{-1}(U) \subset Z$ and $U_Y = \psi^{-1}(U) \subset Y$. Choose a point $z \in Z$ such that $\pi(z) = \xi_0$. Then $\pi_1(U_Z, z)$ is a normal subgroup of $\pi_1(U, \xi_0)$ and we have the following exact sequence

$$0 \longrightarrow \pi_1(U_Z, z) \longrightarrow \pi_1(U, \xi_0) \longrightarrow W \longrightarrow 0.$$

The monodromy of the cover $\psi : U_Y \longrightarrow U$ at the point ξ_0 gives a homomorphism

$$\rho : \pi_1(U, \xi_0) \longrightarrow \mathrm{Aut}(\psi^{-1}(\xi_0)).$$

The kernel $\ker \rho$ equals $\cap_y \pi_1(U_Y, y)$ where y varies over the set $\psi^{-1}(\xi_0)$. We note that $\pi_1(U_Z, z) \subset \pi_1(U_Y, y)$ for any point $y \in \psi^{-1}(\xi_0)$, hence $\pi_1(U_Z, z) \subset \ker \rho$. Therefore the monodromy map ρ factorizes over W

$$\rho : \pi_1(U, \xi_0) \longrightarrow W \longrightarrow \mathrm{Aut}(\psi^{-1}(\xi_0)).$$

Notice that according to our definitions the monodromy action of $\pi_1(U, \xi_0)$ on the fibre $\psi^{-1}(\xi_0)$ is a right action.

Choosing an element in the fibre $\psi^{-1}(\xi_0)$ induces a bijection

$$\{\lambda g_1, \ldots, \lambda g_d\} \xrightarrow{\sim} \psi^{-1}(\xi_0),$$

which is W-equivariant according to the definitions. In the sequel we identify the above sets, i.e. we label the elements of $\psi^{-1}(\xi_0)$ by $\lambda = \lambda g_1, \ldots, \lambda g_d$.

For every point $\xi \in U$ choose a path γ_ξ in U connecting ξ and ξ_0. The path defines a bijection

$$\mu : \psi^{-1}(\xi) \to \psi^{-1}(\xi_0) = \{\lambda = \lambda g_1, \ldots, \lambda g_d\}$$

in the following way: For any $y \in \psi^{-1}(\xi)$ denote by $\tilde{\gamma}_y$ the lift of γ_ξ starting at y. If $\lambda g_j \in \psi^{-1}(\xi_0)$ denotes the end point of $\tilde{\gamma}_y$, define

$$\mu(y) = \lambda g_j.$$

Define

$$K_{U,\lambda} := \{(x,y) \in \psi^{-1}(U) \times \psi^{-1}(U) : \ \psi(x) = \psi(y), \ \ (\mu(x), \mu(y)) - (\lambda, \lambda) - 1 > 0\}.$$

and let $\overline{K}_\lambda$ denote the closure of $K_{U,\lambda}$ in $Y \times Y$.

Lemma 3.2. *The divisor $\overline{K}_\lambda$ is an integral symmetric effective correspondence on the curve Y, canonically associated to the triple $(\pi, \mathbb{S}_\omega, \lambda)$.*

Proof. For the first assertion it suffices to show that $(\lambda g, \lambda) - (\lambda, \lambda) - 1$ is a non-negative integer for all $g \in W \setminus H$, since $(\, ,)$ is a W-invariant scalar product. But $(\lambda g, \lambda) - (\lambda, \lambda) - 1$ is an integer, since λ is a weight. Hence it suffices to show that $(\lambda g, \lambda) > (\lambda, \lambda)$ for every $g \in W \setminus H$. For this note that

$$(\lambda, \lambda) - (\lambda g, \lambda) = \frac{1}{2}[(\lambda g, \lambda g) - 2(\lambda g, \lambda) + (\lambda, \lambda)] = \frac{1}{2}(\lambda g - \lambda, \lambda g - \lambda),$$

which implies the assertion, since $(\, ,)$ is negative definite and $H = \mathrm{Stab}(\lambda) \subset W$.

For the last assertion we have to show that the $K_{U,\lambda}$ does not depend on the choice of ξ_0 the path γ_ξ connecting ξ and ξ_0. This is a consequence of the W-invariance of the form $(\, ,)$ and the above mentioned fact that the monodromy map ρ factorizes via the group W. $\square$

We call $\overline{K}_\lambda$ the *Kanev correspondence* associated to the weight λ, since it was introduced by Kanev in [K1]. Considered as a map $Y \to \mathrm{Div}(Y)$, it is given by

$$\overline{K}_\lambda(y) = \sum_{j=1}^{d} [(\lambda g_j, \mu(y)) - (\lambda, \lambda) - 1] \mu^{-1}(\lambda g_j) + y. \tag{7}$$

Note that y is added, since in the sum y appears with coefficient -1, because $(\lambda g_i, \mu(y)) = (\lambda, \lambda)$ if $\mu(y) = \lambda g_i$. We need the following description of $\overline{K}_\lambda(y)$. According to our construction, for any $y \in \psi^{-1}(U)$, there is a unique integer i_y, $1 \le i_y \le d$ such that $\mu(y) = \lambda g_{i_y}$.

Proposition 3.3. *If $y \in \psi^{-1}(U)$ with $\mu(y) = \lambda g_{i_y}$, then*

$$\overline{K}_\lambda(y) = \sum_{j=2}^{d} [(\lambda g_j, \lambda) - (\lambda, \lambda) - 1] \mu^{-1}(\lambda g_j g_{i_y})$$

Proof. By (7) and using the W-invariance of $(\, ,)$, we have

$$\begin{aligned}
\overline{K}_\lambda(y) &= \sum_{\lambda g_j \neq \mu(y)} [(\lambda g_j, \mu(y)) - (\lambda, \lambda) - 1] \mu^{-1}(\lambda g_j) \\
&= \sum_{\lambda g_j \neq \mu(y)} [(\lambda g_j g_{i_y}^{-1}, \mu(y) g_{i_y}^{-1}) - (\lambda, \lambda) - 1] \mu^{-1}(\lambda g_j) \\
&= \sum_{\lambda g_j \neq \lambda g_{i_y}} [(\lambda g_j g_{i_y}^{-1}, \lambda) - (\lambda, \lambda) - 1] \mu^{-1}(\lambda g_j) \\
&= \sum_{j=2}^{d} [(\lambda g_j, \lambda) - (\lambda, \lambda) - 1] \mu^{-1}(\lambda g_j g_{i_y}).
\end{aligned}$$

$\square$

3.4. **Relations between $\overline{K}_\lambda$ and $\overline{S}_\lambda$.** We want to work out how the Kanev correspondence $\overline{K}_\lambda$ is related to the Schur correspondence $\overline{S}_\lambda$ on the curve Y. For this we need a special choice of the set of representatives $g_1, \dots, g_d$ of the right cosets of the subgroup H of W. In the sequel we choose a set of representatives $\{g_1 = 1, \dots, g_d\}$ such that $\{g_1^{-1}, \dots, g_d^{-1}\}$ is also a set of representatives of the right cosets of H in W. That there is always such a set, is a consequence of the marriage theorem of combinatorics (see [Ha, Theorem 5.1.7]).

Proposition 3.4. *Let $y \in \psi^{-1}(U)$ with $\mu(y) = \lambda g_{i_y}$. Then*

$$\overline{S}_\lambda(y) = |H|^2 \sum_{j=1}^{d} (\lambda g_j, \lambda) \mu^{-1}(\lambda g_j g_{i_y}).$$

Proof. According to Proposition 3.1, $\overline{S}_\lambda(y) = |H|^2 \sum_{j=1}^d (\lambda g_j, \lambda) \varphi(z^{g_j^{-1}})$, where z is a point in the fiber $\varphi^{-1}(y) \subset Z$.
Suppose first that $\mu(y) = \lambda$, i.e. $i_y = 1$. As W acts by right multiplication on the set $\{\lambda g_1, \dots, \lambda g_d\}$, we have $\mu(\varphi(z))g = \mu(\varphi(z^g))$, and hence

$$\varphi(z^{g_j^{-1}}) = \mu^{-1}(\mu(\varphi(z))g_j^{-1}) = \mu^{-1}(\lambda g_j^{-1})$$

for all j. Therefore we get, using the W-invariance of $(\ ,\ )$,

$$\overline{S}_\lambda(y) = |H|^2 \sum_{j=1}^{d} (\lambda g_j, \lambda) \mu^{-1}(\lambda g_j^{-1}) = |H|^2 \sum_{j=1}^{d} (\lambda, \lambda g_j^{-1}) \mu^{-1}(\lambda g_j^{-1}) = |H|^2 \sum_{j=1}^{d} (\lambda g_j, \lambda) \mu^{-1}(\lambda g_j),$$

where for the last equation we used the fact that with $\{g_1, \dots, g_d\}$ also $\{g_1^{-1}, \dots, g_d^{-1}\}$ is a set of representatives of the right cosets of H.
Finally, if $\mu(y) \neq \lambda$, we have $\mu(y) g_{i_y}^{-1} = \lambda$. Hence we can apply the above equation to the bijection $\widetilde{\mu} = g_{i_y}^{-1} \cdot \mu : \psi^{-1}(\psi(y)) \to \{\lambda g_1, \dots, \lambda g_d\}$, which gives

$$\overline{S}_\lambda(y) = |H|^2 \sum_{j=1}^{d} (\lambda g_j, \lambda) \widetilde{\mu}^{-1}(\lambda g_j) = |H|^2 \sum_{j=1}^{d} (\lambda g_j, \lambda) \mu^{-1}(\lambda g_j g_{i_y}).$$

□

With the notation and identifications of above we can state the main result of this section

Theorem 3.5. *The Kanev and Schur correspondence $\overline{K}_\lambda$ and $\overline{S}_\lambda$ on the curve $Y = Z/H$ associated to λ are related as follows*

$$\overline{S}_\lambda = |H|^2 \left(\overline{K}_\lambda - \overline{\Delta} + ((\lambda, \lambda) + 1)\overline{T} \right),$$

where $\overline{\Delta}$ denotes the diagonal in $Y \times Y$ and $\overline{T} = \psi^* \psi_*$ the trace correpondence of the morphism ψ.

Proof. It suffices to show that $[\overline{S}_\lambda + |H|^2(\overline{\Delta} - \overline{K}_\lambda)](y) = [(\lambda,\lambda)+1]|H|^2\overline{T}(y)$ for all $y \in \psi^{-1}(U)$. But applying Propositions 3.3 and 3.4 we have

$$\begin{aligned}[\overline{S}_\lambda + |H|^2(\overline{\Delta} - \overline{K}_\lambda)](y) &= |H|^2[\sum_{j=1}^d (\lambda g_j,\lambda)\mu^{-1}(\lambda g_j g_{i_y}) + y \\ &\qquad - \sum_{j=2}^d [(\lambda g_j,\lambda) - (\lambda,\lambda) - 1]\mu^{-1}(\lambda g_j g_{i_y})] \\ &= |H|^2((\lambda,\lambda)+1)\sum_{j=1}^d \mu^{-1}(\lambda g_j g_{i_y})\end{aligned}$$

But the right multiplication with g_{i_y} permutes only the elements $\lambda g_1, \dots, \lambda g_d$, which implies that $\sum_{j=1}^d \mu^{-1}(\lambda g_j g_{i_y}) = \psi^{-1}\psi(y) = \overline{T}(y)$. This completes the proof of the theorem. □

Corollary 3.6. $\deg \overline{K}_\lambda = 1 - d((\lambda,\lambda)+1)$.

Proof. Note first that $\deg S_\lambda = \sum_{g\in W}(\lambda g,\lambda) = (\sum_{g\in W}\lambda g,\lambda) = 0$, since $\sum_{g\in W}\lambda g$ is W-invariant and thus equal to 0 in V. This implies $\deg \overline{S}_\lambda = 0$ and hence Theorem 3.5 gives the assertion, since $\deg \overline{\Delta} = 1$ and $\deg \overline{T} = d$. □

Corollary 3.7. *We have the relations*

(1) $\overline{T}\,\overline{S}_\lambda = \overline{S}_\lambda \overline{T} = 0$.
(2) $\overline{K}_\lambda \overline{T} = \overline{T}\,\overline{K}_\lambda = \deg \overline{K}_\lambda \cdot \overline{T}$.

Proof. To obtain (1) we consider $T = \pi^*\pi_*$, the trace correspondence of π and note that for all $z \in Z$,

$$TS_\lambda(z) = \sum_{g\in W}(\lambda g,\lambda)\,T(z^g) = \sum_{g\in W}(\lambda g,\lambda)\cdot T(z) = \deg S_\lambda \cdot T(z) = 0$$

and similarly $S_\lambda T = 0$. Now $\overline{T} = \frac{1}{|H|}\varphi_* T$ implies the assertion.

To obtain (2) we compute using $\overline{\Delta}\,\overline{T} = \overline{T}$, $\overline{T}^2 = d\overline{T}$ and Theorem 3.5

$$\overline{K}_\lambda \overline{T} = \frac{1}{|H|^2}\overline{S}_\lambda\overline{T} + \overline{\Delta}\,\overline{T} - ((\lambda,\lambda)+1)\overline{T}^2 = \overline{T} - d((\lambda,\lambda)+1)\overline{T} = \deg \overline{K}_\lambda \cdot \overline{T},$$

where the last equation follows from Corollary 3.6. Similarly Theorem 3.5 gives the equation $\overline{K}_\lambda \overline{T} = \overline{T}\,\overline{K}_\lambda$. □

For the next theorem we need a lemma.

Lemma 3.8. $\overline{S}_\lambda^2 = \overline{e} \cdot \overline{S}_\lambda$, *with* $\overline{e} = \frac{|H|\cdot|W|\cdot(\lambda,\lambda)}{\dim V_\omega} \in \mathbb{Q}$.

Proof. Schur's orthogonality relations imply that $p_\lambda = \frac{1}{e}\cdot S_\lambda$ is an idempotent of the rational group ring $\mathbb{Q}[W]$ or equivalently that $S_\lambda^2 = e \cdot S_\lambda$ with $e = \frac{|W|\cdot(\lambda,\lambda)}{\dim V_\omega}$, see [LR] Proposition 2.3. Note that $e < 0$, since $(\ ,\ )$ is negative definite.

According to [Fu] Proposition 16.1.2 (a), $(\varphi\times\varphi)_* S_\lambda \cdot (\varphi\times\varphi)_* S_\lambda = (\varphi\times\varphi)_*(S_\lambda\cdot S_\lambda)$, which implies

$$\begin{aligned}\overline{S}_\lambda^2 = (\varphi\times\varphi)_*(S_\lambda^2) &= (\varphi\times\varphi)_*(e\cdot S_\lambda) \\ &= (\varphi\times\varphi)_*(e\Delta)\cdot(\varphi\times\varphi)_*(S_\lambda) = |H|e\overline{S}_\lambda.\end{aligned}$$

□

Theorem 3.9. *The Kanev correspondence* $\overline{K}_\lambda$ *satisfies the following cubic equation*
$$(\overline{K}_\lambda - \overline{\Delta})(\overline{K}_\lambda + (q_\lambda - 1)\overline{\Delta})(\overline{K}_\lambda - \deg \overline{K}_\lambda \overline{\Delta}) = 0$$
with $q_\lambda = -\frac{d\cdot(\lambda,\lambda)}{\dim V_\omega} \in \mathbb{N}$.

Proof. Theorem 3.5 and Lemma 3.8 give
$$|H|^4(\overline{K}_\lambda - \overline{\Delta} + a\overline{T})^2 = \overline{e}|H|^2(\overline{K}_\lambda - \overline{\Delta} + a\overline{T})$$
with $a = (\lambda, \lambda) + 1$. Applying Corollary 3.7 and $\overline{T}^2 = d\overline{T}$ as well as $\overline{\Delta}\,\overline{T} = \overline{T}$, this implies that there is a rational number c such that
$$(\overline{K}_\lambda - \overline{\Delta})^2 - \frac{\overline{e}}{|H|^2}(\overline{K}_\lambda - \overline{\Delta}) + c\overline{T} = 0.$$
Multiplying this equation by $\overline{K}_\lambda - \deg \overline{K}_\lambda \overline{\Delta}$ and using Corollary 3.7 again, we get
$$(\overline{K}_\lambda - \overline{\Delta})(\overline{K}_\lambda - \overline{\Delta} - \frac{\overline{e}}{|H|^2}\overline{\Delta})(\overline{K}_\lambda - \deg \overline{K}_\lambda \overline{\Delta}) = 0.$$
This completes the proof of the theorem. □

3.5. **The Prym variety** P_λ**.** We consider the rational correspondence $\overline{s}_\lambda = \frac{1}{|H|^2}\overline{S}_\lambda$. Then $\overline{s}_\lambda$ satisfies the relation $\overline{s}_\lambda^2 = -q_\lambda \overline{s}_\lambda$ and by Theorem 3.5 we have

(8) $$\overline{s}_\lambda = \overline{K}_\lambda - \overline{\Delta} + ((\lambda,\lambda) + 1)\overline{T}.$$

Hence we see that $\overline{s}_\lambda$ is an integral correspondence if and only if (λ, λ) is an integer. This happens e.g. for E_8, but not for E_6 or E_7 – see the tables in section 4.

The correspondence $\overline{s}_\lambda$ induces a rational endomorphism also denoted by $\overline{s}_\lambda \in \mathrm{End}_{\mathbb{Q}}(\mathrm{Jac}(Y))$. We introduce the following abelian subvariety of $\mathrm{Jac}(Y)$, which we will also call *Prym variety*
$$P_\lambda := \mathrm{im}(m\overline{s}_\lambda) \subset \mathrm{Jac}(Y),$$
where m is some integer such that $m\overline{s}_\lambda \in \mathrm{End}(\mathrm{Jac}(Y))$. It is clear that P_λ does not depend on the integer m.

Since $\overline{K}_\lambda - \overline{\Delta}$ is a correspondence on Y it induces an endomorphism $v_\lambda \in \mathrm{End}(\mathrm{Jac}(Y))$. We denote by
$$S = \mathrm{Prym}(Y/X)$$
the Prym variety of the covering $\psi : Y \to X$, see Remark 2.1(ii). Since by Corollary 3.7 $v_\lambda t = tv_\lambda$, we obtain that $v_\lambda(S) \subset S$. Hence v_λ restricts to a symmetric endomorphism
$$u_\lambda \in \mathrm{End}(S) \qquad \text{with} \qquad u_\lambda^2 = q_\lambda u_\lambda.$$

Proposition 3.10. *We have an equality of abelian subvarieties of* S
$$P_\lambda = \mathrm{im}(u_\lambda) \subset S.$$

Proof. Because of (8) the endomorphisms $\overline{s}_\lambda$ and u_λ coincide on S up to a non-zero integer multiple. Moreover the restriction of $\overline{s}_\lambda$ to the subvariety $\psi^*\mathrm{Jac}(X)$ is zero because $\overline{s}_\lambda t = 0$ (Corollary 3.7). The equality now follows since $\mathrm{Jac}(Y)$ and $\mathrm{Jac}(X) \times S$ are isogenous. □

Remark 3.11. Note that the abelian subvariety $P_\lambda \subset S$ only depends on $\lambda \in V_\omega$ and not on the lattice $\mathbb{S}_\omega \subset V_\omega$. Moreover we recall from [Me] Proposition 4.3 (1) that P_λ and $P_{\lambda'}$ are isogenous for any $\lambda, \lambda' \in V_\omega$.

Remark 3.12. If $\psi : Y \to X$ is étale, then the Kanev correspondence $\overline{K}_\lambda$ is fixed-point-free. On the other hand we notice that the induced endomorphism v_λ does not satisfy a quadratic equation — only its restriction u_λ to S — hence we are not in a situation where Kanev's criterion ([K2] or [BL] Theorem 12.9.1) applies.

4. Examples

As our main examples let us work out the invariants of the last section in the case of a Weyl group W of ADE-type. Let $\pi : Z \to X$ be a Galois covering of smooth projective curves with Galois group W. As weights λ we choose the minuscule weights ϖ_i (with the notation of [Bo]) except in the case of type E_8, where we consider the quasi-minuscule weight $\lambda = \varpi_8$. As above let V_λ denote the corresponding absolutely irreducible $\mathbb{Q}$-representation, H the stabilizer subgroup of λ in W and $Y = Z/H$. If $(\ , \ )$ denotes the uniquely determined W-invariant negative definite symmetric form on V_λ as defined in section 3.2, the following tables give the values for the numbers (λ, λ), $d = \deg(Y/X)$, $\dim V_\lambda$, q_λ and the degree of the Kanev-correspondence $\overline{K}_\lambda$. For the definition of the Dynkin index d_λ see [LS] and also Proposition 7.4. The case of a Weyl group of type E is closely related to a del Pezzo surface (see [K1]). The last line of the first table gives the degree of the corresponding surface.

Weyl group of type	$E_4 = A_4$	$E_5 = D_5$	E_6	E_7	E_8
weight λ	ϖ_2	ϖ_4, ϖ_5	ϖ_1, ϖ_6	ϖ_7	ϖ_8
(λ, λ)	$-\frac{6}{5}$	$-\frac{5}{4}$	$-\frac{4}{3}$	$-\frac{3}{2}$	-2
$d = \deg(Y/X)$	10	16	27	56	240
$\dim V_\lambda$	10	16	27	56	248
$q_\lambda = d_\lambda$	3	4	6	12	60
$\deg \overline{K}_\lambda$	3	5	10	29	241
del Pezzo of degree	5	4	3	2	1

Weyl group of type	A_n	D_n
weight λ	$\varpi_i,\ 1 \leq i \leq n$	ϖ_{n-1}, ϖ_n
(λ, λ)	$-\frac{i(n+1-i)}{n+1}$	$-\frac{n}{4}$
$d = \deg(Y/X)$	$\binom{n+1}{i}$	2^{n-1}
$\dim V_\lambda$	$\binom{n+1}{i}$	2^{n-1}
$q_\lambda = d_\lambda$	$\binom{n-1}{i-1}$	2^{n-3}
$\deg \overline{K}_\lambda$	$\binom{n-1}{i-1}n - \binom{n+1}{i} + 1$	$2^{n-3}(n-4) + 1$

Proof. The symmetric form $(\ , \ )$ is given by the negative of the Cartan matrix. This gives the value of (λ, λ) using the explicit form of λ as outlined in [Bo]. The length of the orbit of λ under the action of W gives the degree of Y/X. The dimension of V_λ follows from the fact that λ is a minuscule weight, respectively quasi-minuscule weight in the case of E_8. The values of q_λ and $\deg \overline{K}_\lambda$ are computed by Theorem 3.9 and Corollary 3.6. For the degree of the del Pezzo surface see [K1]. For the Dynkin indices see [LS] Proposition 2.6. $\square$

Remark 4.1. We note that in the examples of the tables $q_\lambda = d_\lambda$. This equality of the two integers is a coincidence, as e.g. for $(G_2, \varpi_1) : d_\lambda = 2$, $q_\lambda = 6$ or for $(F_4, \varpi_4) : d_\lambda = 6$, $q_\lambda = 12$ see [LS] Proposition 2.6 and [K1] page 176.

5. The Donagi-Prym variety

5.1. **Definitions and properties.** Following [Don1] we introduce the compact commutative algebraic group

$$\mathrm{Prym}(\pi, \mathbb{S}_\omega) := \mathrm{Hom}_W(\mathbb{S}_\omega, \mathrm{Pic}(Z)),$$

which we call the Donagi-Prym variety associated to the pair $(\pi, \mathbb{S}_\omega)$, where $\pi : Z \to X$ is a Galois covering with group W and $\mathbb{S}_\omega$ is a right-$\mathbb{Z}[W]$-module. The elements of $\mathrm{Prym}(\pi, \mathbb{S}_\omega)$ are W-equivariant homomorphisms $\phi : \mathbb{S}_\omega \to \mathrm{Pic}(Z)$; for an alternate description see Lemma 6.1. Its connected component $\mathrm{Prym}(\pi, \mathbb{S}_\omega)_0$ containing 0 is an abelian variety. For any element $\lambda \in \mathbb{S}_\omega$ we consider the corresponding evaluation map

$$ev_\lambda : \mathrm{Prym}(\pi, \mathbb{S}_\omega) \longrightarrow \mathrm{Pic}(Z), \qquad \phi \mapsto \phi(\lambda).$$

Proposition 5.1. *We have the equality*

$$\mathrm{Hom}_W(\mathbb{S}_\omega, \mathrm{Pic}(Z)) = \mathrm{Hom}_W(\mathbb{S}_\omega, \mathrm{Jac}(Z)).$$

In particular all connected components of the image $ev_\lambda(\mathrm{Prym}(\pi, \mathbb{S}_\omega))$ are contained in $\mathrm{Jac}(Z)$.

Proof. Consider a homomorphism $\phi \in \mathrm{Hom}_W(\mathbb{S}_\omega, \mathrm{Pic}(Z))$. The composite map $\deg \circ \phi$ is a W-invariant homomorphism from $\mathbb{S}_\omega$ to $\mathbb{Z}$. Since the dual lattice $\mathrm{Hom}_\mathbb{Z}(\mathbb{S}_\omega, \mathbb{Z})$ is an irreducible W-module, we obtain that $\deg \circ \phi = 0$. □

We denote by

$$\Gamma_\lambda = \mathbb{S}_\omega / (\lambda \cdot \mathbb{Z}[W]).$$

This is a finite abelian group since $V_\omega = \mathbb{S}_\omega \otimes_\mathbb{Z} \mathbb{Q}$ is an irreducible W-module.

Proposition 5.2. *The kernel of the evaluation map equals*

$$\ker ev_\lambda = \mathrm{Hom}_W(\Gamma_\lambda, \mathrm{Jac}(Z)).$$

Proof. Let U denote the image of the natural map $\mathbb{Z}[W] \xrightarrow{a} U \xrightarrow{i} \mathbb{S}_\omega$ defined by $a(g) = \lambda g$ for $g \in W$. Then we have the exact sequence of right $\mathbb{Z}[W]$-modules

$$0 \longrightarrow U \xrightarrow{i} \mathbb{S}_\omega \longrightarrow \Gamma_\lambda \longrightarrow 0.$$

Now we apply the left-exact contravariant functor $\mathrm{Hom}_W(\cdot, \mathrm{Jac}(Z))$ and we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_W(\Gamma_\lambda, \mathrm{Jac}(Z)) \longrightarrow \mathrm{Hom}_W(\mathbb{S}_\omega, \mathrm{Jac}(Z)) \xrightarrow{\check{i}} \mathrm{Hom}_W(U, \mathrm{Jac}(Z)).$$

We then apply the same functor to the surjective $\mathbb{Z}[W]$-morphism $a : \mathbb{Z}[W] \longrightarrow U$ and we obtain an injective morphism

$$\check{a} : \mathrm{Hom}_W(U, \mathrm{Jac}(Z)) \longrightarrow \mathrm{Hom}_W(\mathbb{Z}[W], \mathrm{Jac}(Z)) = \mathrm{Jac}(Z).$$

Note that the composite morphism $\check{a} \circ \check{i}$ equals the evaluation map ev_λ and that $\ker ev_\lambda = \ker \check{i}$ because $\check{a}$ is injective. □

5.2. **The image of the evaluation map ev_λ.** It is clear that

$$ev_\lambda(\mathrm{Prym}(\pi, \mathbb{S}_\omega)) \subset \mathrm{Jac}^H(Z),$$

where $H = \mathrm{Stab}(\lambda)$ and $\mathrm{Jac}^H(Z)$ denotes the subvariety of $\mathrm{Jac}(Z)$ parametrizing H-invariant line bundles over Z. We recall the exact sequence of abelian groups ([Dol] Proposition 2.2)

$$0 \longrightarrow H^* \longrightarrow \mathrm{Pic}(H;Z) \xrightarrow{\iota} \mathrm{Pic}^H(Z) \longrightarrow H^2(H, \mathbb{C}^*) \longrightarrow 0,$$

where $\mathrm{Pic}(H;Z)$ denotes the group of H-linearized line bundles over Z (see [Dol] section 1) and $H^* = \mathrm{Hom}(H, \mathbb{C}^*)$ denotes the group of characters of H. The map ι is the map which forgets the H-linearization.

Proposition 5.3. *We have the exact sequence*

$$0 \longrightarrow H^* \longrightarrow \mathrm{Jac}(H;Z) \xrightarrow{\iota} \mathrm{Jac}^H(Z) \longrightarrow H^2(H, \mathbb{C}^*) \longrightarrow 0. \tag{9}$$

Proof. We denote by d the degree of the cover $\psi : Y \to X$. Consider the following diagram, in which the first line is obtained as the kernel of the degree morphism.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Jac}(Y)/H^* & \to & \mathrm{Jac}^H(Z) & \xrightarrow{c} & H^2(H,\mathbb{C}^*) & \\
 & \downarrow & & \downarrow & & \downarrow\cong & \\
0 \to & \mathrm{Pic}(Y)/H^* & \to & \mathrm{Pic}^H(Z) & \xrightarrow{c} & H^2(H,\mathbb{C}^*) & \to 0 \\
 & \downarrow\deg & & \downarrow\deg & & \downarrow 0 & \\
0 \to & \mathbb{Z} & \xrightarrow{\cdot d} & \mathbb{Z} & \to & \mathbb{Z}/d\mathbb{Z} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow\cong & \\
 & 0 & \to & \mathbb{Z}/d\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z}/d\mathbb{Z} & \to 0
\end{array}$$

The snake lemma now implies that the natural homomorphism $c : \mathrm{Jac}^H(Z) \to H^2(H, \mathbb{C}^*)$ is surjective. □

In section 6.5 we will show that for any $\lambda \in \mathbb{S}_\omega$ the evaluation map ev_λ lifts to $\mathrm{Jac}(H;Z)$. Finally we also recall the exact sequence

$$0 \longrightarrow \mathrm{Jac}(Y) \longrightarrow \mathrm{Jac}(H;Z) \longrightarrow \bigoplus_{x \in \mathrm{Br}(\varphi)} \mathbb{Z}/e_x\mathbb{Z} \longrightarrow 0, \tag{10}$$

where x varies over the branch divisor $\mathrm{Br}(\varphi)$ of the Galois covering $\varphi : Z \to Y = Z/H$ and e_x denotes the ramification index of x.

5.3. **The group Γ_λ.** We now compute the finite group Γ_λ in some special cases, which will be worked out in detail in section 8. Given a simple Lie algebra $\mathfrak{g}$ we take for $\mathbb{S}_\omega$ the weight lattice of $\mathfrak{g}$ and denote by $\Lambda_R \subset \mathbb{S}_\omega$ the root lattice. We use the notation of [Bo]. In particular α_i denote the simple roots and ϖ_i denote the fundamental weights of $\mathfrak{g}$.

Lemma 5.4. *In the following cases the group Γ_λ is trivial.*

(1) $W = W(E_8)$, $\lambda = \varpi_8$.
(2) $W = W(E_7)$, $\lambda = \varpi_7$.
(3) $W = W(E_6)$, $\lambda = \varpi_1$ *or* $\lambda = \varpi_6$.
(4) $W = W(D_n)$ *with n odd,* $\lambda = \varpi_{n-1}$ *or* $\lambda = \varpi_n$.
(5) $W = W(A_n)$, $\lambda = \varpi_i$ *with i coprime to* $n+1$.

Proof. In each case we have to show that $\lambda \cdot \mathbb{Z}[W] = \mathbb{S}_\omega$. Let $\alpha \in \Lambda_R$ be a root and let s_α be its associated reflection. We have

$$s_\alpha(\varpi) = \varpi - (\varpi|\alpha)\alpha \qquad \forall \varpi \in \mathbb{S}_\omega.$$

Note that for Lie algebras of type ADE all roots α have same length $(\alpha|\alpha) = 2$. Here $(\cdot|\cdot)$ denotes the Cartan-Killing form on Λ_R.

(1) Clearly the elements ϖ_8 and $s_{\alpha_8}(\varpi_8)$ are in the lattice $\lambda \cdot \mathbb{Z}[W]$. Moreover $s_{\alpha_8}(\varpi_8) = \varpi_8 - \alpha_8$, hence $\alpha_8 \in \lambda \cdot \mathbb{Z}[W]$. The Weyl group W acts transitively on the roots, which implies that $\Lambda_R \subset \lambda \cdot \mathbb{Z}[W]$. For $\mathfrak{g}$ of type E_8, we have $\Lambda_R = \mathbb{S}_\omega$ and we are done.
(2) As before $\varpi_7, s_{\alpha_7}(\varpi_7) \in \lambda \cdot \mathbb{Z}[W]$, hence $\alpha_7 \in \lambda \cdot \mathbb{Z}[W]$. Since W acts transitively on the roots, we have $\Lambda_R \subset \lambda \cdot \mathbb{Z}[W]$. Here $\varpi_7 \in \lambda \cdot \mathbb{Z}[W]$, but $\varpi_7 \notin \Lambda_R$. Since $[\mathbb{S}_\omega : \Lambda_R] = 2$, this implies the assertion.
(3) The computations are similar to the previous case using $(\alpha_1|\varpi_1) = (\alpha_6|\varpi_6) = 1$ and $[\mathbb{S}_\omega : \Lambda_R] = 3$.
(4) Since W acts transitively on the roots, we obtain as before that $\Lambda_R \subset \lambda \cdot \mathbb{Z}[W]$. Since n is odd, we have that $\mathbb{S}_\omega/\Lambda_R \cong \mathbb{Z}/4\mathbb{Z}$ is generated by the class of ϖ_n or ϖ_{n-1}.
(5) As before we obtain that $\Lambda_R \subset \lambda \cdot \mathbb{Z}[W]$. The class of the fundamental weight ϖ_i in the quotient $\mathbb{S}_\omega/\Lambda_R \cong \mathbb{Z}/(n+1)\mathbb{Z}$ equals the class of $i \in \mathbb{Z}/(n+1)\mathbb{Z}$. The assertion then follows.

□

6. Mumford groups

6.1. **Twisted Mumford groups.** Let T be a torus and suppose we are given a left action of the group W on T

$$\sigma : W \longrightarrow \mathrm{Aut}(T).$$

For any $g \in W$ we also denote the automorphism of T by g. This left action σ induces a right action of W on the group of characters $\mathrm{Hom}(T, \mathbb{C}^*)$. We suppose that the representation of W on $\mathrm{Hom}(T, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible, hence $\mathrm{Hom}(T, \mathbb{C}^*)$ is of the form $\mathbb{S}_\omega$ for some $\omega \in \widehat{W}$. Conversely given a lattice $\mathbb{S}_\omega$ there always exists a torus T with a left W-action such that

$$\mathbb{S}_\omega := \mathrm{Hom}(T, \mathbb{C}^*)$$

as W-modules. We recall that W acts from the right on the curve Z.

Let E_T be a principal T-bundle over Z. The group W then acts from the left on the set of principal T-bundles in two different ways: $g \in W$ sends E_T to

(1) g^*E_T, the pull-back of E_T under the automorphism g of Z.
(2) $E_T \times_g^T T$, the T-bundle obtained from E_T by extension of structure group from T to T, where $t \in T$ acts on T by left-multiplication with $g(t) \in T$.

Note that both actions commute. In this paper we will denote the combined left W-action on the set of principal T-bundles by

$$g.E_T = g^*E_T \times_g^T T, \qquad \text{for } g \in W.$$

Since the action of W on Z is a right action, we have $(g_1g_2)^*E_T = g_1^*g_2^*E_T$. This implies that the action $(g, E_T) \mapsto g.E_T$ is a left action.

We say that a T-bundle E_T is *W-invariant* if $E_T \simeq g.E_T$ for all $g \in W$. Here the symbol $\simeq$ denotes a T-equivariant isomorphism. Let $\underline{T}$ denote the sheaf of abelian groups on Z defined by $\underline{T}(U) = \mathrm{Mor}(U,T)$ for any open subset $U \subset Z$. The cohomology group $H^1(Z,\underline{T})$ parametrizes isomorphism classes of T-bundles over Z. We denote by $H^1(Z,\underline{T})^W$ the W-invariant subset, where W acts both on the curve Z and on the torus T.

Lemma 6.1. *There are canonical bijections*

$$\begin{aligned} H^1(Z,\underline{T})^W &= \{W\textit{-invariant } T\textit{-bundles } E_T \textit{ on } Z\} \\ &= \mathrm{Prym}(\pi, \mathbb{S}_\omega). \end{aligned}$$

Proof. The first equality follows from the definition of $H^1(Z,\underline{T})^W$. For the second equality note that the map

$$\Phi : \{T\text{-bundles } E_T \text{ on } Z\} \longrightarrow \mathrm{Hom}(\mathbb{S}_\omega, \mathrm{Pic}(Z))$$
$$\Phi(E_T) = (\phi : \mathbb{S}_\omega \to \mathrm{Pic}(Z),\ \phi(\lambda) = E_T \times^T_\lambda \mathbb{C}^*)$$

is bijective. The inverse map is given by mapping $\phi \in \mathrm{Hom}(\mathbb{S}_\omega, \mathrm{Pic}(Z))$ to $E_T = L_1 \oplus \cdots \oplus L_n$, where $L_i = \phi(\lambda_i)$, $\lambda_1, \ldots, \lambda_n$ being a $\mathbb{Z}$-basis of $\mathbb{S}_\omega$. One can easily show that the inverse map does not depend on the choice of the $\mathbb{Z}$-basis.

We have to show that $\phi = \Phi(E_T)$ is W-equivariant if and only if $E_T \simeq g.E_T$ for all $g \in W$. Since W acts from the right on $\mathbb{S}_\omega$ and Z, the map ϕ is W-equivariant if and only if

$$\phi(\lambda g) = (g^{-1})^* \phi(\lambda) \tag{11}$$

for all $\lambda \in \mathbb{S}_\omega$ and $g \in W$. But

$$\phi(\lambda g) = E_T \times^T_{\lambda g} \mathbb{C}^* = (E_T \times^T_g T) \times^T_\lambda \mathbb{C}^*$$

and

$$(g^{-1})^* \phi(\lambda) = (g^{-1})^* (E_T \times^T_\lambda \mathbb{C}^*).$$

Hence (11) is valid for all $\lambda \in \mathbb{S}_\omega$ if and only if $(g^{-1})^* E_T \simeq E_T \times^T_g T$, which is the case if and only if $E_T \simeq g^* E_T \times^T_g T = g.E_T$. $\square$

Definition 6.2. Given a W-invariant T-bundle E_T over Z we define the *σ-twisted Mumford group of E_T* by

$$\mathcal{G}^\sigma(E_T) := \{(\gamma, g) \mid g \in W,\ \gamma : E_T \longrightarrow g.E_T\},$$

where γ is a T-equivariant isomorphism. The composition law in $\mathcal{G}^\sigma(E_T)$ is given by

$$(\gamma, g).(\mu, g') = ((g.\mu) \circ \gamma, gg'),$$

where $g.\mu : g.E_T \longrightarrow g.(g'.E_T) = (gg').E_T$ denotes the isomorphism obtained by applying $g.$ to μ.

Since the automorphism group $\mathrm{Aut}_T(E_T)$ equals T, we obtain that the group $\mathcal{G}^\sigma(E_T)$ fits into the exact sequence

$$1 \longrightarrow T \longrightarrow \mathcal{G}^\sigma(E_T) \longrightarrow W \longrightarrow 1, \tag{12}$$

where the second map is the map which forgets the isomorphism γ. We note that the induced action by conjugation of W on the normal subgroup T coincides with σ. To simplify notation we denote the σ-twisted Mumford group $\mathcal{G}^\sigma(E_T)$ by N.

Proposition 6.3. *Let E_T be a W-invariant T-bundle over Z with projection $p : E_T \to Z$. Then the natural T-action on E_T extends canonically to a N-action on the variety E_T preserving the fibers of $\pi \circ p : E_T \to X$, i.e. there exists a morphism over X defining a group action*

$$\mu : E_T \times N \longrightarrow E_T,$$

which extends right-multiplication of T on E_T.

Proof. Given $n = (\gamma, g) \in N$ we first define

$$\nu_n := \beta_g \circ \gamma,$$

where β_g is the pull-back of $g : Z \to Z$ via the projection map $p : E_T \times_g^T T \to Z$. This is summarized by the diagram

$$\begin{array}{ccc} E_T & & \\ \downarrow \gamma & \searrow \nu_n & \\ g^*(E_T \times_g^T T) & \xrightarrow{\beta_g} & E_T \times_g^T T \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{g} & Z. \end{array}$$

In order to define the morphism μ, it suffices to define for any $n = (\gamma, g) \in N$ the right-multiplication with n on E_T

$$\mu_n : E_T \longrightarrow E_T \qquad \text{satisfying} \qquad \mu_{nn'} = \mu_{n'} \circ \mu_n \quad \forall n, n' \in N. \tag{13}$$

The set of morphisms $\{\mu_n\}_{n \in N}$ is related to the set of "twisted" morphisms $\{\nu_n\}_{n \in N}$ by composition

$$\mu_n : E_T \xrightarrow{\nu_n} E_T \times_g^T T \xrightarrow{\alpha_g} E_T, \qquad \text{with } \alpha_g \left[\overline{(e,t)}\right] = eg^{-1}(t).$$

Here $\overline{(e,t)}$ denotes the class of $(e,t) \in E_T \times T$ in the quotient $E_T \times_g^T T := E_T \times T/T$. Note that we use the equality $etn = eng^{-1}(t)$ for $e \in E_T$, $n = (\gamma, g) \in N$ and $t \in T$.

It is now straightforward to check that the set $\{\mu_n\}_{n \in N}$ satisfies (13). $\square$

6.2. **The relative case.** In order to construct the abelianization map (see section 7.3) we need to define the twisted Mumford group for families of W-invariant T-bundles.

Let S be a scheme and let $\mathcal{E}_T$ be a family of T-bundles over Z parametrized by S, i.e. a T-bundle over the product $Z \times S$. Let $\pi_S : Z \times S \to S$ denote projection onto the second factor. We say that $\mathcal{E}_T$ is *W-invariant* if for any $g \in W$ and any closed point $s \in S$ there exists an isomorphism

$$\mathcal{E}_{T|Z \times \{s\}} \simeq g.\mathcal{E}_{T|Z \times \{s\}}. \tag{14}$$

The following lemma is the analogue of the see-saw theorem for T-bundles (see [Mu]).

Lemma 6.4. *Let $\mathcal{A}_T$ be a family of T-bundles over Z parametrized by S. Suppose that for any closed point $s \in S$ the T-bundle $\mathcal{A}_{T|Z\times\{s\}}$ is trivial. Then there exists a T-bundle M over S such that $\mathcal{A}_T \simeq \pi_S^* M$.*

Applying the preceding lemma to (14) we obtain that for any $g \in W$ there exists a T-bundle M_g over S such that

$$\mathcal{E}_T \simeq g.\mathcal{E}_T \otimes \pi_S^* M_g.$$

Moreover the map $W \to H^1(S, \underline{T})$, $g \mapsto M_g$ is easily seen to be a group homomorphism.

We now define the σ-twisted Mumford group of the family $\mathcal{E}_T$ by

$$\mathcal{G}^\sigma(\mathcal{E}_T) := \{(\gamma, g) \mid g \in W,\ \gamma : \mathcal{E}_T \longrightarrow g.\mathcal{E}_T \otimes \pi_S^* M_g\},$$

where γ is a T-equivariant isomorphism over $Z \times S$. We easily see that the relative Mumford group fits in the exact sequence

$$1 \longrightarrow \mathrm{Mor}(S, T) \longrightarrow \mathcal{G}^\sigma(\mathcal{E}_T) \longrightarrow W \longrightarrow 1.$$

If S is complete and connected, then $\mathrm{Mor}(S, T) = T$ and for any closed point $s \in S$ the restriction map

$$\mathrm{res}_s :\ \mathcal{G}^\sigma(\mathcal{E}_T) \xrightarrow{\sim} \mathcal{G}^\sigma(E_T), \qquad (\gamma, g) \mapsto (\mathrm{res}_s(\gamma), g) \tag{15}$$

is an isomorphism. Here $E_T := \mathcal{E}_{T|Z\times\{s\}}$.

6.3. **Non-twisted Mumford groups.** Let G be an affine algebraic group.

Definition 6.5. Given a principal G-bundle E_G over Z we define the (non-twisted) Mumford group of E_G by

$$\mathcal{G}^1(E_G) := \{(\gamma, g) \mid g \in W,\ \gamma : E_G \longrightarrow g^* E_G\},$$

where γ is a G-equivariant isomorphism. The composition law in $\mathcal{G}^1(E_G)$ is given by

$$(\gamma, g).(\mu, g') = (g^*(\mu) \circ \gamma, gg').$$

As before, the group $\mathcal{G}^1(E_G)$ fits into the exact sequence

$$1 \longrightarrow \mathrm{Aut}(E_G) \longrightarrow \mathcal{G}^1(E_G) \longrightarrow W \longrightarrow 1. \tag{16}$$

The induced action by conjugation of W on $\mathrm{Aut}(E_G)$ is trivial.

Proposition 6.6. *Let E_T be a W-invariant T-bundle over Z and denote $N = \mathcal{G}^\sigma(E_T)$. Let $E_N := E_T \times^T N$ be the principal N-bundle obtained by extending the structure group from T to N. Then the exact sequence* (16) *for E_N*

$$1 \longrightarrow \mathrm{Aut}(E_N) \longrightarrow \mathcal{G}^1(E_N) \longrightarrow W \longrightarrow 1$$

splits canonically.

Proof. By Proposition 6.3 the variety E_T admits a canonical N-action. Given $e \in E_T$ and $n = (\gamma, g) \in N$ we denote their product by en. We consider the morphism

$$\Phi_n : E_T \times^T N \longrightarrow E_T \times^T N, \qquad \Phi_n\left[\overline{(e, f)}\right] = \overline{(en, n^{-1}f)}$$

with $e \in E_T$ and $f \in N$. Then one easily checks that Φ_n is well-defined and that it makes the following diagram commute

$$\begin{array}{ccc} E_T \times^T N & \xrightarrow{\Phi_n} & E_T \times^T N \\ \downarrow p & & \downarrow p \\ Z & \xrightarrow{g} & Z. \end{array}$$

Hence Φ_n is a lift of the automorphism g to the principal N-bundle $E_T \times^T N$ and we may view Φ_n as an element of $\mathcal{G}^1(E_N)$. Moreover $\Phi_{nn'} = \Phi_{n'} \circ \Phi_n$ and $\Phi_t = \mathrm{id}$ for all $n, n' \in N$ and all $t \in T$. Hence the morphism Φ_n only depends on $g \in W$. This gives the canonical splitting. □

6.4. **Mumford groups and descent.** We recall the relation between the non-twisted Mumford group $\mathcal{G}^1(E_G)$ and descent.

Definition 6.7. *Let E_G be a G-bundle over Z. A W-linearization of the G-bundle E_G is a splitting of the exact sequence* (16)

$$\mathcal{G}^1(E_G) \overset{\curvearrowleft}{\longrightarrow} W.$$

Descent theory gives in the usual way (see [Gr2])

Proposition 6.8. *Let E_G be a W-linearized G-bundle over Z. There exists a G-bundle F_G over X such that*

$$E_G \simeq \pi^* F_G$$

if and only if for any ramification point $z \in Z$ of the covering $\pi : Z \to X$ the stabilizer subgroup $W_z \subset W$ of z acts trivially on the fiber $(E_G)_z$.

Corollary 6.9. *If the covering $\pi : Z \to X$ is étale, then given E_G over Z there is a bijection between the set of W-linearizations on E_G and the set of G-bundles F_G over X such that $E_G \simeq \pi^* F_G$.*

Remark 6.10. We can rephrase Proposition 6.6 by saying that the N-bundle $E_N := E_T \times^T N$ admits a canonical W-linearization. The conditions for descent of E_N to X given in Proposition 6.8 then translate into certain conditions on the W-invariant T-bundle E_T — for the case of cameral coverings, see [DG].

6.5. **The evaluation map ev_λ lifts.**

Proposition 6.11. *For any $\lambda \in \mathbb{S}_\omega$ the evaluation map ev_λ lifts to $\mathrm{Jac}(H; Z)$; i.e. there is a commutative diagram*

$$\begin{array}{ccc} & & \mathrm{Jac}(H; Z) \\ & \tilde{ev}_\lambda \nearrow & \downarrow \iota \\ \mathrm{Prym}(\pi, \mathbb{S}_\omega) & \xrightarrow{ev_\lambda} & \mathrm{Jac}^H(Z). \end{array}$$

Proof. We have to show that the line bundle $L = ev_\lambda(E_T) = E_T \times^T_\lambda \mathbb{C}^*$ admits a canonical H-linearization. We denote by $\lambda = \lambda_1, \ldots, \lambda_d \in \mathbb{S}_\omega$ the W-orbit of the element $\lambda \in \mathbb{S}_\omega$, with $d = [W : H]$ and by $L_i := E_T \times^T_{\lambda_i} \mathbb{C}^* \in \mathrm{Jac}(Z)$ the line bundle obtained through λ_i for $i = 1, \ldots, d$. As before we denote $N = \mathcal{G}^\sigma(E_T)$ and we consider $V = \mathrm{Ind}_T^N(\mathbb{C}_\lambda)$ the induced representation of N from the representation $T \to GL(\mathbb{C}_\lambda)$ given by $\lambda \in \mathrm{Hom}(T, \mathbb{C}^*)$. The restricted representation decomposes as a T-module ([Se] Proposition 15)

$$\mathrm{Res}_T(V) = \left[\bigoplus_{i=1}^{d} \mathbb{C}_{\lambda_i}\right]^{\oplus |H|}$$

which implies that the vector bundle $E_N \times^N V$ decomposes as $\left[\bigoplus_{i=1}^{d} L_i\right]^{\oplus |H|}$. Moreover by Proposition 6.6 E_N admits a canonical W-linearization, hence the decomposable bundle $E_N \times^N V$ also does. Since the subgroup $H \subset W$ preserves the direct summand $L = L_1$, we are done. □

In section 7.3 we will need the next lemma. First we introduce some notation. Given $\lambda \in \mathbb{S}_\omega$ we denote by $\lambda = \lambda_1, \ldots, \lambda_d \in \mathbb{S}_\omega$ the W-orbit of λ and by $L_i := E_T \times^T_{\lambda_i} \mathbb{C}^*$ the associated line bundles for $i = 1, \ldots, d$. We recall (see (10)) that if $\pi : Z \to X$ is étale, then $\mathrm{Jac}(Y) = \mathrm{Jac}(H; Z)$ and we denote by $M \in \mathrm{Jac}(Y)$ the line bundle defined by the relation

$$\varphi^* M = L_1 = ev_\lambda(E_T).$$

Lemma 6.12. *The direct sum $\bigoplus_{i=1}^{d} L_i$ admits a canonical W-linearization and, if $\pi : Z \to X$ is étale, then the W-linearized vector bundle $\bigoplus_{i=1}^{d} L_i$ equals $\pi^*(\psi_* M)$.*

Proof. We have already seen in the proof of Proposition 6.11 that $\bigoplus_{i=1}^{d} L_i$ admits a canonical W-linearization. If $\pi : Z \to X$ is étale, then there exists a vector bundle A over X such that $\pi^* A = \bigoplus_{i=1}^{d} L_i$ as W-linearized vector bundles. By adjunction for the morphism $\psi : Y \to X$ we have $\mathrm{Hom}(A, \psi_* M) = \mathrm{Hom}(\psi^* A, M)$ and since π is étale, we have

$$\mathrm{Hom}(\psi^* A, M) = \mathrm{Hom}(\pi^* A, \varphi^* M)^H = \mathrm{Hom}(\bigoplus_{i=1}^{d} L_i, L_1)^H,$$

where the exponent H denotes H-equivariant homomorphisms. The last space contains the projection onto the first factor, which is H-equivariant. This gives a nonzero homomorphism $\alpha : A \to \psi_* M$.

Similarly one can show using the isomorphism $\mathcal{H}om(\psi_* M, \mathcal{O}_X) = \psi_*(M^{-1})$ that $\mathrm{Hom}(\psi_* M, A) = \mathrm{Hom}(\bigoplus_{i=1}^{d} L_i^{-1}, L_1^{-1})^H$. As before projection onto the second factor gives a nonzero homomorphism $\beta : \psi_* M \to A$ and one easily checks that $\alpha \circ \beta = \mathrm{id}_{\psi_* M}$. Hence α is an isomorphism and we are done. □

6.6. **Relation between the Prym variety P_λ and the Donagi-Prym $\mathrm{Prym}(\pi, \mathbb{S}_\omega)$.** By Proposition 6.11 there exists a morphism

$$\widetilde{ev}_\lambda : \mathrm{Prym}(\pi, \mathbb{S}_\omega) \longrightarrow \mathrm{Jac}(H; Z)$$

and because of the exact sequence (10) the connected component $\mathrm{Prym}(\pi, \mathbb{S}_\omega)_0$ containing 0 is mapped by $\widetilde{ev}_\lambda$ into $\mathrm{Jac}(Y)$.

Proposition 6.13. *We have an equality of abelian subvarieties of* $\mathrm{Jac}(Y)$

$$\widetilde{ev}_\lambda\left(\mathrm{Prym}(\pi, \mathbb{S}_\omega)_0\right) = P_\lambda \subset \mathrm{Jac}(Y).$$

Proof. We consider the natural map of $\mathbb{Z}$-algebras

$$\phi : \bigoplus_{\omega \in \widehat{W}} \mathrm{End}(\mathbb{S}_\omega) \longrightarrow \mathbb{Z}[W].$$

Note that $\phi \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-algebras $\bigoplus_{\omega \in \widehat{W}} \mathrm{End}(V_\omega) \cong \mathbb{Q}[W]$. Taking the tensor product over $\mathbb{Z}[W]$ with the $\mathbb{Z}[W]$-module $\mathrm{Jac}(Z)$ induces an isogeny decomposition of $\mathrm{Jac}(Z)$ (see also [Don1] Formula (5.3))

$$\phi : \bigoplus_{\omega \in \widehat{W}} \mathbb{S}_\omega \otimes_{\mathbb{Z}} \mathrm{Prym}(\pi, \mathbb{S}_\omega) \longrightarrow \mathrm{Jac}(Z) \otimes_{\mathbb{Z}[W]} \mathbb{Z}[W] = \mathrm{Jac}(Z).$$

Moreover the restriction of ϕ to $\lambda \cdot \mathbb{Z} \otimes_{\mathbb{Z}} \mathrm{Prym}(\pi, \mathbb{S}_\omega)$ is the evaluation map ev_λ (section 5.1). With the notation of Lemma 3.8, the multiplication with the idempotent $p_\lambda = \frac{1}{e} S_\lambda \in \mathbb{Q}[W]$ corresponds under $\phi \otimes_Z \mathbb{Q}$ to the projection onto the $\mathbb{Q}$-vector space $\lambda \cdot \mathbb{Q} \otimes_{\mathbb{Q}} V_\omega^* \subset \bigoplus_{\omega \in \widehat{W}} \mathrm{End}(V_\omega)$ (see e.g. [Me] section 4.4). The proposition now follows immediately from the definition of the Prym variety P_λ. □

7. The abelianization map: the étale case

7.1. **Grothendieck's spectral sequence.** In this subsection we recall some facts related to Grothendieck's spectral sequence [Gr1]:

Let W be any finite group — not necessarily a Weyl group, let Z be a curve with a right W-action with quotient $\pi : Z \to X$ and let A be an abelian algebraic group, which is also a W-module. Let $\underline{A}$ denote the W-sheaf of abelian groups defined by $\underline{A}(U) = \mathrm{Mor}(U, A)$ for an open subset $U \subset Z$. Consider the following two left-exact functors

$$\Gamma_Z : \{W\text{-sheaves over } Z\} \to \{W\text{-modules}\}, \qquad \Gamma_Z(\underline{A}) = \Gamma(Z, \underline{A}) = \text{global sections of } \underline{A},$$

$$\Gamma^W : \{W\text{-modules}\} \to \{\text{abelian groups}\}, \qquad \Gamma^W(M) = M^W = W\text{-invariant elements of } M.$$

Consider the composite functor $\Gamma_Z^W = \Gamma^W \circ \Gamma_Z$. Its n-th derived functor, which we denote by $H^n(Z; W; \cdot)$, is computed by Grothendieck's spectral sequences

$$\begin{aligned} E_2^{p,q} &= H^p(W, H^q(Z, \underline{A})) \quad &\Rightarrow \quad E^{p+q} &= H^{p+q}(Z; W; \underline{A}) \\ {}'E_2^{p,q} &= H^p(X, R^q\pi_*^W(\underline{A})) \quad &\Rightarrow \quad E^{p+q} &= H^{p+q}(Z; W; \underline{A}) \end{aligned}$$

and the associated exact sequence of low degree terms of the first spectral sequence is

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{c} E_2^{2,0} \to E^2. \tag{17}$$

In the case W acts on a torus $A = T$ with $\mathbb{S}_\omega = \mathrm{Hom}(T, \mathbb{C}^*)$ this exact sequence becomes using Lemma 6.1

$$0 \to H^1(W, T) \to H^1(Z; W; \underline{T}) \to \mathrm{Prym}(\pi, \mathbb{S}_\omega) \xrightarrow{c} H^2(W, T) \to H^2(Z; W; \underline{T}) \tag{18}$$

One can work out the following description of the homomorphism c. Here we omit the details.

Proposition 7.1. *For any T-bundle $E_T \in \mathrm{Prym}(\pi, \mathbb{S}_\omega)$ the cohomology class $c(E_T) \in H^2(W, T)$ equals the extension class* (12) *of the twisted Mumford group $\mathcal{G}^\sigma(E_T)$ of E_T.*

Lemma 7.2. *If the covering $\pi : Z \to X$ is étale, then $H^2(Z; W; \underline{T}) = 0$.*

Proof. We use the second spectral sequence ${}'E_2^{p,q}$. First we observe that the sheaves $R^1\pi_*^W(\underline{T})$ and $R^2\pi_*^W(\underline{T})$ are supported on the ramification divisor of $\pi : Z \to X$ by [Gr1] Théorème 5.3.1. Hence ${}'E_2^{1,1} = {}' E_2^{0,2} = 0$.

Next we claim that $\pi^*(\pi_*^W(\underline{T})) = \underline{T}$: let $V \subset Z$ be a sufficiently small open subset such that $g(V) \cap V = \emptyset$ for any $g \in W$ and $g \neq e$ — here we use the analytic topology on Z. Then the open subset $\pi^{-1}(\pi(V))$ decomposes as a disjoint union $\coprod_{g\in W} g(V)$. Now the elements of $\pi^*(\pi_*^W(\underline{T}))(V)$ correspond to W-equivariant morphisms $\mathrm{Mor}^W(\pi^{-1}(\pi(V)), T)$. The restriction to V gives a canonical bijection

$$\mathrm{Mor}^W(\coprod_{g\in W} g(V), T) = \mathrm{Mor}(V, T) = \underline{T}(V),$$

which proves the claim.

Finally we observe that $\underline{T} \cong (\underline{\mathbb{C}}^*)^n$ and since $H^2(Z, \underline{\mathbb{C}}^*) = 0$ — this easily follows from the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_Z \to \mathcal{O}_Z^* \to 0$, we obtain that $H^2(Z, \underline{T}) = 0$. We now conclude that ${}'E_2^{2,0} = H^2(X, \pi_*^W(\underline{T})) = 0$, since $H^2(X, \pi_*^W(\underline{T}))$ identifies with the W-invariant subspace of $H^2(Z, \underline{T})$. □

For any $\lambda \in \mathbb{S}_\omega = \mathrm{Hom}(T, \mathbb{C}^*)$ let $H = \mathrm{Stab}(\lambda) \subset W$ and let β_λ be the composite map

$$\beta_\lambda : H^2(W, T) \xrightarrow{res_H} H^2(H, T) \xrightarrow{\lambda} H^2(H, \mathbb{C}^*).$$

If the covering $\pi : Z \to X$ is not necessarily étale, we can say the following on the image of c. The next proposition will not be used in the sequel as we will concentrate on the étale case.

Proposition 7.3. *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Prym}(\pi, \mathbb{S}_\omega) & \xrightarrow{c} & H^2(W, T) \\ \big\downarrow{\scriptstyle ev_\lambda} & & \big\downarrow{\scriptstyle \beta_\lambda} \\ \mathrm{Jac}^H(Z) & \xrightarrow{c'} & H^2(H, \mathbb{C}^*) \end{array}$$

and $\beta_\lambda \circ c = 0$ for any $\lambda \in \mathbb{S}_\omega$.

Proof. In order to prove that the diagram commutes it is sufficient to combine the three exact sequences (17) obtained in the three cases $(W, T), (H, T)$ and $(H, \mathbb{C}^*)$ — here the first factor denotes the finite group and the second the abelian algebraic group A on which the finite group acts. We leave the details to the reader. The claim $\beta_\lambda \circ c = c' \circ ev_\lambda = 0$ is an immediate consequence of Proposition 6.11 and (9). □

7.2. **Moduli stack of G-bundles.** Let G be a simple and simply-connected algebraic group and let $T \subset G$ be a maximal torus, $T \subset N(T)$ the normalizer of the torus and $W = N(T)/T$ the Weyl group of G. Let $\mathbb{S}_\omega := \mathrm{Hom}(T, \mathbb{C}^*)$ be the weight lattice of G.

We denote by $\mathcal{M}_X(G)$ the moduli stack parametrizing principal G-bundles over the curve X. We recall some results from [LS] and [So2] on line bundles over $\mathcal{M}_X(G)$: let $\lambda \in \mathbb{S}_\omega$ be a dominant weight and $\rho_\lambda : G \to \mathrm{SL}(V_\lambda)$ the associated irreducible representation. Then ρ_λ induces a morphism of stacks

$$\widetilde{\rho}_\lambda : \mathcal{M}_X(G) \to \mathcal{M}_X(\mathrm{SL}(m)), \qquad E_G \mapsto E_G \times^G V_\lambda.$$

Here $m = \dim V_\lambda$. We denote by $\mathcal{D}_\lambda = \widetilde{\rho}_\lambda^* \mathcal{D}$ the pull-back of the determinant line bundle $\mathcal{D}$ over $\mathcal{M}_X(\mathrm{SL}(m))$. The next proposition is proved in [LS] and [So2].

Proposition 7.4. *For any simple and simply-connected algebraic group G we have*

(1) *There exists an ample line bundle $\mathcal{L}$ over $\mathcal{M}_X(G)$ such that*

$$\mathrm{Pic}(\mathcal{M}_X(G)) \cong \mathbb{Z} \cdot \mathcal{L}.$$

(2) *For any dominant weight $\lambda \in \mathbb{S}_\omega$, the integer d_λ defined by the relation*

$$\mathcal{D}_\lambda = \mathcal{L}^{\otimes d_\lambda}$$

equals the Dynkin index of the representation ρ_λ of G.

7.3. **The abelianization map Δ_θ.** ¿From now on we assume that the Galois covering $\pi : Z \to X$ is étale with Galois group equal to the Weyl group $W = N(T)/T$. We denote by $n \in H^2(W, T)$ the extension class of $N(T)$. Note that by Lemma 7.2 and by (18) the homomorphism

$$c : \mathrm{Prym}(\pi, \mathbb{S}_\omega) \longrightarrow H^2(W, T)$$

is surjective. We denote by $\mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$ a connected component of the fibre of c over n. Note that all connected components are isomorphic to each other.

As in section 6.2 we consider a universal family $\mathcal{E}_T$ of T-bundles over $Z \times \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$ — note that $\mathcal{E}_T$ is W-invariant — and we choose an isomorphism

$$\theta : \mathcal{G}^\sigma(\mathcal{E}_T) \xrightarrow{\sim} N(T)$$

inducing the identity on the subgroups T of $\mathcal{G}^\sigma(\mathcal{E}_T)$ and $N(T)$ and on the quotient W. The existence of θ follows from (15) and the fact that $\mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$ lies in the fiber of c over n, the extension class of $N(T)$.

Proposition 7.5. *Given an isomorphism θ there exists a morphism*

$$\Delta_\theta : \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n \longrightarrow \mathcal{M}_X(G),$$

such that for any $E_T \in \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$ the G-bundle $\Delta_\theta(E_T)$ satisfies

$$\pi^* \Delta_\theta(E_T) = E_T \times^T G.$$

The map Δ_θ is called the abelianization map.

Proof. The existence of the morphism Δ_θ will follow from the existence of a family of $N(T)$-bundles over X parametrized by $\mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$, or equivalently from the existence of a family of $N(T)$-bundles over Z with a W-linearization parametrized by $\mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$. But this follows from the relative version of Proposition 6.6, which says that the $N(T)$-bundle $\mathcal{E}_T \times_\theta^T N(T)$ admits a canonical W-linearization — here we use the isomorphism θ. □

Remark 7.6. (i): Note that Δ_θ factorizes through $\mathcal{M}_X(N(T))$.
(ii): A priori Δ_θ depends on the choice of θ. Two different choices of θ differ by an element in $\mathrm{Aut}^0(N(T))$, the group of automorphisms of $N(T)$ inducing the identity on T and W. Note that $T \subset \mathrm{Aut}^0(N(T))$ and that $\mathrm{Aut}^0(N(T))/T = H^1(W, T)$. The cohomology groups $H^1(W, T)$ and $H^2(W, T)$ have been computed in [Ma].

7.4. **Direct images of line bundles.** Consider a dominant weight $\lambda \in \mathbb{S}_\omega$ and let V_λ denote the associated irreducible representation of G. We consider the weight space decomposition

$$V_\lambda = \bigoplus_{\mu \in \Theta(\lambda)} V_\lambda^\mu,$$

where V_λ^μ denotes the weight space of V_λ associated to the weight $\mu \in \mathbb{S}_\omega$. The Weyl group W acts on the set $\Theta(\lambda)$ of all weights μ such that $V_\lambda^\mu \neq \{0\}$ and decomposes it into k orbit spaces

$$\Theta(\lambda) = \Theta_1 \cup \Theta_2 \cup \ldots \cup \Theta_k.$$

For each $j = 1, \ldots, k$ we choose a weight $\lambda_j \in \Theta_j$, i.e. $\Theta_j = \lambda_j.W$. We introduce for each j the subgroup $H_j = \mathrm{Stab}(\lambda_j) \subset W$ and the quotient $\psi_j : Z/H_j = Y_j \to X$. Let $n_j = \dim V_\lambda^{\lambda_j}$. Note that $n_j = \dim V_\lambda^\mu$ for any $\mu \in \Theta_j$, that $\lambda_1 = \lambda$ and $n_1 = 1$.

Since π is étale, we have $\mathrm{Jac}(H_j; Z) = \mathrm{Jac}(Y_j)$ and by Proposition 6.11 there exist morphisms

$$\widetilde{ev}_{\lambda_j} : \mathrm{Prym}(\pi, \mathbb{S}_\omega) \longrightarrow \mathrm{Jac}(Y_j).$$

For convenience of notation we introduce the product and the map

$$\mathrm{Jac}(Y_\bullet) := \mathrm{Jac}(Y_1) \times \mathrm{Jac}(Y_2) \times \cdots \times \mathrm{Jac}(Y_k), \qquad \widetilde{ev}_\bullet := (\widetilde{ev}_{\lambda_1}, \widetilde{ev}_{\lambda_2}, \ldots, \widetilde{ev}_{\lambda_k}),$$

and the direct image morphism

$$\psi_\bullet : \mathrm{Jac}(Y_\bullet) \longrightarrow \mathcal{M}_X(\mathrm{GL}(m)), \qquad (M_1, M_2, \ldots, M_k) \mapsto \bigoplus_{j=1}^k [\psi_{j*}(M_j)]^{\oplus n_j}.$$

Proposition 7.7. *We assume π étale. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n & \xrightarrow{\widetilde{ev}_\bullet} & \mathrm{Jac}(Y_\bullet) \\ \big\downarrow \Delta_\theta & & \big\downarrow \psi_\bullet \\ \mathcal{M}_X(G) & \xrightarrow{\tilde{\rho}_\lambda} & \mathcal{M}_X(\mathrm{GL}(m)) \end{array}$$

Proof. Let $E_T \in \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n$. The rank-$m$ vector bundle $\tilde{\rho}_\lambda[\Delta_\theta(E_T)]$ pulls back under π to the decomposable W-linearized vector bundle over Z

$$E_T \times^T V_\lambda = \bigoplus_{\mu \in \Theta(\lambda)} E_T \times^T V_\lambda^\mu = \bigoplus_{j=1}^k \left[\bigoplus_{\mu \in \Theta_j} E_T \times_\mu^T \mathbb{C} \right]^{\oplus n_j}.$$

Clearly the W-linearization preserves the direct summands $\bigoplus_{\mu \in \Theta_j} E_T \times_\mu^T \mathbb{C}$ for $j = 1, \ldots, k$ and by Lemma 6.12 we have the equality

$$\bigoplus_{\mu \in \Theta_j} E_T \times_\mu^T \mathbb{C} = \pi^*(\psi_{j*}(M_j)),$$

as W-linearized vector bundles. Here $M_j = \widetilde{ev}_j(E_T)$. This proves the claim. $\square$

Remark 7.8. Note that the composite map $\psi_\bullet \circ \widetilde{ev}_\bullet = \tilde{\rho}_\lambda \circ \Delta_\theta$ takes values in $\mathcal{M}_X(\mathrm{SL}(m))$.

8. Proof of the main theorem

For the convenience of the reader we recall the set-up of our main result. Let G be a simple and simply-connected algebraic group and $T \subset G$ a maximal torus. Let $W := N(T)/T$ denote the Weyl group of G and $\mathbb{S}_\omega := \mathrm{Hom}(T, \mathbb{C}^*)$ the weight lattice. Let $\pi : Z \to X$ be an étale Galois covering of smooth projective curves with Galois group W. For a dominant weight $\lambda \in \mathbb{S}_\omega$ we consider the Prym variety (see section 3.5)

$$P_\lambda \subset \mathrm{Jac}(Y)$$

with $Y = Z/H$ and $H = \mathrm{Stab}(\lambda)$. Let $\overline{K}_\lambda$ denote the Kanev correspondence on the curve Y. Let L_Y be a line bundle over $\mathrm{Jac}(Y)$ representing the principal polarization. Then we can prove our main result.

Theorem 8.1. *Assume that the following holds:*

- *the W-covering $\pi : Z \to X$ is étale,*
- $q_\lambda = d_\lambda$*,*
- *the group $\Gamma_\lambda = \mathbb{S}_\omega/\lambda \cdot \mathbb{Z}[W]$ is trivial,*
- *the weight λ is minuscule or quasi-minuscule,*
- *the homomorphism $\psi^* : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$ is injective.*

Then

(1) *there exists a line bundle M over P_λ such that*

$$L_Y|P_\lambda = M^{\otimes q_\lambda}.$$

(2) *the type of the polarization M over P_λ equals*

$$K(M) = (\mathbb{Z}/m\mathbb{Z})^{2g}, \qquad \text{with} \qquad m = \frac{\deg(Y/X)}{\gcd(\deg(\overline{K}_\lambda) - 1, \deg(Y/X))},$$

and g denotes the genus of X.

Proof. **(1)** If π is étale, Proposition 6.11 gives a morphism

$$\widetilde{ev}_\lambda : \mathrm{Prym}(\pi, \mathbb{S}_\omega) \longrightarrow \mathrm{Jac}(H; Z) = \mathrm{Jac}(Y).$$

Moreover by Proposition 6.13 we have $\widetilde{ev}_\lambda(\mathrm{Prym}(\pi, \mathbb{S}_\omega)_0) = P_\lambda$. Since we have assumed that Γ_λ is trivial, the evaluation map ev_λ is injective by Proposition 5.2. Hence the map $\widetilde{ev}_\lambda$ is also injective and induces isomorphisms by restriction

$$\widetilde{ev}_\lambda : \mathrm{Prym}(\pi, \mathbb{S}_\omega)_0 \xrightarrow{\sim} P_\lambda \subset \mathrm{Jac}(Y), \qquad \widetilde{ev}_\lambda : \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n \xrightarrow{\sim} T_\alpha(P_\lambda) \subset \mathrm{Jac}(Y),$$

where $T_\alpha(P_\lambda)$ denotes the translate by an element $\alpha \in \mathrm{Jac}(Y)$ of the Prym variety P_λ. In order to show (1) it suffices to show that $L_Y|T_\alpha(P_\lambda)$ is divisible by q_λ.

We first consider the case λ minuscule, i.e. $k = 1$ with the notation of section 7.4. In that case the commutative diagram of Proposition 7.7 simplifies

$$\begin{tikzcd}
\mathrm{Prym}(\pi, \mathbb{S}_\omega)_n \arrow[r, "\widetilde{ev}_\lambda"] \arrow[d, "\Delta_\theta"] & T_\alpha(P_\lambda) \arrow[d, "\psi_*"] \\
\mathcal{M}_X(G) \arrow[r, "\widetilde{\rho}_\lambda"] & \mathcal{M}_X(\mathrm{SL}(m)).
\end{tikzcd}$$

With the notation of section 7.2 we know that $\widetilde{\rho}_\lambda^* \mathcal{D} = \mathcal{L}^{\otimes d_\lambda}$. Hence the line bundle

$$\widetilde{ev}_\lambda^* \left((\psi_*)^* \mathcal{D}\right) = \Delta_\theta^*(\widetilde{\rho}_\lambda^* \mathcal{D}) = (\Delta_\theta^* \mathcal{L})^{\otimes d_\lambda}$$

is divisible by $d_\lambda = q_\lambda$. Since the first horizontal map $\widetilde{ev}_\lambda$ is an isomorphism, we deduce that $(\psi_*)^* \mathcal{D} = L_Y | T_\alpha(P_\lambda)$ is also divisible by q_λ.

In the case λ is quasi-minuscule, we have $k = 2$ and $\lambda_2 = 0$. Hence $H_2 = \mathrm{Stab}(0) = W$ and $\psi_2 = \mathrm{id} : Y_2 = X \to X$. Moreover $\widetilde{ev}_0 : \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n \to \mathrm{Jac}(X)$ is the constant zero map and the commutative diagram of Proposition 7.7 simplifies

$$\begin{array}{ccc} \mathrm{Prym}(\pi, \mathbb{S}_\omega)_n & \xrightarrow{\widetilde{ev}_\lambda} & T_\alpha(P_\lambda) \\ \big\downarrow \Delta_\theta & & \big\downarrow \psi_\bullet \\ \mathcal{M}_X(G) & \xrightarrow{\widetilde{\rho}_\lambda} & \mathcal{M}_X(\mathrm{SL}(m)), \end{array}$$

with $\psi_\bullet(M) = \psi_{1*}(M) \oplus \mathcal{O}_X^{\oplus n_2}$ for $M \in \mathrm{Jac}(Y)$. We then conclude as in the case of a minuscule weight.

(2) Let S denote the usual Prym variety $\mathrm{Prym}(Y/X)$. Then we recall from section 3.5 that $P_\lambda = \mathrm{im}(u_\lambda) \subset S$ and that the endomorphism $u_\lambda \in \mathrm{End}(S)$ satisfies the relation $u_\lambda^2 = q_\lambda u_\lambda$. Hence we can apply Proposition 2.10 (a), which says that the type of the polarization $K(M)$ is given by

$$K(M) = u_\lambda(K(L_Y|S)),$$

where $L_Y|S$ denotes the restriction of the line bundle L_Y to S. Since $\psi^* : \mathrm{Jac}(X) \longrightarrow \mathrm{Jac}(Y)$ is injective, we have — see Remark 2.1 (ii) — the equalities $S = \mathrm{Prym}(Y/X) = \mathrm{im}(d - t)$ and $\psi^*\mathrm{Jac}(X) = \mathrm{im}(t)$, where $t \in \mathrm{End}(\mathrm{Jac}(Y))$ denotes endomorphism associated to the trace correspondence of $\psi : Y \to X$. By [BL, Corollary 12.1.4] or Lemma 2.5 we obtain that

$$K(L_Y|S) = K(L_Y|\mathrm{Jac}(X)) = \mathrm{Jac}(X) \cap S.$$

Moreover $K(L_Y|\mathrm{Jac}(X)) = \psi^*\left[\mathrm{Jac}(X)_d\right] \cong (\mathbb{Z}/d\mathbb{Z})^{2g}$, with $d = \deg(Y/X)$. We deduce from Corollary 3.7(2) that $u_\lambda t = (\deg(\overline{K}_\lambda) - 1)t$ and since $K(L_Y|S) \subset \mathrm{im}(t) = \psi^*\mathrm{Jac}(X)$ we obtain that $u_\lambda(K(L_Y|S))$ equals the image of the multiplication by $\deg(\overline{K}_\lambda) - 1$ in the group $(\mathbb{Z}/d\mathbb{Z})^{2g}$. This proves the assertion (2). □

We deduce from Lemma 5.4 and from the two tables of section 4 the following list of examples satisfying the 5 conditions of Theorem 8.1. This shows the main theorem stated in the introduction.

Corollary 8.2. *The type $K(M) = (\mathbb{Z}/m\mathbb{Z})^{2g}$ of the induced polarization $L_{Y|P_\lambda} = M^{\otimes q_\lambda}$ on the Prym variety P_λ is given by the table.*

Weyl group of type	*weight*	$q_\lambda = d_\lambda$	m
A_n	ϖ_i; $(i, n+1) = 1$	$\binom{n-1}{i-1}$	$n+1$
D_n; *n odd*	ϖ_{n-1}, ϖ_n	2^{n-3}	4
E_6	ϖ_1, ϖ_6	6	3
E_7	ϖ_7	12	2
E_8	ϖ_8	60	1

Proof. We only have to check that $\psi^* : \mathrm{Jac}(X) \to \mathrm{Jac}(Y)$ is injective in the cases mentioned in the table. This follows from the formula $\ker \psi^* = \ker(W^* \to H^*)$, where W^* and H^* denote the groups of characters of W and H, and from a straightforward case-by-case study. □

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