REPRESENTATIONS OF BRAID GROUPS AND GENERALISATIONS

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Abstract. We define and study extensions of Artin's representation and braid monodromy representation to the case of topological and algebraical generalisations of braid groups. In particular we provide faithful representations of braid groups of oriented surfaces with boundary components as (outer) automorphisms of free groups. We give also similar representations for braid groups of non oriented surfaces with boundary components and we show a representation of braid groups of closed surfaces as outer automorphisms of free groups. Finally, we provide faithful representations of Artin-Tits groups of type $\mathcal{D}$ as automorphisms of free groups.

1. Introduction

Let F_n be the free group of rank n with the set of generators $\{x_1, x_2, \dots, x_n\}$. Assume further that $\mathrm{Aut}(F_n)$ is the automorphism group of F_n. The Artin braid group B_n can be represented as a subgroup of $\mathrm{Aut}(F_n)$. This representation, due to Artin himself, is defined associating to any generator σ_i, for $i = 1, 2, \dots, n-1$, of B_n the following automorphism of F_n:

$$\sigma_i : \begin{cases} x_i \longmapsto x_i\, x_{i+1}\, x_i^{-1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases}$$

Moreover (see for instance [10, Theorem 5.1]), any automorphism β of $\mathrm{Aut}(F_n)$ corresponds to an element of B_n if and only if β satisfies the following conditions:

$$\begin{aligned} &i) \quad \beta(x_i) = a_i^{-1}\, x_{s(i)}\, a_i,\ 1 \leq i \leq n, \\ &ii) \quad \beta(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n, \end{aligned}$$

where s is a permutation from the symmetric group S_n and $a_i \in F_n$.

Generalisations of Artin's representation have been provided by Wada [13] and further by Crisp and Paris [5], in order to construct group invariants of oriented links.

Another interesting representation of B_n is the *braid monodromy representation* of B_n into $\mathrm{Aut}(F_{n-1})$ (see last Section), that was proven to be faithful in [6] and [12].

In this paper, we extend Artin's and braid monodromy representations to some generalisations of braid groups.

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In the case of braid groups of oriented surfaces with boundary components our representations are faithful (Theorem 2). Moreover, in the case of surfaces of genus $g \geq 1$, the induced representations of surface braid groups as outer automorphisms hold faithful (Theorem 3).

In the case of closed surfaces, as earlier remarked in [4] for braid groups of the sphere, we cannot extend Artin's representation and we provide a representation in the outer automorphism group of a finitely generated free group.

In the last Section we consider the braid monodromy representation of B_n in $\mathrm{Aut}(F_{n-1})$ and we provide a faithful representation of the n-th Artin-Tits group of type $\mathcal{D}$ in $\mathrm{Aut}(F_n)$ (Proposition 10).

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2. Braid groups of orientable surfaces with boundary components

Surface braids as collections of paths. Let $\mathcal{P} = \{p_1, \dots, p_n\}$ be a set of n distinct points (*punctures*) in the interior of a surface Σ.

A *geometric braid* on Σ based at $\mathcal{P}$ is a collection $(\psi_1, \dots, \psi_n)$ of n disjoint paths (called *strands*) on $\Sigma \times [0,1]$ which run monotonically with $t \in [0,1]$ and such that $\psi_i(0) = (p_i, 0)$ and $\psi_i(1) \in \mathcal{P} \times \{1\}$. Two braids are considered to be equivalent if they are isotopic relatively to the base points. The usual product of paths defines a group structure on the equivalence classes of braids. This group, denoted usually by $B_n(\Sigma)$, does not depend on the choice of $\mathcal{P}$ and it is called braid group on n strands of Σ. The nth braid group of the disk D^2, $B_n(D^2)$, is isomorphic to B_n.

In the following we will denote by $B_n(\Sigma_{g,p})$ the braid group on n strands of an orientable surface of genus g with p boundary components (we set $\Sigma_g = \Sigma_{g,0}$) and by $B_n(N_{g,p})$ the braid group on n strands of a non-orientable surface of genus g with p boundary components (we set $N_g = N_{g,0}$).

In this section we will consider an orientable surface $\Sigma_{g,p}$ of genus $g \geq 0$ and with $p > 0$ boundary components. We set also $n \geq 2$.

We denote by $\sigma_1, ..., \sigma_{n-1}$ the standard generators of the braid group B_n. Since $p > 0$, we can embed a disk in $\Sigma_{g,p}$ and therefore we can consider $\sigma_1, ..., \sigma_{n-1}$ as elements of $B_n(\Sigma_{g,p})$. Let also $a_1, ..., a_g, b_1, ..., b_g, z_1, ..., z_{p-1}$ be the generators of $\pi_1(\Sigma_{g,p})$, where z_i's denote loops around the holes. Assume that the base point of the fundamental group is the startpoint of the first strand. Then each element $\gamma \in \pi_1(\Sigma_{g,p})$ determines an element denoted also by γ in $B_n(\Sigma_{g,p})$, by considering the braid whose first strand is describing the curve γ and other strands are constant.

Let us set $x_{2k-1} = a_k$ and $x_{2k} = b_k$ for $k = 1, \ldots, g$. According to [1] we have that the group $B_n(\Sigma_{g,p})$ admits a presentation with generators:

$$\sigma_1, \ldots, \sigma_{n-1}, x_1, \ldots, x_{2g}, z_1, \ldots, z_{p-1},$$

and defining relations:
– Braid relations:
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ 1 \le i \le n-2,$
$\sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1,\ 1 \le i, j \le n-1,$
– Mixed relations:
(R1) $x_r\sigma_i = \sigma_i x_r$, $i \ne 1$, $1 \le r \le 2g$,
(R2) $(\sigma_1^{-1}x_r\sigma_1^{-1})x_r = x_r(\sigma_1^{-1}x_r\sigma_1^{-1})$, $1 \le r \le 2g$,
(R3) $(\sigma_1^{-1}x_s\sigma_1)x_r = x_r(\sigma_1^{-1}x_s\sigma_1)$, $1 \le s < r \le 2g$, $(s, r) \ne (2m-1, 2m)$,
(R4) $(\sigma_1^{-1}x_{2m-1}\sigma_1^{-1})x_{2m} = x_{2m}(\sigma_1^{-1}x_{2m-1}\sigma_1)$, $1 \le m \le g$,
(R5) $z_j\sigma_i = \sigma_i z_j$, $i \ne 1$, $1 \le j \le p-1$,
(R6) $(\sigma_1^{-1}z_j\sigma_1)x_r = x_r(\sigma_1^{-1}z_j\sigma_1)$, $1 \le r \le 2g$, $1 \le j \ne p-1$,
(R7) $(\sigma_1^{-1}z_j\sigma_1)z_l = z_l(\sigma_1^{-1}z_j\sigma_1)$, $1 \le j < l \le p-1$,
(R8) $(\sigma_1^{-1}z_j\sigma_1^{-1})z_j = z_j(\sigma_1^{-1}z_j\sigma_1)$, $1 \le j \le p-1$.

Associating to any surface braid the corresponding permutation one obtains a surjective homomorphism

$$\pi : B_n(\Sigma_{g,p}) \longrightarrow S_n,$$

such that $\pi(\sigma_i) = (i, i+1)$, $i = 1, \ldots, n-1$, $\pi(x_r) = \pi(z_j) = e$ for $1 \le r \le 2g$ and $1 \le j \le p-1$.

In [1] the second author considered the subgroup $D_n(\Sigma_{g,p}) = \pi^{-1}(S_{n-1})$ and found its generators. We provide a set of defining relations of $D_n(\Sigma_{g,p})$ using the well-known Reidemeister-Schreier's method (see [8, Chap. 2]).

Let $M_n = \{m_l \mid 1 \le l \le n\}$ be the set defined as follows:

$$m_l = \sigma_{n-1} \ldots \sigma_l,\ l = 1, \ldots, n-1,\ m_n = 1.$$

It is easy to prove (see [1]) that $|B_n(\Sigma_{g,p}) : D_n(\Sigma_{g,p})| = n$ and that M_n is a Schreier set of coset representatives of $D_n(\Sigma_{g,p})$ in $B_n(\Sigma_{g,p})$. Define the map $^- : B_n(\Sigma_{g,p}) \longrightarrow M_n$ which takes an element $w \in B_n(\Sigma_{g,p})$ into the representative $\overline{w}$ from M_n. The element $w\overline{w}^{-1}$ belongs to $D_n(\Sigma_{g,p})$ and, by Theorem 2.7 from [8], the group $D_n(\Sigma_{g,p})$ is generated by

$$s_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},$$

where λ runs over the set M_n and a runs over the set of generators of $B_n(\Sigma_{g,p})$.

Case 1. If $a \in \{\sigma_1, \ldots, \sigma_{n-1}\}$, then we find the generators

$$\tau_k = \sigma_{n-1} \ldots \sigma_{k+1}\sigma_k^2\sigma_{k+1}^{-1} \ldots \sigma_{n-1}^{-1},\ k = 1, \ldots, n-2,\ \tau_{n-1} = \sigma_{n-1}^2.$$

Case 2. If $a \in \{x_1, \dots, x_{2g}\}$, then we find the generators

$$w_r = \sigma_{n-1} \dots \sigma_1 x_r \sigma_1^{-1} \dots \sigma_{n-1}^{-1},\ r = 1, \dots, 2g.$$

Case 3. If $a \in \{z_1, \dots, z_{p-1}\}$, then we find the generators

$$\xi_j = \sigma_{n-1} \dots \sigma_1 z_j \sigma_1^{-1} \dots \sigma_{n-1}^{-1},\ j = 1, \dots, p-1.$$

To find defining relations of $D_n(\Sigma_{g,p})$ we define a rewriting process τ. It allows us to rewrite a word which is written in the generators of $B_n(\Sigma_{g,p})$ and to present an element in $D_n(\Sigma_{g,p})$ as a word in the generators of $D_n(\Sigma_{g,p})$. Let us associate to the reduced word

$$u = a_1^{\varepsilon_1}\, a_2^{\varepsilon_2} \dots a_\nu^{\varepsilon_\nu},\ \varepsilon_l = \pm 1,$$

where

$$a_l \in \{\sigma_1, \sigma_2, \dots, \sigma_{n-1},\ x_1, x_2, \dots, x_{2g},\ z_1, z_2, \dots, z_{p-1}\},$$

the word

$$\tau(u) = s_{k_1,a_1}^{\varepsilon_1}\, s_{k_2,a_2}^{\varepsilon_2} \dots s_{k_\nu,a_\nu}^{\varepsilon_\nu}$$

in the generators of $D_n(\Sigma_{g,p})$, where k_j is a representative of the $(j-1)$th initial segment of the word u if $\varepsilon_j = 1$ and k_j is a representative of the jth initial segment of the word u if $\varepsilon_j = -1$. By [8, Theorem 2.9], the group $D_n(\Sigma_{g,p})$ is defined by relations

$$r_{\mu,\lambda} = \tau(\lambda\, r_\mu\, \lambda^{-1}),\ \lambda \in M_n,$$

where r_μ is a defining relation of $B_n(\Sigma_{g,p})$.

Proposition 1. *The group $D_n(\Sigma_{g,p})$ admits a presentation with the generators*

$\sigma_1, \dots, \sigma_{n-2}, x_1, \dots, x_{2g}, z_1, \dots, z_{p-1}, \tau_1, \dots, \tau_{n-1},\ w_1, \dots, w_{2g},\ \xi_1, \dots,\ \xi_{p-1};$
and relations:

– *Braid relations*
(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ 1 \le i \le n-3,$
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i,\ |i-j| > 1,\ 1 \le i, j \le n-2,$
(B3) $\sigma_k^{-1} \tau_l \sigma_k = \tau_l,\ k \ne l-1, l,$
(B4) $\sigma_{l-1}^{-1} \tau_l \sigma_{l-1} = \tau_{l-1},$
(B5) $\sigma_l^{-1} \tau_l \sigma_l = \tau_l \tau_{l+1} \tau_l^{-1},\ l \ne n-1.$
– *Mixed relations*
(R1.1) $x_r \sigma_i = \sigma_i x_r$, $2 \le i \le n-2$, $1 \le r \le 2g$,
(R1.2) $x_r \tau_i = \tau_i x_r$, $2 \le i \le n-1$,
(R1.3) $w_r \sigma_i = \sigma_i w_r$, $1 \le r \le 2g$, $\le i \le n-2$,
(R2.1) $(\sigma_1^{-1} x_r \sigma_1^{-1}) x_r = x_r (\sigma_1^{-1} x_r \sigma_1^{-1})$, $1 \le r \le 2g$,
(R2.2) $x_r^{-1} w_r x_r = \tau_1^{-1} w_r \tau_1$, $1 \le r \le 2g$,

(R2.3) $x_r^{-1}\tau_1 x_r = \tau_1^{-1}w_r\tau_1 w_r^{-1}\tau_1$, $1 \le r \le 2g$,
(R3.1) $(\sigma_1^{-1}x_s\sigma_1)x_r = x_r(\sigma_1^{-1}x_s\sigma_1)$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(R3.2) $x_r^{-1}(\tau_1^{-1}w_s\tau_1)x_r = \tau_1^{-1}w_s\tau_1$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(R3.3) $x_s w_r = w_r x_s$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(R4.1) $(\sigma_1^{-1}x_{2m-1}\sigma_1^{-1})x_{2m} = x_{2m}(\sigma_1^{-1}x_{2m-1}\sigma_1)$, $1 \le m \le g$,
(R4.2) $x_{2m}^{-1}(\tau_1^{-1}w_{2m-1})x_{2m} = \tau_1^{-1}w_{2m-1}\tau_1$, $1 \le m \le g$,
(R4.3) $x_{2m-1}^{-1}w_{2m}x_{2m-1} = \tau_1^{-1}w_{2m}$, $1 \le m \le g$,
(R5.1) $z_j\sigma_i = \sigma_i z_j$, $2 \le i \le n-2$, $1 \le j \le p-1$,
(R5.2) $z_j\tau_i = \tau_i z_j$, $2 \le i \le n-1$, $1 \le j \le p-1$,
(R5.3) $\xi_j\sigma_i = \sigma_i\xi_j$, $1 \le j \le p-1$, $1 \le i \le n-2$,
(R6.1) $(\sigma_1^{-1}z_j\sigma_1)x_r = x_r(\sigma_1^{-1}z_j\sigma_1)$, $1 \le r \le 2g$, $1 \le j \le p-1$,
(R6.2) $x_r^{-1}(\tau_1^{-1}\xi_j\tau_1)x_r = \tau_1^{-1}\xi_j\tau_1$, $1 \le r \le 2g$, $1 \le j \le p-1$,
(R6.3) $z_j w_r = w_r z_j$, $1 \le r \le 2g$, $1 \le j \le p-1$,
(R7.1) $(\sigma_1^{-1}z_j\sigma_1)z_l = z_l(\sigma_1^{-1}z_j\sigma_1)$, $1 \le j < l \le p-1$,
(R7.2) $z_l^{-1}(\tau_1^{-1}\xi_j\tau_1)z_l = \tau_1^{-1}\xi_j\tau_1$, $1 \le j < l \le p-1$,
(R7.3) $z_j\xi_l = \xi_l z_j$, $1 \le j < l \le p-1$,
(R8.1) $(\sigma_1^{-1}z_j\sigma_1^{-1})z_j = z_j(\sigma_1^{-1}z_j\sigma_1^{-1})$, $1 \le j \le p-1$,
(R8.2) $z_j^{-1}(\tau_1^{-1}\xi_j)z_j = \tau_1^{-1}\xi_j$, $1 \le j \le p-1$,
(R8.3) $z_j^{-1}\xi_j z_j = \tau_1^{-1}\xi_j\tau_1$, $1 \le j \le p-1$.

The generators $\sigma_1, \ldots, \sigma_{n-2}, x_1, \ldots, x_{2g}, z_1, \ldots, z_{p-1}$ generate a group isomorphic to $B_{n-1}(\Sigma_{g,p})$ (see also Remark 3.1 from [1]) and it is easy to see that the relations (B1), (B2), (R1.1), (R2.1), ..., (R8.1) are a complet set of relations for $B_{n-1}(\Sigma_{g,p})$. From the other relations we can find the following conjugacy formulae:

(S1) $\tau_l^{\sigma_k} = \tau_l$, $k \ne l-1, l$,
(S2) $\tau_l^{\sigma_{l-1}} = \tau_{l-1}$,
(S3) $\tau_l^{\sigma_l} = \tau_{l+1}^{\tau_l^{-1}}$, $l \ne n-1$,
(S4) $\tau_i^{x_r} = \tau_i$, $2 \le i \le n-1$,
(S5) $w_r^{\sigma_i} = w_r$, $1 \le r \le 2g$, $1 \le i \le n-2$,
(S6) $w_r^{x_r} = w_r^{\tau_1}$,
(S7) $\tau_1^{x_r} = \tau_1^{w_r^{-1}\tau_1}$,
(S8) $w_s^{x_r} = w_s^{[w_r^{-1},\tau_1]}$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(S9) $w_r^{x_s} = w_r$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(S10) $w_{2m-1}^{x_{2m}} = [\tau_1, w_{2m}^{-1}]w_{2m-1}\tau_1$,
(S11) $w_{2m}^{x_{2m-1}} = \tau_1^{-1}w_{2m}$,
(S12) $\tau_i^{z_j} = \tau_i$, $2 \le i \le n-1$, $1 \le j \le p-1$,
(S13) $\xi_j^{\sigma_i} = \xi_j$, $1 \le i \le n-2$, $1 \le j \le p-1$,
(S14) $\xi_j^{x_r} = \xi_j^{[w_r^{-1},\tau_1]}$, $1 \le r \le 2g$, $1 \le j \le p-1$,
(S15) $w_r^{z_j} = w_r$, $1 \le r \le 2g$, $1 \le j \le p-1$,

(S16) $\xi_j^{z_l} = \xi_j^{[\xi_l^{-1}, \tau_1]}$, $1 \leq j < l \leq p-1$,
(S17) $\xi_l^{z_j} = \xi_l$, $1 \leq j < l \leq p-1$,
(S18) $\tau_1^{z_j} = [\tau_1, \xi_j^{-1}]\tau_1$, $1 \leq j \leq p-1$,
(S19) $\xi_j^{z_j} = \xi_j^{\tau_1}$, $1 \leq j \leq p-1$,
where $a^b = a^{-1}ba$ and $[a,b] = a^{-1}b^{-1}ab$.

Let $U_{n-1,g,p}$ be the subgroup of $D_n(\Sigma_{g,p})$ generated by $\{\tau_1, \dots \tau_{n-1}, w_1, \dots, w_{2g}, \xi_1, \dots, \xi_{p-1}\}$.

Proposition 2. *The group $U_{n-1,g,p}$ is a normal subgroup of $D_n(\Sigma_{g,p})$ and it is a free group of rank $n+p+2g-2$.*

Proof. The statement was proven in [1, Section.2] using the interpretation of $U_{n-1,g,p}$ as the fundamental group of the surface $\Sigma_{g,p}$ with $n-1$ points removed. □

From relations (S1)...(S19) we deduce that $B_{n-1}(\Sigma_{g,p})$ acts on $U_{n-1,g,p}$ by conjugacy.

Theorem 1. *The group $B_{n-1}(\Sigma_{g,p})$ with $n \geq 3$, $g \geq 0$ and $p > 0$ acts by conjugacy on the free group $U_{n-1,g,p}$. Therefore we have a representation $\rho_U : B_{n-1}(\Sigma_{g,p}) \to Aut(U_{n-1,g,p})$ defined algebraically as follows:*
– Generators σ_i, $i = 1, \dots, n-2$:

$$\sigma_i : \begin{cases} \tau_i \longmapsto \tau_{i+1}^{\tau_i^{-1}} \, ; \\ \tau_{i+1} \longmapsto \tau_i \, ; \\ \tau_l \longmapsto \tau_l, \ l \neq i, i+1, \, ; \\ w_r \longmapsto w_r, \ 1 \leq r \leq 2g \, ; \\ \xi_j \longmapsto \xi_j, \ 1 \leq j \leq p-1. \end{cases}$$

– Generators x_r, $r = 1, \dots, 2g$:

$$x_r : \begin{cases} \tau_1 \longmapsto \tau_1^{w_r^{-1}\tau_1} \, ; \\ \tau_i \longmapsto \tau_i, \ 2 \leq i \leq n \, ; \\ w_s \longmapsto w_s^{[w_r^{-1}, \tau_1]}, \ s < r, \ (s,r) \neq (2m-1, 2m) \, ; \\ w_{r-1} \longmapsto [\tau_1, w_r^{-1}] w_{r-1} \tau_1, \ \textit{if}\, r = 2m \, ; \\ w_r \longmapsto w_r^{\tau_1} \, ; \\ w_s \longmapsto w_s, \ r < s, \ (r,s) \neq (2m-1, 2m) \, ; \\ w_{r+1} \longmapsto \tau_1^{-1} w_{r+1}, \ \textit{if}\, r = 2m-1 \, ; \\ \xi_j \longmapsto \xi_j^{[w_r^{-1}, \tau_1]}, \ 1 \leq j \leq p-1 \, . \end{cases}$$

– Generators z_j, $j = 1, \dots, p-1$:

$$z_j : \begin{cases} \tau_1 \longmapsto \tau_1^{\xi_j^{-1}\tau_1} \, ; \\ \tau_i \longmapsto \tau_i, \ 2 \leq i \leq n \, ; \\ w_r^{z_j} \longmapsto w_r, \ 1 \leq r \leq 2g) \, ; \\ \xi_l^{z_j} \longmapsto \xi_l^{[\xi_j^{-1}, \tau_1]}, \ 1 \leq l < j \leq p-1 \, ; \\ \xi_j^{z_j} \longmapsto \xi_j^{\tau_1} \, ; \\ \xi_l^{z_j} \longmapsto \xi_l, \ 1 \leq j < l \leq p-1 \, . \end{cases}$$

In the following we outline a proof of the faithfulness of the representation ρ_U of $B_{n-1}(\Sigma_{g,p})$ given in Theorem 1 using the interpretation of surface braids as mapping classes.

First, we recall that the mapping class group of a surface $\Sigma_{g,p}$, let us denote it by $\mathcal{M}_{g,p}$, is the group of isotopy classes of orientation-preserving self-homeomorphisms which fix the boundary components pointwise.

Let $\mathcal{P} = \{p_1, \dots, p_n\}$ be a set of n distinct points (*punctures*) in the interior of the surface $\Sigma_{g,p}$. The punctured mapping class group of $\Sigma_{g,p}$ relative to $\mathcal{P}$ is defined to be the group of isotopy classes of orientation-preserving self-homeomorphisms which fix the boundary components pointwise, and which fix $\mathcal{P}$ setwise. This group, that we will denote by $\mathcal{M}_{g,p}^n$, does not depend on the choice of $\mathcal{P}$, but just on its cardinal.

We recall also that a simple closed curve C is *essential* if either it does not bound a disk or it bounds a disk containing at least two punctures.

Finally, we denote T_C as a Dehn twist along a simple closed curve C. Let C and D be two simple closed curves bounding an annulus containing only the puncture p_j. We shall say that the multitwist $T_C T_D^{-1}$ is a *j-bounding pair braid*, also called *spin map* in [4].

Surface braids as mapping classes. Let $g, p \geq 0$ and let $\psi_{n,0} : \mathcal{M}_{g,p}^n \to \mathcal{M}_{g,p}$ be the homomorphism induced by the map which forgets the set $\mathcal{P}$. When $p = 0$, according to a well-known result of Birman [4, Chapter 4.1], the group $B_n(\Sigma_g)$ is isomorphic to $\ker \psi_{n,0}$ if $g > 1$. The statement of Birman's theorem concerns the case of closed surfaces, but the proof extends naturally to the case of surfaces with boundary components and the group $B_n(\Sigma_{g,p})$ is isomorphic to $\ker \psi_{n,0}$ if $g \geq 1$ and $p > 0$.

Geometrically the correspondence between $\ker \psi_{n,0}$ and $B_n(\Sigma_{g,p})$ is realized as follows: given a homeomorphism h of $\Sigma_{g,p}$ isotopic to the identity Id of $\Sigma_{g,p}$ and fixing boundary components pointwise and $\mathcal{P}$ setwise, the track of the punctures $p_1, \dots, p_n$ under an isotopy from h to Id is the geometric braid corresponding to the homeomorphism h.

Theorem 2. *Let $g \geq 0$, $p > 0$ and $n \geq 3$. The representation $\rho_U : B_{n-1}(\Sigma_{g,p}) \to \mathrm{Aut}(U_{n-1,g,p})$ is faithful.*

Proof. First we prove the claim when g is greater or equal then 1. According to Birman's result we can represent the group $D_n(\Sigma_{g,p})$ as a (normal) subgroup of $\mathcal{M}_{g,p}^n$, more precisely as the subgroup of mapping classes in $\ker \psi_{n,0}$ sending the nth puncture into itself (see also Remark 18 from [3]). In particular generators $\tau_1, \dots, \tau_{n-1}, \ w_1, \dots, w_{2g}, \ \xi_1, \dots, \xi_{p-1}$, of $D_n(\Sigma_{g,p})$ (see Proposition 1) correspond to n-bounding pair braids in $\mathcal{M}_{g,p}^n$.

Let $\psi_{n,n-1} : \mathcal{M}_{g,p}^n \to \mathcal{M}_{g,p}^{n-1}$ be the homomorphism forgetting the last puncture. Now, let us consider the exact sequence associate to the restriction of $\psi_{n,n-1}$ to $D_n(\Sigma_{g,p})$. The image of $D_n(\Sigma_{g,p})$ by $\psi_{n,n-1}$ coincides with $\ker \psi_{n-1,0}$ which is isomorphic to $B_{n-1}(\Sigma_{g,p})$. On the other hand, $\ker \psi_{n,n-1} \cap D_n(\Sigma_{g,p})$ is the subgroup of $\mathcal{M}_{g,p}^n$ generated by n-bounding pair braids $\tau_1, \dots, \tau_{n-1}, \ w_1, \dots, w_{2g}, \ \xi_1, \dots, \xi_{p-1}$ and therefore it is isomorphic to $U_{n-1,g,p}$.

The group $B_{n-1}(\Sigma_{g,p})$ embeds naturally in $B_n(\Sigma_{g,p})$ by sending generators of $B_{n-1}(\Sigma_{g,p})$ in corresponding ones of $B_n(\Sigma_{g,p})$ and therefore $B_{n-1}(\Sigma_{g,p})$ can be considered also as a subgroup of $\mathcal{M}_{g,p}^n$. Moreover, the action considered in Theorem 1 corresponds to the action by conjugacy of $B_{n-1}(\Sigma_{g,p})$, seen as a subgroup of $\mathcal{M}_{g,p}^n$, on $U_{n-1,g,p}$. Since $U_{n-1,g,p}$ is a normal subgroup of $\mathcal{M}^n_{g,p}$ we can define a map $\Theta : \mathrm{Inn}(\mathcal{M}^n_{g,p}) \to \mathrm{Aut}(U_{n-1,g,p})$. We prove that Θ is injective and therefore that the action by conjugacy of $B_{n-1}(\Sigma_{g,p})$ on $U_{n-1,g,p}$ is faithful. Let $g \in \mathrm{Inn}(\mathcal{M}^n_{g,p})$ such that $\Theta(g) = 1$ and C be an essential curve. We can associate to the curve C another simple closed curve D such that they bound an annulus containing only the puncture p_n. The mapping class $T_C T_D^{-1}$ is then a n-bounding pair braid.

Since $\Theta(g) = 1$, then $g\, T_C T_D^{-1} g^{-1} = T_C T_D^{-1}$ and from a simple argument on the index of intersection of curves (see for instance Proposition 2.10 of [2]) one can easily deduce that $g(C) = C$. Since $g(C) = C$ for any essential curve C it follows that g is isotopic to the identity (Lemma 5.1 and Theorem 5.3 of [7]). Therefore Θ is injective and in particular the representation defined in Theorem 1 is faithful when $g \geq 1$.

The only case left is when the genus is equal to zero. We recall that, for $p > 0$, the group $B_n(\Sigma_{0,p})$ is isomorphic to the subgroup $B_{n+p-1,p-1}$ of B_{n+p-1} fixing the last $p-1$ strands. In this case our representation coincides with Artin representation of B_{n+p-1} in $\mathrm{Aut}(F_{n+p-1})$ restricted to the subgroup $B_{n+p-1,p-1}$. □

Corollary 1. *The group $B_m(\Sigma_{g,p})$ is residually finite for $m \geq 1$, $g \geq 0$ and $p > 0$.*

Proof. Baumslag and Smirnov proved (see for instance [9, Theorem 4.8]) that any finitely presented group which is isomorphic to a subgroup of automorphisms of a free group of finite rank is residually finite. Therefore for $m > 1$ the statement is a Corollary of Theorem 2. In the case $m = 1$, the group $B_1(\Sigma_{g,p})$ is isomorphic to the fundamental group of $\Sigma_{g,p}$ which is free and therefore residually finite. □

Remark also that from Theorem 2 one can derive, using Fox derivatives, a Burau representation for $B_n(\Sigma_{g,p})$. Since the restriction on B_n of such representation coincides with the usual Burau representation of B_n, the Burau representation for $B_n(\Sigma_{g,p})$ obtained from Theorem 2 is not faithful for $n \geq 5$.

3. Surface braids as outer automorphisms of free groups

As recalled in the Introduction, any element β of $B_n \subset \mathrm{Aut}(F_n)$ fixes the product $x_1 x_2 \ldots x_n$ of generators of F_n. We prove a similar statement for the group $B_n(\Sigma_{g,p})$, for $p > 0$.

Proposition 3. *Let $p > 0$ and let $U_{n-1,g,p}$ be the free group of rank $n + p + 2g - 2$ defined above. Any element β in $B_{n-1}(\Sigma_{g,p}) \subset \mathrm{Aut}(U_{n-1,g,p})$ fixes the product*

$$A = \tau_{n-1}^{-1} \ldots \tau_2^{-1} \tau_1^{-1} \xi_1 \xi_2 \ldots \xi_{p-1} [w_1^{-1}, w_2] \ldots [w_{2g-1}^{-1}, w_{2g}].$$

Proof. In order to prove the claim it suffices to verify that any generator of $B_{n-1}(\Sigma_{g,p})$, considered as an automorphism of $U_{n-1,g,p}$, fixes the element A.

Case 1: generators σ_i, $1 \leq i \leq n-2$. The group B_{n-1} is a subgroup of $B_{n-1}(\Sigma_{g,p})$ and by Artin's theorem B_{n-1} is a subgroup of $\mathrm{Aut}(F_{n-1})$, $F_{n-1} = \langle \tau_1, \tau_2, \ldots, \tau_{n-1} \rangle$, and fixes the product $\tau_1 \tau_2 \ldots \tau_{n-1}$. Hence, the generator σ_i also fixes the product $\tau_{n-1}^{-1} \ldots \tau_2^{-1} \tau_1^{-1}$.

From Theorem 1 it follows that

$$w_r^{\sigma_i} = w_r, 1 \leq r \leq 2g;\ \xi_j^{\sigma_i} = \xi_j, 1 \leq j \leq p-1.$$

Hence, $A^{\sigma_i} = A$.

Case 2: generators x_r, $1 \leq r \leq 2g$. By Theorem 1 we have

$$(\tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1})^{x_r} = \tau_{n-1}^{-1} \dots \tau_2^{-1} (\tau_1^{-1} w_r \tau_1^{-1} w_r^{-1} \tau_1)$$

and

$$(\xi_1 \dots \xi_{p-1})^{x_r} = (\xi_1 \dots \xi_{p-1})^{[w_r^{-1}, \tau_1]}.$$

Let us consider

$$([w_1^{-1}, w_2] \dots [w_{2g-1}^{-1}, w_{2g}])^{x_r}.$$

We will distinguish two cases: $r = 2m-1$ and $r = 2m$, where $m = 1, \dots g$. In the first case we have the following equalities:

$$([w_1^{-1}, w_2] \dots [w_{2m-3}^{-1}, w_{2m-2}])^{x_{2m-1}} = ([w_1^{-1}, w_2] \dots [w_{2m-3}^{-1}, w_{2m-2}])^{[w_{2m-1}^{-1}, \tau_1]},$$

$$[w_{2m-1}^{-1}, w_{2m}]^{x_{2m-1}} = [w_{2m-1}^{-\tau_1}, \tau_1^{-1} w_{2m}] = [\tau_1, w_{2m-1}^{-1}][w_{2m-1}^{-1}, w_{2m}],$$

$$([w_{2m+1}^{-1}, w_{2m+2}] \dots [w_{2g-1}^{-1}, w_{2g}])^{x_{2m-1}} = [w_{2m+1}^{-1}, w_{2m+2}] \dots [w_{2g-1}^{-1}, w_{2g}].$$

In the second case we have the following equalities:

$$([w_1^{-1}, w_2] \dots [w_{2m-3}^{-1}, w_{2m-2}])^{x_{2m}} = ([w_1^{-1}, w_2] \dots [w_{2m-3}^{-1}, w_{2m-2}])^{[w_{2m}^{-1}, \tau_1]},$$

$$[w_{2m-1}^{-1}, w_{2m}]^{x_{2m}} = [\tau_1^{-1} w_{2m-1}^{-1} [w_{2m}^{-1}, \tau_1], w_{2m}^{\tau_1}] = [\tau_1, w_{2m}^{-1}][w_{2m-1}^{-1}, w_{2m}],$$

$$([w_{2m+1}^{-1}, w_{2m+2}] \dots [w_{2g-1}^{-1}, w_{2g}])^{x_{2m}} = [w_{2m+1}^{-1}, w_{2m+2}] \dots [w_{2g-1}^{-1}, w_{2g}].$$

In both cases we have

$$([w_1^{-1}, w_2] \dots [w_{2g-1}^{-1}, w_{2g}])^{x_r} = [\tau_1, w_r^{-1}][w_1^{-1}, w_2] \dots [w_{2g-1}^{-1}, w_{2g}].$$

Hence, the element x_r acts by conjugation on A as follows

$$A^{x_r} = \tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1} [w_r^{-1}, \tau_1] (\xi_1 \dots \xi_{p-1})^{[w_r^{-1}, \tau_1]} [\tau_1, w_r^{-1}][w_1^{-1}, w_2] \dots$$
$[w_{2g-1}^{-1}, w_{2g}]$,

and then it is easy to check that $A^{x_r} = A$.

Case 3: generators z_j, $1 \leq j \leq p-1$. By Theorem 1 we have

$$(\tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1})^{z_j} = \tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1} [\xi_j^{-1}, \tau_1],$$

$$(\xi_1 \dots \xi_{p-1})^{z_j} = (\xi_1 \dots \xi_{j-1})^{[\xi_j^{-1}, \tau_1]} \xi_j^{\tau_1} \xi_{j+1} \dots \xi_{p-1} = [\xi_j^{-1}, \tau_1]^{-1} \xi_1 \dots \xi_{p-1},$$

and

$$([w_1^{-1}, w_2] \dots [w_{2g-1}^{-1}, w_{2g}])^{z_j} = [w_1^{-1}, w_2] \dots [w_{2g-1}^{-1}, w_{2g}].$$

Hence, $A^{z_j} = A$ and the Proposition follows. □

We recall that $\rho_U : B_{n-1}(\Sigma_{g,p}) \to \mathrm{Aut}(U_{n-1,g,p})$ is the representation of $B_{n-1}(\Sigma_{g,p})$ defined in Theorem 1 and let $p : \mathrm{Aut}(U_{n-1,g,p}) \to \mathrm{Out}(U_{n-1,g,p})$ be the canonical projection.

Theorem 3. *The representation $p \circ \rho_U : B_{n-1}(\Sigma_{g,p}) \to \mathrm{Out}(U_{n-1,g,p})$ is faithful for $p > 0$ when $g > 0$ and for $p > 2$ when $g = 0$.*

Proof. Since the representation $\rho_U : B_{n-1}(\Sigma_{g,p}) \to \mathrm{Aut}(U_{n-1,g,p})$ is faithful (Theorem 2), we can identify $B_{n-1}(\Sigma_{g,p})$ with $\rho_U(B_{n-1}(\Sigma_{g,p}))$.

Now suppose that there exists $\beta \in B_{n-1}(\Sigma_{g,p}) \cap \mathrm{Inn}(U_{n-1,g,p})$. From Proposition 3 one deduces that β is a conjugation by a power m of

$$A = \tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1} \xi_1 \xi_2 \dots \xi_{p-1} [w_1^{-1}, w_2] \dots [w_{2g-1}^{-1}, w_{2g}].$$

Now, let g be a generator of $B_{n-1}(\Sigma_{g,p})$. Since all elements of $B_{n-1}(\Sigma_{g,p})$ fix the element A we deduce the following equalities:

$$g^{-1}(\beta(g(x))) = g^{-1}(A^m g(x) A^{-m}) = A^m x A^{-m} = \beta(x),$$

for any $x \in U_{n-1,g,p}$. One deduces that $g^{-1}\beta g = \beta$ for any generator g of $B_{n-1}(\Sigma_{g,p})$ and therefore β belongs to the center of $B_{n-1}(\Sigma_{g,p})$. Since $B_{n-1}(\Sigma_{g,p})$ (with $n \geq 2$) has trivial center for $p > 0$ when $g > 0$ [11] and for $p > 2$ when $g = 0$, the intersection $B_{n-1}(\Sigma_{g,p}) \cap \mathrm{Inn}(U_{n-1,g,p})$ is trivial and the claim follows. □

Remark 1. *Let $\phi : B_n \to \mathrm{Out}(F_n)$ the representation obtained composing Artin representation of B_n in $\mathrm{Aut}(F_n)$ with the canonical projection of $\mathrm{Aut}(F_n)$ in $\mathrm{Out}(F_n)$. Such representation is not faithful and it is easy to see that the kernel is the center of B_n.*

4. Surface braid groups of non-orientable surfaces with boundary components

Let $N_{g,p}$ be a non-orientable surface of genus $g \geq 1$, with $p > 0$ boundary components. Let $\sigma_1, \dots, \sigma_{n-1}$ be the usual generators of B_n and $a_1, \dots a_g, z_1, \dots z_{p-1}$ be the usual generators of the fundamental group of $N_{g,p}$. As in previous section we can consider $\sigma_1, \dots, \sigma_{n-1}$ and $a_1, \dots a_g, z_1, \dots z_{p-1}$ as elements of $B_n(N_{g,p})$. According to [1] the group $B_n(N_{g,p})$ admits a presentation with generators:

$$\sigma_1, \dots \sigma_{n-1}, a_1, \dots a_g, z_1, \dots z_{p-1},$$

and relations:
– Braid relations:
$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ 1 \leq i \leq n-2,$
$\sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| > 1, \ 1 \leq i, j \leq n-1,$
– Mixed relations:

(R1) $a_r\sigma_i = \sigma_i a_r$, $i \neq 1$, $1 \leq r \leq g$,
(R2) $\sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1$, $1 \leq r \leq g$,
(R3) $(\sigma_1^{-1} a_s \sigma_1) a_r = a_r (\sigma_1^{-1} a_s \sigma_1)$, $1 \leq s < r \leq g$,
(R4) $z_j \sigma_i = \sigma_i z_j$, $i \neq 1$, $1 \leq j \leq p-1$,
(R5) $(\sigma_1^{-1} z_i \sigma_1) a_r = a_r (\sigma_1^{-1} z_i \sigma_1)$, $1 \leq r \leq g$, $1 \leq i \leq p-1$, $n > 1$,
(R6) $(\sigma_1^{-1} z_j \sigma_1) z_l = z_l (\sigma_1^{-1} z_j \sigma_1)$, $1 \leq j < l \leq p-1$,
(R7) $(\sigma_1^{-1} z_j \sigma_1^{-1}) z_j = z_j (\sigma_1^{-1} z_j \sigma_1^{-1})$, $1 \leq j \leq p-1$.

As in previous section let us consider the natural projection of $\pi : B_n(N_{g,p}) \longrightarrow S_n$, which associates to any braid the corresponding permutation. This projection map σ_i in the corresponding transposition and generators $a_1, \ldots, a_g, z_1, \ldots, z_{p-1}$ into the identity. As before, let $D_n(N_{g,p}) = \pi^{-1}(S_{n-1})$ and let $m_l = \sigma_{n-1} \ldots \sigma_l$, $l = 1, \ldots, n-1$, $m_n = 1$. The set $M_n = \{m_l \mid 1 \leq l \leq n\}$ is a Schreier set of coset representatives of $D_n(N_{g,p})$ in $B_n(N_{g,p})$ and the group $D_n(N_{g,p})$ is generated by

$$s_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},$$

where λ runs over the set M_n and a runs over the set of generators of $B_n(N_{g,p})$.

Case 1. If $a \in \{\sigma_1, \ldots, \sigma_{n-1}\}$, then we find the generators

$$\tau_k = \sigma_{n-1} \ldots \sigma_{k+1} \sigma_k^2 \sigma_{k+1}^{-1} \ldots \sigma_{n-1}^{-1}, \ k = 1, \ldots, n-2, \ \tau_{n-1} = \sigma_{n-1}^2.$$

Case 2. If $a \in \{a_1, \ldots, a_g\}$, then we find the generators

$$w_r = \sigma_{n-1} \ldots \sigma_1 a_r \sigma_1^{-1} \ldots \sigma_{n-1}^{-1}, \ r = 1, \ldots, g.$$

Case 3. If $a \in \{z_1, \ldots, z_{p-1}\}$, then we find the generators

$$\xi_j = \sigma_{n-1} \ldots \sigma_1 z_j \sigma_1^{-1} \ldots \sigma_{n-1}^{-1}, \ j = 1, \ldots, p-1.$$

Using the same argument as in the orientable case one can find the following group presentation for $D_n(N_{g,p})$.

Proposition 4. *The group $D_n(N_{g,p})$, $n \geq 1$ admits a presentation with the generators*

$$\sigma_1, \ldots, \sigma_{n-2}, a_1, \ldots, a_g, z_1, \ldots, z_{p-1}, \tau_1, \ldots, \tau_{n-1}, \ w_1, \ldots, w_g, \ \xi_1, \ldots, \xi_{p-1},$$

and the following relations:
– Braid relations:

(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, $1 \leq i \leq n-3$,
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i-j| > 1$, $1 \leq i, j \leq n-2$,
(B3) $\sigma_k^{-1} \tau_l \sigma_k = \tau_l$, $k \neq l-1, l$,

(B4) $\sigma_{l-1}^{-1}\tau_l\sigma_{l-1} = \tau_{l-1}$,
(B5) $\sigma_l^{-1}\tau_l\sigma_l = \tau_l\tau_{l+1}\tau_l^{-1}, \ l \neq n-1$.
– *Mixed relations:*

(R1.1) $a_r\sigma_i = \sigma_i a_r$, $2 \leq i \leq n-2$, $1 \leq r \leq g$,
(R1.2) $a_r\tau_i = \tau_i a_r$, $2 \leq i \leq n-1$,
(R1.3) $w_r\sigma_i = \sigma_i w_r$, $1 \leq r \leq g$, $1 \leq i \leq n-2$,
(R2.1) $(\sigma_1^{-1}a_r\sigma_1^{-1})a_r = a_r(\sigma_1^{-1}a_r\sigma_1)$, $1 \leq r \leq g$,
(R2.2) $a_r^{-1}(\tau_1^{-1}w_r)a_r = \tau_1^{-1}w_r\tau_1$, $1 \leq r \leq g$,
(R2.3) $a_r^{-1}w_r a_r = \tau_1^{-1}w_r$, $1 \leq r \leq g$,
(R3.1) $(\sigma_1^{-1}a_s\sigma_1)a_r = a_r(\sigma_1^{-1}a_s\sigma_1)$, $1 \leq s < r \leq g$,
(R3.2) $a_r^{-1}(\tau_1^{-1}w_s\tau_1)a_r = \tau_1^{-1}w_s\tau_1$, $1 \leq s < r \leq g$,
(R3.3) $a_s w_r = w_r a_s$, $1 \leq s < r \leq g$,
(R4.1) $z_j\sigma_i = \sigma_i z_j$, $2 \leq i \leq n-2$, $1 \leq j \leq p-1$,
(R4.2) $z_j\tau_i = \tau_i z_j$, $2 \leq i \leq n-1$, $1 \leq j \leq p-1$,
(R4.3) $\xi_j\sigma_i = \sigma_i\xi_j$, $1 \leq j \leq p-1$, $1 \leq i \leq n-2$,
(R5.1) $(\sigma_1^{-1}z_j\sigma_1)a_r = a_r(\sigma_1^{-1}z_j\sigma_1)$, $1 \leq r \leq g$, $1 \leq j \leq p-1$,
(R5.2) $a_r^{-1}(\tau_1^{-1}\xi_j\tau_1)a_r = \tau_1^{-1}\xi_j\tau_1$, $1 \leq r \leq g$, $1 \leq j \leq p-1$,
(R5.3) $z_j w_r = w_r z_j$, $1 \leq r \leq g$, $1 \leq j \leq p-1$,
(R6.1) $(\sigma_1^{-1}z_j\sigma_1)z_l = z_l(\sigma_1^{-1}z_j\sigma_1)$, $1 \leq j < l \leq p-1$,
(R6.2) $z_l^{-1}(\tau_1^{-1}\xi_j\tau_1)z_l = \tau_1^{-1}\xi_j\tau_1$, $1 \leq j < l \leq p-1$,
(R6.3) $z_j\xi_l = \xi_l z_j$, $1 \leq j < l \leq p-1$,
(R7.1) $(\sigma_1^{-1}z_j\sigma_1^{-1})z_j = z_j(\sigma_1^{-1}z_j\sigma_1^{-1})$, $1 \leq j \leq p-1$,
(R7.2) $z_j^{-1}(\tau_1^{-1}\xi_j)z_j = \tau_1^{-1}\xi_j$, $1 \leq j \leq p-1$,
(R7.3) $z_j^{-1}\xi_j z_j = \tau_1^{-1}\xi_j\tau_1$, $1 \leq j \leq p-1$.

It is easy to see that the relations (B1), (B2), (R1.1), (R2.1), ..., (R7.1) are defining relations of $B_{n-1}(N_{g,p})$. From the other relations we can find the following conjugacy formulae:
(S1) $\tau_l^{\sigma_k} = \tau_l$, $k \neq l-1, l$,
(S2) $\tau_l^{\sigma_{l-1}} = \tau_{l-1}$,
(S3) $\tau_l^{\sigma_l} = \tau_{l+1}^{\tau_l^{-1}}$, $l \neq n-1$,
(S4) $\tau_i^{a_r} = \tau_i$, $2 \leq i \leq n-1$,
(S5) $w_r^{\sigma_i} = w_r$, $1 \leq r \leq g$, $1 \leq i \leq n-2$,
(S6) $\tau_1^{a_r} = (\tau_1^{-1})^{w_r^{-1}\tau_1}$,
(S7) $w_r^{a_r} = \tau_1^{-1}w_r$, $1 \leq r \leq g$,
(S8) $w_s^{a_r} = w_s^{w_r\tau_1 w_r^{-1}\tau_1}$, $1 \leq s < r \leq g$,
(S9) $w_r^{a_s} = w_r$, $1 \leq s < r \leq g$,
(S10) $\tau_i^{z_j} = \tau_i$, $2 \leq i \leq n-1$, $1 \leq j \leq p-1$,
(S11) $\xi_j^{\sigma_i} = \xi_j$, $1 \leq i \leq n-2$, $1 \leq j \leq p-1$,
(S12) $\xi_j^{a_r} = \xi_j^{w_r\tau_1 w_r^{-1}\tau_1}$, $1 \leq r \leq g$, $1 \leq j \leq p-1$,
(S13) $w_r^{z_j} = w_r$, $1 \leq r \leq g$, $1 \leq j \leq p-1$,

(S14) $\xi_j^{z_l} = \xi_j^{[\xi_l^{-1},\tau_1]}$, $1 \leq j < l \leq p-1$,
(S15) $\xi_l^{z_j} = \xi_l$, $1 \leq j < l \leq p-1$,
(S16) $\tau_1^{z_j} = [\tau_1, \xi_j^{-1}]\tau_1$, $1 \leq j \leq p-1$,
(S17) $\xi_j^{z_j} = \xi_j^{\tau_1}$, $1 \leq j \leq p-1$.

Consider the subgroup

$$W_{n,g,p} = \langle \tau_1, \ldots, \tau_{n-1}, w_1, \ldots, w_g, \xi_1, \ldots, \xi_{p-1} \rangle$$

of $D_{n+1}(N_{g,p})$. The group $W_{n,g,p}$ is free (see [1]) and from (S1)–(S17) we see that $B_n(N_{p,g})$ acts on $W_{n,g,p}$ by conjugacy.

Theorem 4. *The group $B_n(N_{g,p})$, for $n > 1$, $p \geq 1$ and $g \geq 1$ acts by conjugation on the free group $W_{n,g,p}$ and the induced representation $\rho_W : B_n(N_{g,p}) \to \mathrm{Aut}(W_{n,g,p})$ is defined as follows:*
– *Generators σ_i, $i = 1, \ldots, n-1$:*

$$\sigma_i : \begin{cases} \tau_i \longmapsto \tau_{i+1}^{\tau_i^{-1}} ; \\ \tau_{i+1} \longmapsto \tau_i ; \\ \tau_l \longmapsto \tau_l, \; l \neq i, i+1 ; \\ w_r \longmapsto w_r, \; 1 \leq r \leq g ; \\ \xi_j \longmapsto \xi_j, \; 1 \leq j \leq p-1 . \end{cases}$$

– *Generators a_r, $r = 1, \ldots, g$:*

$$a_r : \begin{cases} \tau_1 \longmapsto (\tau_1^{-1})^{w_r^{-1}\tau_1} ; \\ \tau_i \longmapsto \tau_i, \; 2 \leq i \leq n ; \\ w_s \longmapsto w_s^{w_r\tau_1 w_r^{-1}\tau_1}, \; 1 \leq s < r \leq g ; \\ w_r \longmapsto \tau_1^{-1} w_r ; \\ w_s \longmapsto w_s, \; 1 \leq r < s \leq g ; \\ \xi_j \longmapsto \xi_j^{w_r\tau_1 w_r^{-1}\tau_1}, \; 1 \leq j \leq p-1. \end{cases}$$

– *Generators z_j, $j = 1, \ldots, p-1$:*

$$z_j : \begin{cases} \tau_1 \longmapsto [\tau_1, \xi_j^{-1}]\tau_1 ; \\ \tau_i \longmapsto \tau_i, \; 2 \leq i \leq n ; \\ w_r \longmapsto w_r, \; 1 \leq r \leq g ; \\ \xi_l \longmapsto \xi_l^{[\xi_j^{-1},\tau_1]}, \; 1 \leq l < j \leq p-1 ; \\ \xi_j \longmapsto \xi_j^{\tau_1} ; \\ \xi_l \longmapsto \xi_l, \; 1 \leq j < l \leq p-1. \end{cases}$$

We don't know if the representation $\rho_W : B_n(N_{g,p}) \to \mathrm{Aut}(W_{n,g,p})$ is faithful or not.

Proposition 5. *Let β be in $B_n(N_{g,p})$. The element $\rho_W(\beta)$ in $\mathrm{Aut}(W_{n,g,p})$ fixes the product*

$$A = \tau_{n-1}^{-1} \ldots \tau_2^{-1}\, \tau_1^{-1}\, \xi_1\, \xi_2 \ldots \xi_{p-1}\, w_1^2\, w_2^2 \ldots w_g^2.$$

Proof. As above it is enough to prove that images of generators of $B_n(N_{g,p})$ fix A.
Case 1: generators σ_i, $1 \leq i \leq n-1$. The group B_n is a subgroup of $B_n(N_{g,p})$ and by Artin's theorem B_n is a subgroup of $\mathrm{Aut}(F_n)$, $F_n = \langle \tau_1, \tau_2, \dots, \tau_{n-1} \rangle$ and fixes the product $\tau_1 \tau_2 \dots \tau_{n-1}$. Hence, $\rho_W(\sigma_i)$ also fixes the product $\tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1}$.

From Theorem 4 it follows that

$$w_r^{\sigma_i} = w_r, 1 \leq r \leq g; \ \xi_j^{\sigma_i} = \xi_j, 1 \leq j \leq p-1.$$

Hence, $A^{\sigma_i} = A$.

Case 2: generators a_r, $1 \leq r \leq g$. By Theorem 4 we have that

$$(\tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1})^{a_r} = \tau_{n-1}^{-1} \dots \tau_2^{-1} (\tau_1^{w_r^{-1} \tau_1}),$$

and

$$(\xi_1 \dots \xi_{p-1})^{a_r} = (\xi_1 \dots \xi_{p-1})^{w_r \tau_1 w_r^{-1} \tau_1},$$

and

$$(w_1^2 \dots w_g^2)^{a_r} = (w_1^2 \dots w_{r-1}^2)^{w_r \tau_1 w_r^{-1} \tau_1} (\tau_1^{-1} w_r \tau_1^{-1} w_r)(w_{r+1}^2 \dots w_g^2).$$

Hence, the element a_r acts on A as follows:

$$A^{a_r} = \tau_{n-1}^{-1} \dots \tau_2^{-1} \, \tau_1^{-1} \, \xi_1 \dots \xi_{p-1} \, w_1^2 \, w_2^2 \dots w_g^2,$$

and $A^{a_r} = A$.
Case 3. Generators z_j, $1 \leq j \leq p-1$. By Theorem 4 we deduce that

$$(\tau_{n-1}^{-1} \dots \tau_2^{-1} \tau_1^{-1})^{z_j} = \tau_{n-1}^{-1} \dots \tau_2^{-1} \, (\tau_1^{-1} \, \xi_j \, \tau_1^{-1} \, \xi_j^{-1} \, \tau_1),$$

$$(\xi_1 \dots \xi_{p-1})^{z_j} = (\xi_1 \dots \xi_{j-1})^{\xi_j \tau_1^{-1} \xi_j^{-1} \tau_1} \, (\xi_j^{\tau_1}) \, \xi_{j+1} \dots \xi_{p-1},$$

and

$$(w_1^2 \, w_2^2 \dots w_g^2)^{z_j} = w_1^2 \, w_2^2 \dots w_g^2.$$

Hence, $A^{z_j} = A$. □

5. Braid groups of closed surfaces

Let $\Sigma_g = \Sigma_{g,0}$ be a closed orientable surface of genus $g \geq 1$. The group $B_n(\Sigma_g)$, $n > 1$ admits a group presentation with generators:

$$\sigma_1, \dots \sigma_{n-1}, x_1, \dots x_{2g},$$

and relations:
– Braid relations:
$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ 1 \leq i \leq n-2,$
$\sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| > 1, \ 1 \leq i, j \leq n-1,$
– Mixed relations:
(R1) $x_r \sigma_i = \sigma_i x_r$, $i \neq 1$, $1 \leq r \leq 2g$,
(R2) $(\sigma_1^{-1} x_r \sigma_1^{-1}) x_r = x_r (\sigma_1^{-1} x_r \sigma_1^{-1})$, $1 \leq r \leq 2g$,

(R3) $(\sigma_1^{-1}x_s\sigma_1)x_r = x_r(\sigma_1^{-1}x_s\sigma_1)$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(R4) $(\sigma_1^{-1}x_{2m-1}\sigma_1^{-1})x_{2m} = x_{2m}(\sigma_1^{-1}x_{2m-1}\sigma_1)$, $1 \le m \le g$,
(TR5) $[x_1^{-1}, x_2][x_3^{-1}, x_4] \dots [x_{2g-1}^{-1}, x_{2g}] = \sigma_1\sigma_2 \dots \sigma_{n-1}^2 \dots \sigma_2\sigma_1$.

As before, let π be the natural projection of $B_n(\Sigma_g)$ in S_n, let $D_n(\Sigma_g) = \pi^{-1}(S_{n-1})$ and $m_l = \sigma_{n-1} \dots \sigma_l$, $l = 1, \dots, n-1$, $m_n = 1$. The set $M_n = \{m_l \mid 1 \le l \le n\}$ is a Schreier set of coset representatives of $D_n(\Sigma_g)$ in $B_n(\Sigma_g)$ and $D_n(\Sigma_g)$ is generated by

$$\sigma_1, \sigma_2, \dots, \sigma_{n-2}, \tau_1, \tau_2, \dots, \tau_{n-1}, x_1, x_2, \dots, x_{2g}, w_1, w_2, \dots, w_{2g},$$

where

$$\tau_k = \sigma_{n-1} \dots \sigma_{k+1}\sigma_k^2\sigma_{k+1}^{-1} \dots \sigma_{n-1}^{-1},\ k = 1, \dots, n-2,\ \tau_{n-1} = \sigma_{n-1}^2,$$
$$w_r = \sigma_{n-1} \dots \sigma_1 x_r \sigma_1^{-1} \dots \sigma_{n-1}^{-1},\ r = 1, \dots, 2g.$$

Proposition 6. *The group $D_n(\Sigma_g)$, $n > 1$, admits a group presentation with generators:*

$$\sigma_1, \dots, \sigma_{n-2}, x_1, \dots, x_{2g}, \tau_1, \dots, \tau_{n-1},\ w_1, \dots, w_{2g},$$

and relations:
– Braid relations:
(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \le i \le n-3$,
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j| > 1$, $1 \le i, j \le n-2$,
(B3) $\sigma_k^{-1}\tau_l\sigma_k = \tau_l$, $k \ne l-1, l$,
(B4) $\sigma_{l-1}^{-1}\tau_l\sigma_{l-1} = \tau_{l-1}$,
(B5) $\sigma_l^{-1}\tau_l\sigma_l = \tau_l\tau_{l+1}\tau_l^{-1}$, $l \ne n-1$.
– Mixed relations:
(R1.1) $x_r\sigma_i = \sigma_i x_r$, $2 \le i \le n-2$, $1 \le r \le 2g$,
(R1.2) $x_r\tau_i = \tau_i x_r$, $2 \le i \le n-1$,
(R1.3) $w_r\sigma_i = \sigma_i w_r$, $1 \le r \le 2g$, $1 \le i \le n-2$,
(R2.1) $(\sigma_1^{-1}x_r\sigma_1^{-1})x_r = x_r(\sigma_1^{-1}x_r\sigma_1^{-1})$, $1 \le r \le 2g$,
(R2.2) $x_r^{-1}w_r x_r = \tau_1^{-1}w_r\tau_1$, $1 \le r \le 2g$,
(R2.3) $x_r^{-1}\tau_1 x_r = \tau_1^{-1}w_r\tau_1 w_r^{-1}\tau_1$, $1 \le r \le 2g$,
(R3.1) $(\sigma_1^{-1}x_s\sigma_1)x_r = x_r(\sigma_1^{-1}x_s\sigma_1)$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(R3.2) $x_r^{-1}(\tau_1^{-1}w_s\tau_1)x_r = \tau_1^{-1}w_s\tau_1$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(R3.3) $x_s w_r = w_r x_s$, $1 \le s < r \le 2g$, $(s,r) \ne (2m-1, 2m)$,
(R4.1) $(\sigma_1^{-1}x_{2m-1}\sigma_1^{-1})x_{2m} = x_{2m}(\sigma_1^{-1}x_{2m-1}\sigma_1)$, $1 \le m \le g$,
(R4.2) $x_{2m}^{-1}(\tau_1^{-1}w_{2m-1})x_{2m} = \tau_1^{-1}w_{2m-1}\tau_1$, $1 \le m \le g$,
(R4.3) $x_{2m-1}^{-1}w_{2m}x_{2m-1} = \tau_1^{-1}w_{2m}$, $1 \le m \le g$,
(RT.1) $[x_1^{-1}, x_2][x_3^{-1}, x_4] \dots [x_{2g-1}^{-1}, x_{2g}] = \sigma_1\sigma_2 \dots \sigma_{n-3}\sigma_{n-2}^2\sigma_{n-3} \dots \sigma_2\sigma_1\tau_1$,
(RT.2) $[w_1^{-1}, w_2][w_3^{-1}, w_4] \dots [w_{2g-1}^{-1}, w_{2g}] = \tau_1\tau_2 \dots \tau_{n-1}$.

Consider the subgroup $V_{n-1,g}$ of $D_n(\Sigma_g)$ generated by $\{\tau_2, \dots, \tau_{n-1}, w_1, \dots, w_{2g}\}$. The group $V_{n-1,g}$ is free and normal in $D_n(\Sigma_g)$ [1]. Using relation (RT.2) one finds that

$$\tau_1 = [w_1^{-1}, w_2][w_3^{-1}, w_4] \dots [w_{2g-1}^{-1}, w_{2g}]\tau_{n-1}^{-1} \dots \tau_2^{-1}$$

and therefore $\tau_1 \in V_n$. Also, consider the subgroup

$$\overline{B}_{n-1}(\Sigma_g) = \langle \sigma_1, \dots, \sigma_{n-2}, x_1, \dots, x_{2g} \rangle$$

of $D_n(\Sigma_g)$. The group $\overline{B}_{n-1}(\Sigma_g)$ is not isomorphic to $B_{n-1}(\Sigma_g)$.

In fact, one can prove that the relations (B1), (B2), (R1.1), (R2.1), (R3.1), (R4.1) and (RT.1) are defining relations of $\overline{B}_{n-1}(\Sigma_g)$ and therefore we can deduce that $B_{n-1}(\Sigma_g) \simeq \overline{B}_{n-1}(\Sigma_g)/\langle\langle \tau_1 \rangle\rangle$. From the other relations we can see that $\overline{B}_{n-1}(\Sigma_g)$ acts on V_n by conjugacy. Hence, the following theorem holds.

Proposition 7. *There exists a representation $\rho_V : \overline{B}_{n-1}(\Sigma_g) \to \mathrm{Aut}(V_{n-1,g})$, induced by the action by conjugacy of $\overline{B}_{n-1}(\Sigma_g)$ on the free group $V_{n-1,g} = \langle \tau_2, \dots, \tau_{n-1}, w_1, \dots, w_{2g} \rangle$ given algebraically as follows:*

–Action on generators $\tau_2, \dots, \tau_{n-1}$:
(S1) $\tau_l^{\sigma_k} = \tau_l$, $k \neq l-1, l$,

(S2) $\tau_l^{\sigma_{l-1}} = \tau_{l-1}$,

(S3) $\tau_l^{\sigma_l} = \tau_{l+1}^{\tau_l^{-1}}$, $l \neq n-1$,

(S4) $\tau_i^{x_r} = \tau_i$, $2 \leq i \leq n-1$, $1 \leq r \leq 2g$,

–Action on generators $w_1, \dots, w_{2g}$:
(S5) $w_r^{\sigma_i} = w_r$, $1 \leq r \leq 2g$, $1 \leq i \leq n-2$,

(S6) $w_r^{x_r} = w_r^{\tau_1}$, $1 \leq r \leq 2g$,

(S7) $\tau_1^{x_r} = \tau_1^{w_r^{-1}\tau_1}$, $1 \leq r \leq 2g$,

(S8) $w_s^{x_r} = w_s^{[w_r^{-1}, \tau_1]}$, $1 \leq s < r \leq 2g$, $(s, r) \neq (2m-1, 2m)$,

(S9) $w_r^{x_s} = w_r$, $1 \leq s < r \leq 2g$, $(s, r) \neq (2m-1, 2m)$,

(S10) $w_{2m-1}^{x_{2m}} = [\tau_1, w_{2m}^{-1}] w_{2m-1} \tau_1$,

(S11) $w_{2m}^{x_{2m-1}} = \tau_1^{-1} w_{2m}$,
where

$$\tau_1 = [w_1^{-1}, w_2][w_3^{-1}, w_4] \dots [w_{2g-1}^{-1}, w_{2g}] \tau_{n-1}^{-1} \dots \tau_2^{-1}.$$

From Proposition 7 we get a representation of $B_{n-1}(\Sigma_g)$ in $\mathrm{Aut}(V_{n-1,g})/\langle\langle t \rangle\rangle$, where $t \in \mathrm{Aut}(V_{n-1,g})$ is the conjugation by the word

$$[w_1^{-1}, w_2] \dots [w_{2g-1}^{-1}, w_{2g}] \tau_{n-1}^{-1} \dots \tau_2^{-1}.$$

Therefore, we have the following result:

Proposition 8. *There exists a representation* $\rho_{\widetilde{V}} : B_{n-1}(\Sigma_g) \to \mathrm{Out}(V_{n-1,g})$*, given algebraically as follows:*

–Action (up to conjugacy) on generators $\tau_2, \ldots, \tau_{n-1}$*:*
(S1) $\tau_l^{\sigma_k} = \tau_l$, $k \neq l-1, l$,

(S2) $\tau_l^{\sigma_{l-1}} = \tau_{l-1}$,

(S3) $\tau_l^{\sigma_l} = \tau_{l+1}^{\tau_l^{-1}}$, $l \neq n-1$,

(S4) $\tau_i^{x_r} = \tau_i$, $2 \leq i \leq n-1$, $1 \leq r \leq 2g$,

–Action (up to conjugacy) on generators $w_1, \ldots, w_{2g}$*:*
(S5) $w_r^{\sigma_i} = w_r$, $1 \leq r \leq 2g$, $1 \leq i \leq n-2$,

(S6) $w_r^{x_r} = w_r^{\tau_1}$, $1 \leq r \leq 2g$,

(S7) $\tau_1^{x_r} = \tau_1^{w_r^{-1}\tau_1}$, $1 \leq r \leq 2g$,

(S8) $w_s^{x_r} = w_s^{[w_r^{-1}, \tau_1]}$, $1 \leq s < r \leq 2g$, $(s, r) \neq (2m-1, 2m)$,

(S9) $w_r^{x_s} = w_r$, $1 \leq s < r \leq 2g$, $(s, r) \neq (2m-1, 2m)$,

(S10) $w_{2m-1}^{x_{2m}} = [\tau_1, w_{2m}^{-1}] w_{2m-1} \tau_1$,

(S11) $w_{2m}^{x_{2m-1}} = \tau_1^{-1} w_{2m}$,
where

$$\tau_1 = [w_1^{-1}, w_2][w_3^{-1}, w_4] \ldots [w_{2g-1}^{-1}, w_{2g}] \tau_{n-1}^{-1} \ldots \tau_2^{-1}.$$

We don't know if the representation $\rho_{\widetilde{V}}$ of $B_{n-1}(\Sigma_g)$ is faithful or not.

6. Artin-Tits groups

We recall that classical braid groups are also called Artin-Tits groups of type $\mathcal{A}$. More precisely, let (W, S) be a Coxeter system and let us denote the order of the element st in W by $m_{s,t}$ (for $s, t \in S$). Let $A(W)$ be the group defined by the following group presentation:

$$A(W) = \langle S \mid \underbrace{st\cdots}_{m_{s,t}} = \underbrace{ts\cdots}_{m_{s,t}} \text{ for any } s \neq t \in S \text{ with } m_{s,t} < +\infty \rangle .$$

The group $A(W)$ is the Artin-Tits group associated to W. The group $A(W)$ is said to be of spherical type if W is finite. There exists three infinite families of finite Coxeter groups usually denoted respectively as of type $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$. Artin braid groups correspond to the family of Artin-Tits groups associated to Coxeter groups of type $\mathcal{A}$ and braid groups of the annulus coincide with Artin-Tits groups associated to Coxeter groups of type $\mathcal{B}$. In this Section we give a faithful representation of the remaining infinite family of spherical Artin-Tits groups, associated to Coxeter groups of type $\mathcal{D}$.

First let us denote by $A(\mathcal{D}_n)$ the n-th Artin-Tits group of type $\mathcal{D}$ with the group presentation provided by its Coxeter graph (see Figure 1), where vertices $\delta_1, \ldots, \delta_n$ are the generators of the group and two generators δ_i, δ_j verify the relation $\delta_i\delta_j\delta_i = \delta_j\delta_i\delta_j$ if they are related by an edge and elsewhere they commute.

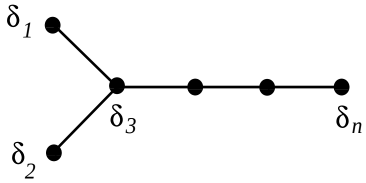

FIGURE 1.

We denote by $\pi_D : A(\mathcal{D}_n) \to B_n$ the epimorphism defined by $\pi_D(\delta_1) = \pi_D(\delta_2) = \sigma_1$ and $\pi_D(\delta_i) = \sigma_{i-1}$ for $i = 3, \ldots, n$, which admits the section $s_D : B_n \to A(\mathcal{D}_n)$ defined by $s_D(\sigma_i) = \delta_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Proposition 9. [6, Proposition 2.3]

(1) *The representation $\rho_D : B_n \to Aut(F_{n-1})$ given algebraically by:*

$$\rho_D(\sigma_1) : \begin{cases} x_1 \to x_1, \\ x_j \to x_1^{-1}x_j, j \neq i, \end{cases}$$

and for $2 \leq i \leq n-1$,

$$\rho_D(\sigma_i) : \begin{cases} x_{i-1} \longmapsto x_i, \\ x_i \longmapsto x_i x_{i-1}^{-1} x_i, \\ x_j \longmapsto x_j, \qquad j \neq i-1, i. \end{cases}$$

is well defined and faithful.

(2) *The group $A(\mathcal{D}_n)$ is isomorphic to $F_{n-1} \rtimes_{\rho_D} B_n$, where the projection on the second factor is π_D and the section $B_n \to F_{n-1} \rtimes_{\rho_D} B_n$ is just s_D.*

(3) *In particular, $\ker \pi_D = F_{n-1}$ is freely generated by $\lambda_1, \ldots, \lambda_{n-1}$, where $\lambda_1 = \delta_1\delta_2^{-1}$ and $\lambda_i = (\delta_{i+1} \cdots \delta_3)(\delta_1\delta_2^{-1})(\delta_{i+1} \cdots \delta_3)^{-1}$ for $i = 2, \ldots, n-1$.*

The first item of Proposition 9 was already established in [12] by topological means and the third item is actually proven in the proof of Proposition 2.3 in [6].

From Proposition 9 we can deduce a faithful representation of $A(\mathcal{D}_n)$ into $\mathrm{Aut}(F_n)$.

Proposition 10. *The representation $\iota : A(\mathcal{D}_n) \to \mathrm{Aut}(F_n)$ given algebraically by:*

$$\iota(\delta_1) : \begin{cases} x_1 \longmapsto x_1, & \\ x_j \longmapsto x_j x_1^{-1}, & j \neq 1, n, \\ x_n \longmapsto x_1 x_n x_1^{-1}, & \end{cases}$$

$$\iota(\delta_2) : \begin{cases} x_1 \longmapsto x_1, & \\ x_j \longmapsto x_1^{-1} x_j, & j \neq 1, n, \\ x_n \longmapsto x_n, & \end{cases}$$

and for $3 \leq i \leq n$,

$$\iota(\delta_i) : \begin{cases} x_{i-2} \longmapsto x_{i-1}, & \\ x_{i-1} \longmapsto x_{i-1} x_{i-2}^{-1} x_{i-1}, & \\ x_j \longmapsto x_j, & j \neq i-2, i-1. \end{cases}$$

is well defined and faithful.

Proof. From Proposition 9 it follows that $F_{n-1} \rtimes_{\rho_D} B_n$ is isomorphic to $A(\mathcal{D}_n)$ through the morphism $\chi : F_{n-1} \rtimes_{\rho_D} B_n \to A(\mathcal{D}_n)$ defined algebraically as follows: $\chi(x_i) = \lambda_i$ for any generator x_i of F_{n-1} and $\chi(\sigma_i) = \delta_{i+1}$ for any generator σ_i of B_n.

Remark that $F_{n-1} \rtimes_{\rho_D} B_n$ is a subgroup of $F_{n-1} \rtimes \mathrm{Aut}(F_{n-1})$. Now, considering the natural inclusion of $\mathrm{Aut}(F_{n-1})$ into $\mathrm{Aut}(F_n)$ leaving the generator x_n invariant and the action by conjugacy of F_{n-1} on F_n, we obtain a morphism $\phi : F_{n-1} \rtimes_{\rho_D} B_n \to \mathrm{Aut}(F_n)$, which can be easily proved to be injective.

By direct calculation one can verify that $\iota(\lambda_i)$ is the conjugacy by x_i for $i = 1, \ldots, n-1$; moreover, since $\iota \circ \chi$ restricted to B_n coincides with the representation of B_n into $\mathrm{Aut}(F_n)$ obtained composing the map ρ_D with the natural inclusion of $\mathrm{Aut}(F_{n-1})$ into $\mathrm{Aut}(F_n)$ leaving the generator x_n invariant, we can deduce that $\iota \circ \chi = \phi$ and therefore the claim follows. □

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