Range of Brownian motion with drift

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Abstract

Let $(B_\delta(t)\,;\,t \geq 0)$ be a Brownian motion with drift $\delta > 0$, starting at 0. Let us define by induction $S_1 = -\inf_{t\geq 0} B_\delta(t)$, ρ_1 the last time such that $B_\delta(\rho_1) = -S_1$, $S_2 = \sup_{0\leq t\leq \rho_1} B_\delta(t)$, ρ_2 the last time such that $B_\delta(\rho_2) = S_2$ and so on. Setting $A_k = S_k + S_{k+1}\,;\; k \geq 1$, we compute the law of $(A_1, \cdots, A_k)$ and the distribution of $((B_\delta(t+\rho_l) - B_\delta(\rho_l)\,;\, 0 \leq t \leq \rho_{l-1} - \rho_l))_{2\leq l\leq k}$ for any $k \geq 2$, conditionally to $(A_1, \cdots, A_k)$. We determine the law of the range $R_\delta(t)$ of $(B_\delta(s)\,;\, s \geq 0)$ at time t, and the first range time $\theta_\delta(a)$ (i.e. $\theta_\delta(a) = \inf\{t > 0\,;\, R_\delta(t) > a\}$). We also investigate the asymptotic behaviour of $\theta_\delta(a)$ (resp. $R_\delta(t)$) as $a \to \infty$ (resp. $t \to \infty$).

Key words : Range Process, Enlargement of filtration, Brownian motion with drift.

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1 Introduction

1) The range concerning one-dimensional Markov chains or random walks has been already investigated by [5] and [4]. See for instance [15] for a short survey. The aim of this paper is to study the range of a Brownian motion with drift ; this process being the prototype of transient diffusions. The range $(R^{(X)}(t)\,;\, t \geq 0)$ associated with a continuous process $(X_t\,;\, t \geq 0)$ is the process :

$$R^{(X)}(t) = \sup_{0\leq u,v\leq t} (X_v - X_u) = \sup_{0\leq u\leq t} X_u - \inf_{0\leq u\leq t} X_u. \tag{1.1}$$

When (X_t) is a one-dimensional Brownian motion started at 0, Feller [4] has computed the density function of $R^{(X)}(t)$, using the fact that the joint distribution of $\sup_{0\leq u\leq t} X_u$ and $\inf_{0\leq u\leq t} X_u$ is explicitly known. Unfortunately the result is expressed as the sum of a series, and the result cannot be generalized to diffusions since the joint distribution of the maximum and the minimum is in general unknown.

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To go further we observe that $t \mapsto R^{(X)}(t)$ is a non-decreasing, continuous function starting at 0. Therefore we can define its right continuous inverse :

$$\theta^{(X)}(a) = \inf \left\{ t \geq 0; R^{(X)}(t) > a \right\}. \tag{1.2}$$

Formally it is equivalent to deal with $(\theta^{(X)}(a)\,;\,a \geq 0)$ either $(R^{(X)}(t)\,;\,t \geq 0)$, since we have

$$\left\{ R^{(X)}(t) < a \right\} = \left\{ \theta^{(X)}(a) > t \right\}. \tag{1.3}$$

It is actually more convenient to work with $(\theta^{(X)}(a)\,;\,a \geq 0)$. For instance, if (X_t) is a Brownian motion, then for any $a > 0$, the Laplace transform of the r.v. $\theta^{(X)}(a)$ can be computed, see [7], [13]. Moreover $(\theta^{(X)}(a)\,;\,a \geq 0)$ has independent increments.

Assume that (X_t) is a diffusion process and for simplicity $X_0 = 0$. It is proved (Theorem 4, [13]) that the process $(X_t; 0 \leq t \leq \theta^{(X)}(Ua))$ is distributed as $(X_t; 0 \leq t \leq T^{(X)}(aU) \wedge T^{(X)}(a(U-1))$, where $a > 0$, $T^{(X)}(c)$ is the first hitting time of level c, and U denotes a r.v. uniformly distributed on $[0, 1]$, independent of the underlying process (X_t). This property has been generalized by [11]. The Laplace transform of $T^{(X)}(c) \wedge T^{(X)}(d)$ can be expressed besides through eigenfunctions associated with the generator of (X_t). Consequently previous result gives an analytic expression for the Laplace transform of $\theta^{(X)}(a)$.

2) Let $(B(t)\,;\,t \geq 0)$ be a one-dimensional Brownian motion started at 0, and $(S_t\,;\,t \geq 0)$ its unilateral maximum : $S_t = \sup_{0 \leq u \leq t} B(u)$. The process $(S_t\,;\,t \geq 0)$ gives rise to an excursion theory. More precisely it can be proved (see for instance [10], Chap. XII, section 2.) that the family of processes $\left\{ \left(x - B(T^{(B)}(x_-) + t)\,;\, 0 \leq t \leq T^{(B)}(x) - T^{(B)}(x_-) \right); x > 0 \right\}$ is Poisson point process whose characteristic measure is explicit.

It seems natural to ask for a similar question replacing the one-sided maximum by the range. Results in this direction has been obtained in [6], [8] and [14]. In these decompositions, the fact that the Brownian motion is recurrent plays a crucial role. We shall be a little bit more precise below. This leads naturally to investigate the case of transient diffusions. To obtain explicit results, we restrict our attention to the case of Brownian motion with drift.

Let $(B_\delta(t)\,;\,t \geq 0)$ be the Brownian motion with drift δ : $B_\delta(t) = B(t) + \delta\, t$. For simplicity, we note $(R_\delta(t)\,;\,t \geq 0)$ the range of $(B_\delta(t))$ instead of $(R^{(B_\delta)}(t)\,;\,t \geq 0)$:

$$R_\delta(t) = \sup_{0 \leq s,u \leq t} (B_\delta(s) - B_\delta(u)) = \sup_{0 \leq s \leq t} B_\delta(s) - \inf_{0 \leq s \leq t} B_\delta(s). \tag{1.4}$$

The right continuous inverse of $(R_\delta(t)\,;\,t \geq 0)$ is noted $(\theta_\delta(a)\,;\,a \geq 0)$:

$$\theta_\delta(a) = \inf \{ t \geq 0\,;\, R_\delta(t) > a \}. \tag{1.5}$$

3) Let us explain why the transient case is drastically different from the recurrent case. Let Σ_δ be the random set :

$$\Sigma_\delta = \{ a \geq 0\,;\, B_\delta(\theta_\delta(a)) B_\delta(\theta_\delta(a-)) < 0 \} \tag{1.6}$$

where $\theta_\delta(a-)$ is the left limit of θ_δ at a.

When $\delta = 0$, for any $a > 0$, the two sets $\Sigma_0 \cap (0, a]$ and $\Sigma_0 \cap [a, +\infty)$ are infinite. This generates a difficulty to enumerate the points in Σ_0. However a decomposition of the Brownian path $(B_0(t)\,;\, t \geq 0)$ has been given [8, 14] through Σ_0, and three dimensional Bessel processes.

Suppose $\delta > 0$. Since $B_\delta(t)$ drifts to infinity, as $t \to +\infty$, for any $a > 0$ the random set $\Sigma_\delta \cap [a, +\infty)$ is finite and enables an explicit description of Σ_δ. Let us first introduce the minimum $-S_1$ of $(B_\delta(t)\,;\, t \geq 0)$, and ρ_1 the last time such that $B_\delta(\rho_1) = -S_1$. Secondly let S_2 be the maximum of $B_\delta(t)$, t running over $[0, \rho_1]$, and ρ_2 be the last time in $[0, \rho_1]$ such that $B_\delta(\rho_2) = S_2$, and so on, see (2.1) for details. Then

$$\Sigma_\delta = \{A_n\,;\, n \geq 1\} \tag{1.7}$$

where $A_n = S_n + S_{n+1}$.

We compute the density function of $(S_1, \cdots, S_n)$ in Proposition 2.1. We also obtain an enlighted description of the law of $(A_1, \cdots, A_n)$ (cf Theorem 2.2). Let $\psi : \mathbb{R} \to (-\infty, 1)$ be the function :

$$\psi(x) = \frac{e^x - 1 - x}{e^x - 1} \quad \text{if} \quad x \neq 0, \quad \text{and} \quad \psi(0) = 0. \tag{1.8}$$

Then ψ is one-to-one from $\mathbb{R}$ to $(-\infty, 1)$ and :

$$\left(\psi(2\delta A_1), \frac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)}, \ldots, \frac{\psi((-1)^{n+1} 2\delta A_n)}{\psi((-1)^{n+1} 2\delta A_{n-1})}\right) \overset{d}{=} (U_1, U_2, \ldots, U_n) \tag{1.9}$$

where $U_1, \cdots, U_n$ are i.i.d. r.v.'s, uniformly distributed on $[0, 1]$.

We are also able to determine the law of the sequence of processes $(|B_\delta(t + \rho_n) - B_\delta(\rho_n)|\,;\, 0 \leq t \leq \rho_{n-1} - \rho_n)_{n \geq 2}$. The result being technical, we do not state it in the Introduction. A complete formulation can be found in Theorem 2.4. We would like to note that three dimensional Bessel processes coming from the decomposition of the Brownian motion have to be replaced by non negative diffusions distributed as the process $Z^{(\delta)}$ where

$$Z^{(\delta)}(t) = B_t + \delta \int_0^t \coth(\delta Z^{(\delta)}(s))\, ds. \tag{1.10}$$

Heuristically, $(Z^{(\delta)}(t)\,;\, t \geq 0)$ is the process $(B_\delta(t)\,;\, t \geq 0)$ conditioned to be positive, see [17]. Taking formally the limit $\delta \to 0$ in (1.10), we recover the three-dimensional Bessel process started at 0.

4) In Section 4, we focus on the law of $R_\delta(t)$ (resp. $\theta_\delta(a)$) where $t > 0$ (resp. $a > 0$) is fixed. We compute the two distribution functions of $R_\delta(t)$ and $\theta_\delta(a)$. We partially recover the result of [2]. A path decomposition of $(B_\delta(t)\,;\, 0 \leq t \leq \theta_\delta(a))$ (Proposition 4.1) allows us to determine the Laplace transform of $\theta_\delta(a)$. Then, by a straightforward approach, it is easy to obtain the asymptotic behaviour of $\theta_\delta(a)$, $a \to +\infty$. We prove two limit results : the first one looks like a Law of

Large Numbers and the second one is similar to the Central Limit Theorem. More precisely :

$$\frac{\theta_\delta(a)}{a} \xrightarrow[a\to\infty]{a.s.} \frac{1}{\delta}, \tag{1.11}$$

$$\sqrt{a}\left(\frac{\theta_\delta(a)}{a} - \frac{1}{\delta}\right) \xrightarrow[a\to\infty]{d} \mathcal{N}\left(0, 1/\delta^3\right), \tag{1.12}$$

where $\mathcal{N}\left(0, 1/\delta^3\right)$ denote the Gaussian distribution with 0-mean and variance $1/\delta^3$.

2 Notations and main results

We keep the notations introduced in the Introduction. In particular $(B_{\pm\delta}(t)\,;\, t \geq 0)$ is a Brownian motion with constant drift $\pm\delta$ starting at 0. Throughout this paper, we suppose that $\delta > 0$.

Let $-S_1$ denote the absolute (random) minimum of $(B_\delta(t)\,;\, t \geq 0)$ and ρ_1 the last (random) time when it is reached. The decomposition of the Brownian path $t \mapsto B_\delta(t)$ may be continued by induction on $k \geq 2$, as follows :

$$\begin{cases} S_k &= \sup\limits_{t\in[0,\rho_{k-1}]} (-1)^k B_\delta(t), \\ \rho_k &= \sup\{t \in [0, \rho_{k-1}]\ ,\ B_\delta(t) = (-1)^k\, S_k\}. \end{cases} \tag{2.1}$$

Note that if we set $\rho_0 = \infty$, then relations (2.1) are still valid for $k = 1$.

Since $B_\delta(t)$ goes to $+\infty$, as $t \to \infty$, and $t \mapsto B_\delta(t)$ is a continuous function, then the random times $(\rho_k)_{k\geq 1}$ are well defined. This does not mean that $\{t \in [0, \rho_{k-1}]; B_\delta(t) = (-1)^k\, S_k\}$ is reduced to the singleton $\{\rho_k\}$. We actually prove that this property holds.

We start with the law of $(S_1, \cdots, S_k)\,;\, k \geq 1$.

Proposition 2.1 *Suppose $\delta > 0$. Let $(S_k\,;\, k \geq 1)$ be defined by (2.1). Then*

1. *The law of S_1 is exponential of parameter 2δ (i.e. with density function $2\delta e^{-2\delta x}\mathbb{1}_{\{x\geq 0\}}$).*
2. *Conditionally to $\{S_1 = x_1\}$, S_2 has density function :*

$$\delta e^{\delta x_1}\,\mathrm{sh}(\delta x_1)\frac{1}{\mathrm{sh}^2(\delta(x + x_1))}\mathbb{1}_{\{x\geq 0\}}. \tag{2.2}$$

3. *For every $k \geq 3$, conditionally to $\{S_1 = x_1, \ldots, S_{k-1} = x_{k-1}\}$, S_k has density function :*

$$\delta\frac{\mathrm{sh}\,\delta x_{k-1}\,\mathrm{sh}\,\delta(x_{k-1} + x_{k-2})}{\mathrm{sh}\,\delta x_{k-2}}\frac{1}{\mathrm{sh}^2\,\delta(x_{k-1} + x)}\mathbb{1}_{\{0\leq x\leq x_{k-2}\}}.$$

Proof The proof of Proposition 2.1 is postponed in Section 3. □

Note that $x_k + x_{k-1}$ appears in the conditional density function of S_k. This leads us to introduce

$$A_k = S_k + S_{k+1} \quad ; \quad k \geq 1. \tag{2.3}$$

$(A_k)_{k\geq 1}$ is the sequence of maximal ranges associated with $(B_\delta(t)\,;\, t \geq 0)$.

Theorem 2.2 *Suppose $\delta > 0$. Then*

1. *$(A_1, \cdots, A_n)$ has a density function given by*

$$\frac{\delta^n}{2}\left[\prod_{k=1}^{n-1}\frac{e^{(-1)^k\delta a_k}}{\mathrm{sh}(\delta a_k)}\right]\frac{e^{(-1)^n 2\delta a_n} - 1 + (-1)^{n+1}2\delta a_n}{\mathrm{sh}^2\,\delta a_n}\mathbb{1}_{\{0\leq a_n\leq\cdots\leq a_1\}}. \tag{2.4}$$

2. *Let : $\psi : \mathbb{R} \to (-\infty, 1)$*

$$\psi(x) = \frac{e^x - 1 - x}{e^x - 1} \quad ; \quad x \neq 0 \quad \textit{and} \quad \psi(0) = 0.$$

ψ is one-to-one from $\mathbb{R}$ to $(-\infty, 1)$ and we have

$$\left(\psi(2\delta A_1), \frac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)}, \ldots, \frac{\psi((-1)^{n+1}2\delta A_n)}{\psi((-1)^{n+1}2\delta A_{n-1})}\right) \stackrel{d}{=} (U_1, U_2, \ldots, U_n) \tag{2.5}$$

where $U_1, \cdots, U_n$ are i.i.d. r.v.'s, uniformly distributed on $[0, 1]$.

Proof The proof is postponed in Section 3 □

Remark 2.3 1) Recall that Σ_δ is defined by (1.6). In [14], it is proved that if $\delta = 0$, then Σ_0 is a one-dimensional Poisson point process (P.p.p.) with characteristic measure $\nu(da) = \frac{1}{a}\mathbb{1}_{\{a>0\}}da$. Concerning P.p.p. we refer to [10] (chapter XII).

If $\delta > 0$, we claim that Σ_δ is not a P.p.p.. Using (2.5), by tedious calculations we prove

$$\mathbb{E}\left[\sum_{n\geq 1} f(A_n)\right] = \delta^2\int_0^\infty f(x)\frac{x}{\mathrm{sh}^2(\delta x)}dx, \tag{2.6}$$

for any positive Borel function f.

Suppose that Σ_δ is a P.p.p. with characteristic measure ν_δ, (2.6) implies that :

$$\nu_\delta(dx) = \frac{\delta^2 x}{\mathrm{sh}^2(\delta x)}\mathbb{1}_{\{x>0\}}dx. \tag{2.7}$$

A straightforward calculation gives :

$$\nu_\delta\left([b,+\infty)\right) = \delta b \coth(\delta b) - \log(\mathrm{sh}(\delta b)) - \log 2\ ,\ b>0. \tag{2.8}$$

Consequently, $N = \sum_{n\geq 1} \mathbb{1}_{\{A_n \geq b\}}$ is a Poisson random variable with parameter $\nu_\delta\left([b,+\infty)\right)$.

But $n \mapsto A_n$ is a decreasing sequence, then

$$\mathbb{P}(N=0) = \mathbb{P}(A_1 < b) = \Psi(2\delta b).$$

This generates a contradiction since

$$\mathbb{P}(N=0) = \exp(-\nu_\delta([b,+\infty))) \neq \Psi(2\delta b).$$

2) Due to (2.4), it is easy to check that $(A_{2n-1}, A_{2n}; n\geq 1)$ is a Markov chain, which takes its values in $\left\{(x,y)\in\mathbb{R}^2; 0<y<x\right\}$, with transition probability density function :

$$\mathbb{P}\left((u,v);(x,y)\right) = \frac{4\delta^2}{\Psi(2\delta v)}\frac{1}{e^{2\delta x}-1}\frac{e^{2\delta y}}{e^{2\delta y}-1}\Psi(2\delta y)\mathbb{1}_{\{0<y<x<v\}}. \tag{2.9}$$

Note that this quantity dœs not depend on u.

3) If we set

$$X_n = \psi\left((-1)^{n+1}A_n\right)\ ;\ n\geq 1,$$

then $(X_n\,;\,n\geq 1)$ is a Markov chain on $(-\infty,1)$, with initial distribution the uniform distribution on $[0,1]$, with transition probability kernel :

$$K(x,f) = \mathbb{E}\left[f(U\psi(-\psi^{-1}(x)))\right] \tag{2.10}$$

where U denotes a *r.v.* uniformly distributed in $[0,1]$.

We now give the law of the sequence of processes $((B_\delta(t+\rho_k)-B_\delta(\rho_k)\,;\,t\in[0,\rho_{k-1}-\rho_k]))_{k\geq 1}$. The result makes use of the process $Z^{(\delta)}$ defined by (1.10).

Theorem 2.4 *Let $\delta>0$ and $k\geq 2$.*
Conditionally to $S_1=x_1,\ldots,S_k=x_k$,

1) $(B_\delta(t)\,;\,0\leq t\leq\rho_k)$, $(B_\delta(t+\rho_1)-B_\delta(\rho_1)\,;\,t\geq 0)$, $(B_\delta(t+\rho_2)-B_\delta(\rho_2)\,;\,0\leq t\leq\rho_1-\rho_2)$, ..., $(B_\delta(t+\rho_k)-B_\delta(\rho_k)\,;\,0\leq t\leq\rho_{k-1}-\rho_k)$ are independent;

2) $(B_\delta(t)\,;\,0\leq t\leq\rho_k)$ has the law of Brownian motion with drift $(-1)^k\delta$ stopped at the first hitting time of level $(-1)^k x_k$, and conditioned not to hit $(-1)^{k+1}x_{k-1}$;

3) For any $2\leq l\leq k$, $(|B_\delta(t+\rho_l)-B_\delta(\rho_l)|\,;\,0\leq t\leq\rho_{l-1}-\rho_l)$ is a process with the same law as $Z^{(\delta)}$ stopped at the first moment it reaches the level x_l+x_{l-1}.

3 Proofs of Theorems 2.2, 2.4 and Proposition 2.1

We keep the notations introduced in Sections 1 and 2.

3.1 Proof of Theorem 2.2

1) Formula (2.4) is a direct consequence of Proposition 2.1. Indeed, let F be a test function. We have :

$$\begin{aligned}\mathbb{E}\left[F(A_1,\dots,A_n)\right] &= \mathbb{E}\left[F(S_1+S_2,\dots,S_n+S_{n+1})\right]\\ &= \int_{\mathbb{R}^{n+1}} F(x_1+x_2,\dots,x_n+x_{n+1}) 2\delta^{n+1} \frac{e^{-\delta x_1}\operatorname{sh}\delta x_n}{\prod_{k=1}^{n-1}\operatorname{sh}\delta(x_{k+1}+x_k)}\\ &\times \frac{1}{\operatorname{sh}^2\delta(x_{n+1}+x_n)} \mathbb{1}_{\{x_1\geq x_3\geq\cdots\geq 0\}}\mathbb{1}_{\{x_2\geq x_4\geq\cdots\geq 0\}} dx_1\dots dx_{n+1}.\end{aligned}$$

We use the following change of variables :

$$\left\{\begin{array}{lcl} x_1 &=& x_1\\ a_1 &=& x_1+x_2\\ \dots & & \\ a_n &=& x_n+x_{n+1}.\end{array}\right. \tag{3.1}$$

Then,

$$\begin{aligned} x_k - x_{k+2} &= a_k - a_{k+1}\,;\ 1\leq k\leq n-1\\ x_n &= \alpha + (-1)^{n+1}x_1\\ x_{n+1} &= a_n - (\alpha+(-1)^{n+1}x_1)\\ \text{where}\qquad \alpha &= a_{n-1}-a_{n-2}+\cdots+(-1)^n a_1.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[F(A_1,\dots,A_n)\right] = \int_{\mathbb{R}^n} F(a_1,\dots,a_n) &\frac{2\delta^n}{\left(\prod_{k=1}^{n-1}\operatorname{sh}(\delta a_k)\right)\operatorname{sh}^2\delta a_n}\\ &\times \mathbb{1}_{\{0\leq a_n\leq a_{n-1}\leq\cdots\leq a_1\}} I(\alpha) da_1\dots da_n,\end{aligned}$$

where

$$I(\alpha) = \int_0^\infty \delta e^{-\delta x_1}\operatorname{sh}\delta(\alpha+(-1)^{n+1}x_1)\mathbb{1}_{\{0\leq\alpha+(-1)^{n+1}x_1\leq a_n\}}\,dx_1.$$

Suppose for instance that n is even. Setting $y=\alpha-x_1$ in order to obtain

$$\begin{aligned} I(\alpha) &= e^{-\delta\alpha}\delta\int_0^{a_n} e^{\delta y}\operatorname{sh}\delta y\,dy\\ &= \frac{e^{-\delta\alpha}}{4}\left(e^{2\delta a_n}-1-2\delta a_n\right).\end{aligned}$$

Then (2.4) follows immediately.

2) The density function h_1 of A_1 is

$$h_1(x) = \frac{\delta}{2\,\mathrm{sh}^2\,\delta x}\left(e^{-2\delta x} - 1 + 2\delta x\right). \tag{3.2}$$

Let H_1 denote the distribution function of A_1

$$H_1(x) = \int_0^x h_1(y)\,dy. \tag{3.3}$$

We have

$$\int_0^x \frac{e^{2y} - 1 - 2y}{\mathrm{sh}^2\,y}dy = \frac{e^x}{\mathrm{sh}\,x}\left(e^{-2x} - 1 + 2x\right). \tag{3.4}$$

Consequently

$$H_1(x) = \psi(2\delta x). \tag{3.5}$$

So

$$A_1 \stackrel{d}{=} \frac{1}{2\delta}\psi^{-1}(U_1), \tag{3.6}$$

where U_1 is a uniform random variable on $[0,1]$.

We compute now the density $h_2^{(x)}$ of A_2 conditionally to $A_1 = x$.

$$\begin{aligned} h_2^{(x)}(y) &= \frac{\delta^2}{2}\frac{e^{2\delta y} - 1 - 2\delta y}{\mathrm{sh}^2\,\delta y}\frac{e^{-\delta x}}{\mathrm{sh}\,\delta x}\mathbb{1}_{\{0\le y\le x\}}\frac{2\,\mathrm{sh}^2\,\delta x}{\delta}\frac{1}{e^{-2\delta x} - 1 + 2\delta x} \\ &= \delta\frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x}\frac{e^{2\delta y} - 1 - 2\delta y}{\mathrm{sh}^2\,\delta y}\mathbb{1}_{\{0\le y\le x\}}. \end{aligned} \tag{3.7}$$

Let $H_2^{(x)}$ denote the associated distribution function

$$H_2^{(x)}(y) = \frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x}\delta\int_0^y \frac{e^{2\delta t} - 1 - 2\delta t}{\mathrm{sh}^2\,\delta t}dt. \tag{3.8}$$

Relation (3.4) implies that :

$$\begin{aligned} H_2^{(x)}(y) &= \frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x}\frac{e^{\delta y}}{\mathrm{sh}\,\delta y}\left(e^{-2\delta y} - 1 + 2\delta y\right) \\ &= \frac{\psi(-2\delta y)}{\psi(-2\delta x)}. \end{aligned}$$

So, we have proved that $\left(\psi(2\delta A_1), \dfrac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)}\right)$ is distributed as (U_1, U_2), where U_1 and U_2 are two independent r.v.'s, uniformly distributed on $[0,1]$.

Reasoning by induction, identity (2.5) can be proved by the same way via (2.4).

3.2 Proofs of Theorem 2.4 and Proposition 2.1

In our approach, Theorem 2.4 and Proposition 2.1 are a direct consequence of Proposition 3.2 stated below. Therefore, we first focus on this key result, and we prove Theorem 2.4 and Proposition 2.1 at the end of this subsection.

The description of the laws of $(B_\delta(t)\,;\,0 \le t \le \rho_1)$ and $(B_\delta(t) - B_\delta(\rho_1)\,;\,t \ge \rho_1)$ is given by the well-known theorem of Williams [17].

Proposition 3.1 **([17])**

1) The law of S_1 is exponential with parameter 2δ (i.e. its density function is $2\delta e^{-2\delta x}\mathbb{1}_{\{x\ge 0\}}$).

2) Conditionally to $S_1 = x_1$,

a. $(B_\delta(t)\,;\,t \le \rho_1)$ and $(B_\delta(t+\rho_1) - B_\delta(\rho_1)\,;\,t \ge 0)$ are independent processes

b. $(B_\delta(t)\,;\,t \le \rho_1)$ is a process with the same law as a $B_{-\delta}$ stopped at its first hitting time of $-x_1$

c. $(B_\delta(t+\rho_1) - B_\delta(\rho_1)\,;\,t \ge 0)$ is a (positive) process distributed as $Z^{(\delta)}$.

To obtain the decomposition of the Brownian motion with drift given in Theorem 2.4, Proposition 3.1 leads us to study $(B_{-\delta}(t)\,;\,0 \le t \le T^{-\delta}_{-a})$ conditionally to $\max\limits_{0\le t\le T^{-\delta}_{-a}} B_{-\delta}(t)$ where :

$$T^{-\delta}_{-a} = \inf\{t \ge 0\ ,\ B_{-\delta}(t) < -a\}\ \ ,\ a > 0. \tag{3.9}$$

Since $(-B_{-\delta}(t)\,;\,0 \le t \le T^{-\delta}_{-a})$ and $(B_\delta(t)\,;\,0 \le t \le T^{\delta}_{a})$ have the same law, it is equivalent to determine the distribution of $(B_\delta(t)\,;\,0 \le t \le T^{\delta}_{a})$ conditionally to

$$S = -\inf_{t\in[0,T^\delta_a]} B_\delta(t), \tag{3.10}$$

T^δ_a being the first hitting time of level a :

$$T^\delta_a = \inf\{t \ge 0\ ,\ B_\delta(t) > a\}\ \ ,\ a > 0.$$

Let ρ be the random time :

$$\rho = \sup\{t \in [0, T^\delta_a], B_\delta(t) = -S\}. \tag{3.11}$$

Time ρ plays a central role in our approach, as shows the following proposition.

Proposition 3.2 *Let S and ρ be r.v.'s defined by* (3.10) *and* (3.11)*. Then,*

1) S has a density function φ

$$\varphi(x) = \delta e^{\delta a}\,\mathrm{sh}\,\delta a \frac{1}{\mathrm{sh}^2\,\delta(x+a)}\mathbb{1}_{\{x\ge 0\}}. \tag{3.12}$$

2) Conditionally to $S = b$,

a. $(B_\delta(t)\,;\, 0 \le t \le \rho)$ *and* $(B_\delta(t+\rho) - B_\delta(\rho)\,;\, 0 \le t \le T_a^\delta - \rho)$ *are independent,*

b. $(B_\delta(t)\,;\, 0 \le t \le \rho)$ *is a process with the same law as a Brownian motion with drift* $-\delta$*, stopped at its first hitting time of level* $-b$ *and conditioned to stay less than* a*,*

c. $(B_\delta(t+\rho) - B_\delta(\rho)\,;\, 0 \le t \le T_a^\delta - \rho)$ *is a (positive) process distributed as the process* $Z^{(\delta)}$*, stopped at its first hitting time of level* $a + b$

Before starting the proof of Proposition 3.2, we would like to explain our approach.

Let $(\mathcal{F}_t)$ be the natural filtration generated by $(B(t)\,;\, t \ge 0)$. Unfortunately the random time ρ is not stopping time. However ρ is a last exit time : $\rho = \sup\{t \in [0, T_a^\delta]; B_\delta(t) = \sup_{0 \le u \le t} B_\delta(u)\}$. This leads us to apply the theory of enlargement of filtrations (See Protter [9] Ch. VI).

Let $(\mathcal{F}_t^\rho)$ be the smallest right-continuous filtration containing $(\mathcal{F}_t)$ (i.e. $\mathcal{F}_t \subset \mathcal{F}_t^\rho$, for any $t \ge 0$) such that ρ is a $(\mathcal{F}_t^\rho)$-stopping time. Let :

$$Y_t^\rho = \mathbb{P}\left(\rho \le t | \mathcal{F}_t\right), \tag{3.13}$$

$(Y_t^\rho; t \ge 0)$ is the optional projection of $\left(\mathbb{1}_{\{\rho \le t\}}; t \ge 0\right)$ (cf [9] p 371).
It is easy to check that $(Y_t^\rho\,;\, t \ge 0)$ is a $(\mathcal{F}_t)$-submartingale. Let :

$$Y_t^\rho = M_t^\rho + A_t^\rho, \tag{3.14}$$

be its Doob-Meyer decomposition, where $(M_t^\rho; t \ge 0)$ denotes the martingale part, $(A_t^\rho; t \ge 0)$ is a non-decreasing and adapted process such that $A_0^\rho = 0$. The processes $(Y_t^\rho; t \ge 0)$ and $(A_t^\rho; t \ge 0)$ will be given in Lemma 3.3.
To determine the law of $\big(B_\delta(s); s \le \rho\big)$, we need the following result.
Let $(U_t; t \ge 0)$ be a non-negative and $(\mathcal{F}_t)$-adapted process then ([9] p 371) :

$$\mathbb{E}[U_\rho] = \mathbb{E}\left[\int_0^\infty U_t dA_t^\rho\right]. \tag{3.15}$$

Note that $S = -\min_{0 \le s \le \rho} B_\delta(s)$. Taking $U_t = F\left(B_\delta(s); s \le t\right) f(-\min_{0 \le s \le t} B_\delta(s))$, where F and f are measurable and non negative, we get :

$$\mathbb{E}\left[F\left(B_\delta(s); s \le \rho\right) f(S)\right] = \mathbb{E}\left[\int_0^\infty F\left(B_\delta(s); s \le t\right) f(-\min_{0 \le s \le t} B_\delta(s)) dA_t^\rho\right].$$

Since $(A_t^\rho; t \ge 0)$ is explicitly known, previous identity allows us to determine the law of $\big(B_\delta(s); s \le \rho\big)$ conditionally to S, see Lemma 3.5.
To obtain the law of $(B_\delta(t+\rho) - B_\delta(\rho)\,;\, 0 \le t \le T_a^\delta - \rho)$, we use ([9] Theorem 18, p 375) the following property :

$$\overline{B}(t) = B(t) - \chi(t) \ \text{ is a } (\mathcal{F}_t^\rho) - \text{Brownian motion} \tag{3.16}$$

where :

$$\chi(t) = -\int_0^{t\wedge\rho} \frac{1}{1-Y_s^\rho} d\left\langle B, M^\rho\right\rangle_s + \int_0^t \mathbb{1}_{\{\rho<s\}} \frac{1}{Y_s^\rho} d\left\langle B, M^\rho\right\rangle_s . \qquad (3.17)$$

This allows us to prove that $(B_\delta(t+\rho) - B_\delta(\rho)\,;\, 0 \le t \le T_a^\delta - \rho)$ solves a stochastic differential equation of type (1.10).

Lemma 3.3 *Let $(Y_t^\rho)_{t\ge 0}$ be the process defined by (3.13).*

1. *We have :*

$$Y_t^\rho = \frac{e^{2\delta \underline{B_\delta}(t\wedge T_a^\delta)} - e^{-2\delta B_\delta(t\wedge T_a^\delta)}}{e^{2\delta \underline{B_\delta}(t\wedge T_a^\delta)} - e^{-2\delta a}} \;; t \ge 0, \qquad (3.18)$$

where $\underline{B_\delta}(t) = -\inf_{s\le t} B_\delta(s)$.

2. *$(Y_t^\rho)_{t\ge 0}$ is a $(\mathcal{F}_t)$ sub-martingale with Doob-Meyer decomposition (3.14) and*

$$A_t^\rho = \ln\left(\frac{e^{2\delta \underline{B_\delta}(t\wedge T_a^\delta)} - e^{-2\delta a}}{1 - e^{-2\delta a}}\right) \qquad (3.19)$$

$$M_t^\rho = 2\delta \int_0^{t\wedge T_a^\delta} \frac{1}{e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta a}} e^{-2\delta B_\delta(s)}\, dB(s). \qquad (3.20)$$

Proof a) Recall a classical result concerning hitting times of Brownian motion with drift (see for instance Borodin and Salminen [1] formula 2.1.2 p. 295)

$$\mathbb{P}\left(T_b^\delta < T_a^\delta \middle| B_\delta(0) = x\right) = \frac{e^{-2\delta x} - e^{-2\delta a}}{e^{-2\delta b} - e^{-2\delta a}}, \quad \text{for } x \text{ between } a \text{ and } b. \qquad (3.21)$$

b) Let $t > 0$ fixed. We have :

$$\mathbb{1}_{\{\rho\le t\}} = \mathbb{1}_{\{\rho\le t\le T_a^\delta\}} + \mathbb{1}_{\{T_a^\delta \le t\}}. \qquad (3.22)$$

If $t \le T_a^\delta$, $\rho \le t$ means that after the time t, B_δ hits a before $-\underline{B_\delta}(t)$. Consequently :

$$\mathbb{P}(\rho \le t \le T_a^\delta | \mathcal{F}_t) = \frac{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta B_\delta(t)}}{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} \mathbb{1}_{\{t\le T_a^\delta\}}. \qquad (3.23)$$

$\widetilde{Y}^\rho$ denotes the process :

$$\widetilde{Y}_t^\rho = \frac{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta B_\delta(t)}}{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} \text{ for } t \ge 0. \qquad (3.24)$$

Note that $\widetilde{Y}_{T_a^\delta}^\rho = 1$, therefore : $Y_t^\rho = \widetilde{Y}_{t\wedge T_a^\delta}^\rho$.

c) We know that $e^{-2\delta B_\delta(t)} = e^{-2\delta B(t)-2\delta^2 t}$ is a martingale. So using the classical rules of stochastic calculus we get :

$$\begin{aligned} d\widetilde{Y}_t^\rho &= \frac{1}{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} e^{-2\delta B_\delta(t)} 2\delta dB(t) \\ &+ \frac{e^{-2\delta B_\delta(t)} - e^{-2\delta a}}{\left(e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}\right)^2} 2\delta e^{2\delta \underline{B_\delta}(t)} d\underline{B_\delta}(t). \end{aligned} \tag{3.25}$$

Since the support of the random measure $d\underline{B_\delta}$ is included in $\{t \geq 0\,;\, B_\delta(t) = -\underline{B_\delta}(t)\}$, then (3.19) and (3.20) follow immediately. □

The following result will be useful in the proof of Lemma 3.5 below.

Proposition 3.4 *Let $x > 0$ and $y > 0$. Then $(B_\delta(t)\,;\, 0 \leq t \leq T_x^\delta)$ conditioned by $\{T_x^\delta < T_{-y}^\delta\}$ is distributed as $(B_{-\delta}(t)\,;\, 0 \leq t \leq T_x^{-\delta})$ conditioned by $\{T_x^{-\delta} < T_{-y}^{-\delta}\}$.*

Proof Let F be a test function. We have :

$$\begin{aligned} A &= \mathbb{E}\left[F\left(B(t) + \delta t\,;\, 0 \leq t \leq T_x^\delta\right) | T_x^\delta < T_{-y}^\delta\right] \\ &= \mathbb{E}\left[F\left(B(t) + \delta t\,;\, , 0 \leq t \leq T_x^\delta\right) \mathbb{1}_{\{T_x^\delta < T_{-y}^\delta\}}\right] \frac{1}{\mathbb{P}(T_x^\delta < T_{-y}^\delta)}. \end{aligned}$$

The r.v. $F\left(B(t) + \delta t\,;\, 0 \leq t \leq T_x^\delta\right) \mathbb{1}_{\{T_x^\delta < T_{-y}^\delta\}}$ being $\mathcal{F}_{T_x^\delta}$ measurable, and $T_x^\delta < \infty$ a.s., Girsanov's theorem and (3.21) imply :

$$\begin{aligned} A &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp(\delta B(T_x^0) - \frac{\delta^2}{2} T_x^0)\right] \frac{e^{-2\delta x} - e^{2\delta y}}{1 - e^{2\delta y}}, \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp(-\frac{\delta^2}{2} T_x^0)\right] \frac{e^{-\delta x} - e^{2\delta y + \delta x}}{1 - e^{2\delta y}}. \end{aligned}$$

Replacing δ by $-\delta$, we obtain similarly

$$\begin{aligned} \tilde{A} &= \mathbb{E}\left[F\left(B(t) - \delta t, 0 \leq t \leq T_x^{-\delta}\right) | T_x^{-\delta} < T_{-y}^{-\delta}\right], \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp(-\delta B(T_x^0) - \frac{\delta^2}{2} T_x^0)\right] \\ &\quad \times \frac{e^{2\delta x} - e^{-2\delta y}}{1 - e^{-2\delta y}}, \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp(-\frac{\delta^2}{2} T_x^0)\right] \frac{e^{\delta x} - e^{-2\delta y - \delta x}}{1 - e^{-2\delta y}}. \end{aligned}$$

□

We are now able to describe the law of $(B_\delta(t); 0 \leq t \leq \rho)$ conditionally to S.

Lemma 3.5 *1) The r.v. S has a density function φ given by (3.12).*

2) Conditionally to $S = b$, $(B_\delta(t)\,;\, 0 \leq t \leq \rho)$ is a process with the same law as a Brownian motion with drift $-\delta$, stopped at its first hitting time of level $-b$ and conditioned to stay less than a.

Proof of Lemma 3.5 1) Let f and F be two non-negative test functions and $\Delta = \mathbb{E}\left[f(S)F(B_\delta(s), 0 \leq s \leq \rho)\right]$.
We have :

$$\Delta = \mathbb{E}\left[f(\underline{B_\delta}(\rho))F(B_\delta(s), 0 \leq s \leq \rho)\right].$$

Setting $U_t = f(\underline{B_\delta}(t))F(B_\delta(s), 0 \leq s \leq t)$, we have : $\Delta = \mathbb{E}\left[U_\rho\right]$. Since $(U_t, t \geq 0)$ is a predictable process, property (3.15) implies that :

$$\Delta = \mathbb{E}\left[\int_0^\infty U_t \, dA_t^\rho\right].$$

Using (3.19), we obtain

$$\Delta = 2\delta\mathbb{E}\left[\int_0^{T_a^\delta} f(\underline{B_\delta}(t))F(B_\delta(s), 0 \leq s \leq t)\frac{e^{2\delta \underline{B_\delta}(t)}}{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} d\underline{B_\delta}(t)\right].$$

In particular, if $F = 1$, the *change of variable* : “$x = \underline{B_\delta}(t)$” yields to :

$$\begin{aligned}
\mathbb{E}\left[f(S)\right] &= 2\delta\mathbb{E}\left[\int_0^S f(x)\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\, dx\right], \\
&= 2\delta\mathbb{E}\left[\int_0^\infty f(x)\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\mathbb{P}(S \geq x)\, dx\right].
\end{aligned} \tag{3.26}$$

As a result

$$\mathbb{P}(S \in dx) = 2\delta\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\mathbb{P}(S \geq x)\, dx.$$

Hence, S has a density function φ and :

$$\varphi(x) = 2\delta\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\int_x^\infty \varphi(s)\, ds. \tag{3.27}$$

(3.27) may be written as a linear differential equation with respect to $\Phi(x) = \int_x^\infty \varphi(s)\, ds$. It is easy to solve explicitly since $\Phi(0) = 1$:

$$\Phi(x) = \int_x^\infty \varphi(s)\, ds = \frac{1 - e^{-2\delta a}}{e^{2\delta x} - e^{-2\delta a}}. \tag{3.28}$$

This implies (3.12).

2) We now study the law of $(B_\delta(t)\,;\,0\le t\le\rho)$. $(T^\delta_{-x}\,;\,x\ge 0)$ being the right continuous inverse of $(\underline{B_\delta}(t)\,;\,t\ge 0)$, then

$$\begin{aligned}\Delta &= 2\delta\int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s),0\le s\le T^\delta_{-x})\mathbb{1}_{\{T^\delta_{-x}<T^\delta_a\}}\right]\frac{e^{2\delta x}}{e^{2\delta x}-e^{-2\delta a}}\,dx,\\ &= 2\delta\int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s),0\le s\le T^\delta_{-x})|T^\delta_{-x}<T^\delta_a\right]\\ &\qquad\times\frac{e^{2\delta x}}{e^{2\delta x}-e^{-2\delta a}}\,\mathbb{P}(T^\delta_{-x}<T^\delta_a)\,dx,\\ &= \int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s),0\le s\le T^\delta_{-x})|T^\delta_{-x}<T^\delta_a\right]\varphi(x)\,dx.\end{aligned}$$

We now use the result of Proposition 3.4 to obtain the law of $(B_\delta(t)\,;\,0\le t\le\rho)$ conditionally to S. □

Proof of Proposition 3.2

It remains to determine the law of $(B_\delta(t+\rho)-B_\delta(\rho),0\le t\le T^\delta_a-\rho)$ and to prove that this process is independent of $(B_\delta(t);0\le t\le\rho)$.

Let $(\mathcal{F}^\rho_t)_{t\ge 0}$ denote the smallest filtration including $(\mathcal{F}_t)_{t\ge 0}$ for which ρ is a stopping time. ρ is an honest time since, for any $t>0$, on $\{\rho<t\}$, ρ coincides with a $\mathcal{F}_t$-measurable r.v. (see [9] p 370) That allows to use (3.16). Combining (3.17), (3.20) with (3.18), we obtain

$$\begin{aligned}\chi(t) &= -\int_0^{t\wedge\rho}\frac{(e^{2\delta\underline{B_\delta}(s)}-e^{-2\delta a})2\delta e^{-2\delta B_\delta(s)}}{(e^{-2\delta B_\delta(s)}-e^{-2\delta a})(e^{2\delta\underline{B_\delta}(s)}-e^{-2\delta a})}ds\\ &+\int_0^{t\wedge T^\delta_a}\mathbb{1}_{\{\rho<s\}}\frac{(e^{2\delta\underline{B_\delta}(s)}-e^{-2\delta a})2\delta e^{-2\delta B_\delta(s)}}{(e^{2\delta\underline{B_\delta}(s)}-e^{-2\delta B_\delta(s)})(e^{2\delta\underline{B_\delta}(s)}-e^{-2\delta a})}ds,\end{aligned}$$

$$\chi(t)=-\int_0^{t\wedge\rho}\frac{2\delta e^{-2\delta B_\delta(s)}}{e^{-2\delta B_\delta(s)}-e^{-2\delta a}}\,ds+\int_0^{t\wedge T^\delta_a}\mathbb{1}_{\{\rho<s\}}\frac{2\delta e^{-2\delta B_\delta(s)}}{e^{2\delta\underline{B_\delta}(s)}-e^{-2\delta B_\delta(s)}}\,ds.$$

Since $\underline{B_\delta}(s)=S$, then for any $\rho\le s\le T^\delta_a$, we have

$$\chi(t+\rho)-\chi(\rho)=2\delta\int_\rho^{(t+\rho)\wedge T^\delta_a}\frac{e^{-2\delta B_\delta(s)}}{e^{2\delta S}-e^{-2\delta B_\delta(s)}}\,ds;\ 0\le t\le T^\delta_a-\rho.$$

Let $(W(t))$ and $\left(\widehat{Z}(t)\right)$ be the processes

$$W(t)=\bar{B}(t+\rho)-\bar{B}(\rho)\,;\,t\ge 0,\quad \widehat{Z}(t)=B_\delta(\rho+t)-B_\delta(\rho)=B_\delta(\rho+t)+S\,;\,0\le t\le T^\delta_a-\rho.$$

According to (3.16), for any $t \in [0, T_a^\delta - \rho]$ we have :

$$\begin{aligned}\widehat{Z}(t) &= W(t) + \delta t + 2\delta \int_0^t \frac{e^{-2\delta B_\delta(s+\rho)}}{e^{2\delta S} - e^{-2\delta B_\delta(s+\rho)}}\, ds.\\ &= W(t) + \delta \int_0^t \coth(\delta \widehat{Z}(s))ds.\end{aligned}$$

We know that $\big(\bar{B}(t)\big)$ is a $\big(\mathcal{F}^\rho_{\rho+t}\big)$- Brownian motion. Therefore $(W(t))$ is independent of $\mathcal{F}^\rho_\rho$. Since $S = -B_\delta(\rho)$ and $(B_\delta(t); 0 \le t \le \rho)$ are $\mathcal{F}^\rho_\rho$-measurable, then $(W(t))$ is independent of S and $(B_\delta(t); 0 \le t \le \rho)$. Furthermore, $T_a^\delta - \rho = \inf\{t \ge 0\,;\, \widehat{Z}(t) = a + b\}$, with $b = S$. To conclude the proof of Proposition 3.2 (part 1 and 3), we use the fact that stochastic differential equation (1.10) admits a unique non negative strong solution. We can prove this property with a similar method as for a Bessel process (we apply Yamada Watanabe Theorem to the square of $Z^{(\delta)}$). □

Remark 3.6 Conditionally to $\{S \le z\}$, the density of S is :

$$\frac{\delta \operatorname{sh} \delta a \operatorname{sh} \delta(z+a)}{\operatorname{sh} \delta z} \frac{1}{\operatorname{sh}^2 \delta(x+a)} \mathbb{1}_{\{0 \le x \le z\}}. \tag{3.29}$$

Formula (3.29) is really a direct consequence of (3.28).

Proofs of Theorem 2.4 and Proposition 2.1
The processes $(-B_\delta(t)\,;\, t \ge 0)$ and $(B_{-\delta}(t)\,;\, t \ge 0)$ have the same law. Consequently Proposition 3.2 admits a version where $(B_\delta(t)\,;\, 0 \le t \le T_a^\delta)$ (resp. S) is replaced by $(B_{-\delta}(t)\,;\, 0 \le t \le T_{-a}^{-\delta})$ (resp. $\sup\limits_{0 \le t \le T_{-a}^{-\delta}} B_{-\delta}(t)$). Then Theorem 2.4 and Proposition 2.1 can be proved by induction on k. □

4 First range time, range at a fixed time, asymptotic results

In this section, we focus on B_δ stopped at its first range time $\theta_\delta(a)$ defined by (1.5). The first result concerns the law of $(B_\delta(t)\,;\, 0 \le t \le \theta_\delta(a))$. We determine the joint law of $(B_\delta(t)\,;\, 0 \le t \le \tilde{\theta_\delta}(a))$ and $(B_\delta(t + \tilde{\theta_\delta}(a)) - B_\delta(\tilde{\theta_\delta}(a))\,;\, 0 \le \theta_\delta(a) - \tilde{\theta_\delta}(a))$ where

$$\tilde{\theta_\delta}(a) = \begin{cases} \sup\{t \le \theta_\delta(a)\,;\, B_\delta(t) = \inf\limits_{0 \le u \le \theta_\delta(a)} B_\delta(u)\} \text{ if } B_\delta(\theta_\delta(a)) > 0 \\ \\ \sup\{t \le \theta_\delta(a)\,;\, B_\delta(t) = \sup\limits_{0 \le u \le \theta_\delta(a)} B_\delta(u)\} \text{ otherwise .} \end{cases} \tag{4.1}$$

Proposition 4.1 *1. The r.v. $B_\delta(\tilde{\theta_\delta}(a))$ has density function f :*

$$f(u) = \frac{2\delta}{(1-e^{-2\delta a})^2}(e^{2\delta u} - e^{-2\delta a})\mathbb{1}_{\{-a\le u<0\}} + \frac{2\delta}{(e^{2\delta a}-1)^2}(e^{2\delta a} - e^{2\delta u})\mathbb{1}_{\{0\le u\le a\}}. \tag{4.2}$$

2. Conditionally to $\{B_\delta(\tilde{\theta_\delta}(a)) = u\}$,

a. the processes $(B_\delta(s)\,;\,0\le s\le \tilde{\theta_\delta}(a))$ and $(B_\delta(s+\tilde{\theta_\delta}(a)) - u\,;\,0\le s\le \theta(a)-\tilde{\theta_\delta}(a))$ are independent.

b. $(B_\delta(s)\,;\,0\le s\le\tilde{\theta_\delta}(a))$ has the same law as a Brownian motion with drift δ stopped at its first hitting time of level u, conditioned by hitting u before $u - sgn(u)a$.

c. $(|B_\delta(s+\tilde{\theta_\delta}(a)) - u|\,;\,0\le s\le\theta(a)-\tilde{\theta_\delta}(a))$ has the same law as the process $Z^{(\delta)}$ defined by (1.10), stopped at its first hitting time of level a.

A similar result was obtained by Vallois [13] when $\delta = 0$.

As for the proof of Proposition 3.2, a straightforward approach may be developed using enlargement of filtrations, since $\tilde{\theta_\delta}(a)$ is an honest time. More precisely, the optional projection of $\left(\mathbb{1}_{\{\tilde{\theta_\delta}(a)\le t\}}; t\ge 0\right)$ can be decomposed as :

$$\mathbb{P}\left(\tilde{\theta_\delta}(a)\le t|\mathcal{F}_t\right) = M_t + A_t,$$

where $(M_t; t\ge 0)$ is a martingale and :

$$A_t = \frac{2\delta e^{-\delta a}}{e^{\delta a}-e^{-\delta a}}\overline{B_\delta}(t\wedge\theta(a)) - \frac{2\delta e^{\delta a}}{e^{\delta a}-e^{-\delta a}}\underline{B_\delta}(t\wedge\theta(a)). \tag{4.3}$$

The details are left to the reader.

Proposition 4.1 allows us to determine the Laplace transform of $\theta_\delta(a)$. Let $\sigma_\delta(a)$ be the stopping time

$$\sigma_\delta(a) = \inf\left\{s\ge 0\ ,\ Z^{(\delta)}(s) > a\right\}. \tag{4.4}$$

Proposition 4.2 *The Laplace transforms of $\tilde{\theta_\delta}(a)$, $\sigma_\delta(a)$ and $\theta_\delta(a)$ are given by :*

$$\mathbb{E}\left[e^{-\lambda\tilde{\theta_\delta}(a)}\right] = \frac{\delta}{\lambda}\left(\sqrt{2\lambda+\delta^2}\coth(a\sqrt{2\lambda+\delta^2})\coth(\delta a) - \delta - \frac{\sqrt{2\lambda+\delta^2}}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})\,\mathrm{sh}(\delta a)}\right), \tag{4.5}$$

$$\mathbb{E}\left[e^{-\lambda\sigma_\delta(a)}\right] = \frac{\sqrt{2\lambda+\delta^2}}{\delta}\frac{\operatorname{sh}(\delta a)}{\operatorname{sh}(a\sqrt{2\lambda+\delta^2})}, \tag{4.6}$$

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda\theta(a)}\right] = \frac{\sqrt{2\lambda+\delta^2}}{\lambda}\Bigg[&\frac{\sqrt{2\lambda+\delta^2}\operatorname{ch}(a\sqrt{2\lambda+\delta^2})\ \operatorname{ch}(\delta a)}{\operatorname{sh}^2(a\sqrt{2\lambda+\delta^2})}\\ &-\frac{\delta\operatorname{sh}(\delta a)}{\operatorname{sh}(a\sqrt{2\lambda+\delta^2})}-\frac{\sqrt{2\lambda+\delta^2}}{\operatorname{sh}^2(a\sqrt{2\lambda+\delta^2})}\Bigg]\end{aligned} \tag{4.7}$$

where $\lambda \geq 0$.

Remark 4.3 1) In [16], using approximation by random walks, it is proved :

$$\mathbb{E}\left[\theta_\delta(a)\right] = \frac{a^2}{2}f(\delta a), \tag{4.8}$$

where

$$f(x) = \frac{1}{x^2}(x - x\coth x + 1)(x + x\coth x - 1)\,;\ x > 0. \tag{4.9}$$

Moreover f is decreasing, in particular :

$$\mathbb{E}\left[\theta_\delta(a)\right] \leq \mathbb{E}\left[\theta_0(a)\right] = \frac{a^2}{2}. \tag{4.10}$$

The variance of $\theta_\delta(a)$ is computed :

$$\operatorname{Var}\left[\theta_\delta(a)\right] = \frac{a^4}{12}g_1(\delta a)g_2(\delta a), \tag{4.11}$$

with

$$g_1(x) = \frac{3\operatorname{sh}^2 x - x^2}{x^2\operatorname{sh}^2 x}\ ;\ g_2(x) = \frac{x^2\coth^2 x + 4x\coth x - 5 - x^2}{x^2}\ ;\ x > 0.$$

Moreover,

$$\operatorname{Var}\left(\theta_\delta(a)\right) \leq \operatorname{Var}\left(\theta_0(a)\right) = \frac{a^4}{12}.$$

2) It is possible to prove Propositions 4.1 and 4.2 using Theorems 2.2, 2.4 and the identity : $\Omega = \cup_{n\geq 0}\left\{\rho_{n+1} \leq \tilde{\theta}_\delta(a) < \rho_n\right\}$, where the sequence $(\rho_n)_{n\geq 0}$ is defined at the beginning of Section 2. It is however easier and shorter to prove directly Propositions 4.1 and 4.2.

Proof of Proposition 4.2
We only give the main ideas of the proof, the details being left to the reader.

Let f_λ be the function :

$$f_\lambda(x) = \frac{\delta}{\sqrt{2\lambda+\delta^2}}\frac{\operatorname{sh}(x\sqrt{2\lambda+\delta^2})}{\operatorname{sh}(\delta x)}\ ;\ x > 0. \tag{4.12}$$

It is easy to check that f_λ is an eigenfunction of the infinitesimal generator associated with $Z^{(\delta)}$:

$$\frac{1}{2}f''_\lambda(x) + \delta\coth(\delta x)f'_\lambda(x) = \lambda f_\lambda(x)\,;\; x > 0. \tag{4.13}$$

The function f_λ being locally bounded on $[0, +\infty)$, then

$$\mathbb{E}_0\left[\exp(-\lambda\sigma_\delta(a))\right] = \frac{f_\lambda(0)}{f_\lambda(a)}. \tag{4.14}$$

This proves (4.6).

Recall (cf formulas 2.1.4 and 2.2.4 p. 295 of [1])

$$\mathbb{E}\left[e^{-\lambda T_u^\delta}|T_u^\delta < T_{u+a}^\delta\right] = \frac{\mathrm{sh}((a+u)\sqrt{2\lambda+\delta^2})}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})}\frac{\mathrm{sh}(\delta a)}{\mathrm{sh}(\delta(a+u))}, \tag{4.15}$$

$$\mathbb{E}\left[e^{-\lambda T_u^\delta}|T_u^\delta < T_{u-a}^\delta\right] = \frac{\mathrm{sh}((a-u)\sqrt{2\lambda+\delta^2})}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})}\frac{\mathrm{sh}(\delta a)}{\mathrm{sh}(\delta(a-u))}. \tag{4.16}$$

(4.5) is a direct consequence of (3.21) and Proposition 4.1. □

By inversion of the Laplace transform of $\theta_\delta(a)$, the authors in [2] have computed the probability density function of this r.v.

We develop an alternative approach based on the knowledge (c.f. [1], formula 1.15.8 (1) p. 271) of the joint distribution of $(B_\delta(t), R_\delta(t))$. By tedious calculations (see Section 5), we determine the probability distribution function of $R_\delta(t)$.

Relation (1.3) allows us to obtain the probability distribution function of $\theta_\delta(a)$ and the rate of decay of $\mathbb{P}(\theta_\delta(a) > t)$, as $t \to \infty$.

Proposition 4.4 *Let $a > 0$, $t > 0$ and $C_k = k^2\pi^2 + a^2\delta^2$, $k \in \mathbb{N}$. Then :*

$$\mathbb{P}\left(R_\delta(t) < a\right) = \sum_{k=1}^{\infty}\frac{4k^2\pi^2}{C_k^2}\exp\left(-\frac{C_k t}{2a^2}\right)\left\{\left(1-(-1)^k\,\mathrm{ch}(\delta a)\right)\right.$$
$$\left.\times\left(1+\frac{k^2\pi^2 t}{a^2}-\frac{4a^2\delta^2}{C_k}\right)-(-1)^k a\delta\,\mathrm{sh}(\delta a)\right\}. \tag{4.17}$$

$$\mathbb{P}\left(\theta_\delta(a) > t\right) \underset{t\to\infty}{\sim} \frac{4\pi^4\left(1+\mathrm{ch}(\delta a)\right)}{a^2\left(\pi^2+a^2\delta^2\right)}t\exp\left(-\frac{\pi^2+a^2\delta^2}{2a^2}t\right). \tag{4.18}$$

Remark 4.5 1) Formula (4.17) has been obtained in [12], the approach being different.

2) Taking the a-derivative in (4.17) gives the density function of $R_\delta(t)$.

3) Relation (1.3) implies that :

$$\mathbb{P}\left(\theta_\delta(a) > t\right) = \sum_{k=1}^{\infty} \frac{4k^2\pi^2}{C_k^2} \exp\left(-\frac{C_k t}{2a^2}\right) \left\{ \left(1 - (-1)^k \operatorname{ch}(\delta a)\right) \times \left(1 + \frac{k^2\pi^2 t}{a^2} - \frac{4a^2\delta^2}{C_k}\right) - (-1)^k a\delta \operatorname{sh}(\delta a) \right\}. \tag{4.19}$$

Again, taking the t-derivative in (4.19), we obtain the density function of $\theta_\delta(a)$. However that series expension is more complicated than (4.19) (c.f. also Theorem 9 of [2]) : it is more convenient to use probability distribution function instead of probability density function.

The law of $\theta_\delta(a)$ or $R_\delta(t)$ being complicated, it seems natural to consider the asymptotic behaviour of $\theta_\delta(a)$ (resp. $R_\delta(t)$) when a goes to $+\infty$ (resp. $t \to +\infty$).

Proposition 4.6 *The asymptotic comportments of $\theta_\delta(a)$ and $R_\delta(t)$ are given by the following convergence results :*

$$\frac{\theta_\delta(a)}{a} \xrightarrow[a\to\infty]{a.s.} \frac{1}{\delta}, \tag{4.20}$$

$$\frac{R_\delta(t)}{t} \xrightarrow[t\to\infty]{a.s.} \delta, \tag{4.21}$$

$$\delta^{3/2}\sqrt{a}\left(\frac{\theta_\delta(a)}{a} - \frac{1}{\delta}\right) \xrightarrow[a\to\infty]{d} \mathcal{N}\left(0,1\right), \tag{4.22}$$

$$\sqrt{t}\left(\frac{R_\delta(t)}{t} - \delta\right) \xrightarrow[t\to\infty]{d} \mathcal{N}(0,1). \tag{4.23}$$

Proof 1) We first examine (4.20). By (4.7), we have :

$$\mathbb{E}\left[e^{-\frac{\lambda}{a}\theta_\delta(a)}\right] = \frac{\sqrt{\frac{2\lambda}{a}+\delta^2}}{\frac{\lambda}{a}} \left[\frac{\sqrt{\frac{2\lambda}{a}+\delta^2}\operatorname{ch}(a\sqrt{\frac{2\lambda}{a}+\delta^2})\operatorname{ch}(\delta a)}{\operatorname{sh}^2(a\sqrt{\frac{2\lambda}{a}+\delta^2})} - \frac{\delta\operatorname{sh}(\delta a)}{\operatorname{sh}(a\sqrt{\frac{2\lambda}{a}+\delta^2})} - \frac{\sqrt{\frac{2\lambda}{a}+\delta^2}}{\operatorname{sh}^2(a\sqrt{\frac{2\lambda}{a}+\delta^2})}\right].$$

$$\mathbb{E}\left[e^{-\frac{\lambda\theta_\delta(a)}{a}}\right] = \frac{a\left(\delta + \frac{\lambda}{\delta a} + o\left(\frac{1}{a}\right)\right)}{\lambda} \left[\frac{\left(\delta + \frac{\lambda}{\delta a} + o\left(\frac{1}{a}\right)\right)\operatorname{ch}(\delta a + \frac{\lambda}{\delta} + o(1))\operatorname{ch}(\delta a)}{\operatorname{sh}^2(\delta a + \frac{\lambda}{\delta} + o(1))} - \frac{\delta\operatorname{sh}(\delta a)}{\operatorname{sh}(\delta a + \frac{\lambda}{\delta} + o(1))} + \frac{\delta + \frac{\lambda}{\delta a} + o\left(\frac{1}{a}\right)}{\operatorname{sh}^2(\delta a + \frac{\lambda}{\delta} + o(1))}\right].$$

The limit of the two first terms in the bracket is easy to determine :

$$\frac{\operatorname{sh}(\delta a)}{\operatorname{sh}(\delta a + \frac{\lambda}{\delta} + o(1))} = \frac{e^{\delta a} - e^{-\delta a}}{e^{\delta a + \frac{\lambda}{\delta} + o(1)} - e^{-\delta a - \frac{\lambda}{\delta} + o(1)}} \underset{a \to \infty}{\longrightarrow} e^{-\frac{\lambda}{\delta}},$$

$$\frac{\operatorname{ch}(\delta a + \frac{\lambda}{\delta} + o(1)) \operatorname{ch}(\delta a)}{\operatorname{sh}^2(\delta a + \frac{\lambda}{\delta} + o(1))} \underset{a \to \infty}{\longrightarrow} e^{-\frac{\lambda}{\delta}}.$$

As for the third term, it may be neglected (being equivalent to $4\delta \exp(-2\delta a)$).

Hence $\theta_\delta(a)/a$ converges in distribution to the constant $1/\delta$. This implies that $\theta_\delta(a)/a$ converges in probability to $1/\delta$. Since $(R_\delta(t)\,;\, t \geq 0)$ is the right continuous inverse of $(\theta_\delta(a)\,;\, a \geq 0)$, $R_\delta(t)/t$ converges in probability to δ.

$(R_\delta(t)\,;\, t \geq 0)$ is a subadditive process (cf [3], example 6.2 p.320, in the discrete case). The subadditive ergodic theorem implies that $R_\delta(t)/t$ converges a.s., as $t \to \infty$. As a result, $R_\delta(t)/t$ converges a.s. towards δ when $t \to \infty$.

2) To prove (4.22), we use the characteristic function of $\theta_\delta(a)$. This function can be explicitly determined through (4.7) and analytic continuation argument. In the discrete case (cf proof of Theorem 20 of [15]) a detailed approach is developed.

3) We claim that $\left(\frac{R_\delta(t)}{t} - \delta\right)\sqrt{t}$ converges in distribution to $\mathcal{N}(0,1)$ when $t \to +\infty$.

Let $a > 0$ and $t > 0$ such that $a\sqrt{t} + \delta t > 0$. We have :

$$p = \mathbb{P}\left(\left(\frac{R_\delta(t)}{t} - \delta\right)\sqrt{t} < a\right) = \mathbb{P}\left(R_\delta(t) < s\right),$$

where $s = a\sqrt{t} + \delta t$.

Property (1.3) implies that

$$p = \mathbb{P}\left(\theta_\delta(s) > t\right) = \mathbb{P}\left(\left(\frac{\theta_\delta(s)}{s} - \frac{1}{\delta}\right)\sqrt{s} > u(t)\right),$$

where

$$u(t) = \left(\frac{t}{s} - \frac{1}{\delta}\right)\sqrt{s} = -\frac{a}{\delta\sqrt{\delta + \frac{a}{\sqrt{t}}}}.$$

Since $t \to +\infty$ implies $s \to +\infty$, and $u(t) \underset{t \to \infty}{\sim} -\frac{a}{\delta^{3/2}}$, (4.23) follows immediately. □

5 Proof of Proposition 4.4

In order to calculate the probability distribution function of $R_\delta(a)$, we perform the following calculations. Beginning with the formula 1.15.8 (1) p. 271

of [1] :

$$\mathbb{P}\left(R_\delta(t) < a\right) = \frac{1}{\sqrt{2\pi t}} e^{-\dfrac{\delta^2 t}{2}} \sum_{k\in\mathbb{Z}} \mu_k,$$

with

$$\mu_k = \int_{-a}^{a} \left(2k+1-\frac{2k(a-|z|)(|z|+2ka)}{t}\right)\exp\left(\delta z - \frac{(|z|+2ka)^2}{2t}\right) dz.$$

Recalling the Poisson formula

$$\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{(x+2ky)^2}{2t}\right) = \frac{1}{2y}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{k^2\pi^2}{2y^2}t\right)\exp\left(\frac{ik\pi x}{y}\right). \quad (5.1)$$

Let us compute the y-derivative and the x-derivative :

$$\begin{aligned}\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\frac{2k}{t}(x+2ky)\exp\left(-\frac{(x+2ky)^2}{2t}\right) &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{k^2\pi^2}{2y^2}t\right) \\ &\times\exp\left(\frac{ik\pi x}{y}\right)\left(\frac{1}{y^2}-\frac{k^2\pi^2 t}{y^4}+\frac{ik\pi x}{y^3}\right),\end{aligned} \quad (5.2)$$

$$\begin{aligned}\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\left(\frac{x+2ky}{t}\right)\exp\left(-\frac{(x+2ky)^2}{2t}\right) &= -\frac{1}{2y}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{k^2\pi^2}{2y^2}t\right) \\ &\times\exp\left(\frac{ik\pi x}{y}\right)\frac{ik\pi}{y}.\end{aligned} \quad (5.3)$$

Replacing x with $|z|$ and y with a in $\frac{t}{y}\times(5.3)+(1-\frac{x}{y})\times(5.1)$, we obtain

$$\begin{aligned}\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}(2k+1)\exp\left(-\frac{(|z|+2ka)^2}{2t}\right) &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{k^2\pi^2 t}{2a^2}\right) \\ &\times\exp\left(\frac{ik\pi|z|}{a}\right)\left(-\frac{ik\pi t}{2a^3}-\frac{|z|-a}{a^2}\right).\end{aligned} \quad (5.4)$$

Replacing x with $|z|$ and y with a in $-(y-x)\times(5.2)$ and adding (5.4), we obtain $\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\mu_k = \frac{1}{2}\sum_{k\in\mathbb{Z}} I_k$, where

$$\begin{aligned}I_k &= \int_{-a}^{a} e^{\delta z}\exp\left(-\frac{k^2\pi^2 t}{2a^2}\right)\exp\left(\frac{ik\pi|z|}{a}\right) \\ &\times\left(-\frac{ik\pi t}{a^3}+(|z|-a)\left(\frac{ik\pi|z|}{a^3}-\frac{k^2\pi^2 t}{a^4}\right)\right)dz.\end{aligned} \quad (5.5)$$

Let us introduce :

$$
\begin{aligned}
C_1(\delta) &= \int_0^a \exp\left(\delta z + \frac{ik\pi z}{a}\right) dz = \frac{a}{a\delta + ik\pi}\left[(-1)^k e^{\delta a} - 1\right], \\
C_2(\delta) &= \int_0^a (z-a)\exp\left(\delta z + \frac{ik\pi z}{a}\right) dz = \frac{a^2}{a\delta + ik\pi} + \frac{a^2(1-(-1)^k e^{\delta a})}{(a\delta + ik\pi)^2}, \\
C_3(\delta) &= \int_0^a z(z-a)\exp\left(\delta z + \frac{ik\pi z}{a}\right) dz \\
&= -\frac{a^3\left(1+(-1)^k e^{\delta a}\right)}{(a\delta + ik\pi)^2} + \frac{2a^3\left((-1)^k e^{\delta a} - 1\right)}{(a\delta + ik\pi)^3}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
I_k \exp\left(\frac{k^2\pi^2 t}{2a^2}\right) &= -\frac{ik\pi t}{a^3}(C_1(\delta) + C_1(-\delta)) - \frac{k^2\pi^2 t}{a^4}(C_2(\delta) + C_2(-\delta)) \\
&\quad + \frac{ik\pi}{a^3}(C_3(\delta) + C_3(-\delta)).
\end{aligned} \tag{5.6}
$$

We decompose $C_i(\delta) + C_i(-\delta)$, $1 \le i \le 3$, as follows :

$$
\begin{aligned}
C_1(\delta) + C_1(-\delta) &= -aA_1 + (-1)^k A_2, \\
C_2(\delta) + C_2(-\delta) &= a^2 A_1 + a^2 A_3 - (-1)^k a^2 A_4, \\
C_3(\delta) + C_3(-\delta) &= -a^3 A_3 - (-1)^k a^3 A_4 - 2a^3 A_5 + (-1)^k 2a^3 A_6,
\end{aligned}
$$

where we have set :

$$
\begin{aligned}
A_1 &= \frac{1}{\delta a + ik\pi} + \frac{1}{-\delta a + ik\pi} = \frac{-2ik\pi}{C_k} \\
A_2 &= \frac{e^{\delta a}}{\delta a + ik\pi} + \frac{e^{-\delta a}}{-\delta a + ik\pi} = \frac{2\,\mathrm{sh}(\delta a)a\delta - 2\,\mathrm{ch}(\delta a)ik\pi}{C_k} \\
A_3 &= \frac{1}{(\delta a + ik\pi)^2} + \frac{1}{(-\delta a + ik\pi)^2} = \frac{2(a^2\delta^2 - k^2\pi^2)}{C_k^2} \\
A_4 &= \frac{e^{\delta a}}{(\delta a + ik\pi)^2} + \frac{e^{-\delta a}}{(-\delta a + ik\pi)^2} \\
&= \frac{2\,\mathrm{ch}(\delta a)(a^2\delta^2 - k^2\pi^2) - 4ia\delta k\pi\,\mathrm{sh}(\delta a)}{C_k^2} \\
A_5 &= \frac{1}{(\delta a + ik\pi)^3} + \frac{1}{(-\delta a + ik\pi)^3} = \frac{-6\delta^2 a^2 ik\pi + 2ik^3\pi^3}{C_k^3} \\
A_6 &= \frac{e^{\delta a}}{(\delta a + ik\pi)^3} + \frac{e^{-\delta a}}{(-\delta a + ik\pi)^3}, \\
&= \frac{(2\delta^3 a^3 - 6k^2\pi^2\delta a)\,\mathrm{sh}(\delta a) + i(-6\delta^2 a^2 k\pi + 2k^3\pi^3)\,\mathrm{ch}(\delta a)}{C_k^3}.
\end{aligned}
$$

Coming back to (5.4), we easily obtain successively :

$$\begin{aligned}
I_k \exp\left(\frac{k^2\pi^2 t}{2a^2}\right) &= \left(\frac{ik\pi t}{a^2} - \frac{k^2\pi^2 t}{a^2}\right) A_1 - (-1)^k \frac{ik\pi t}{a^2} A_2 \\
&\quad - \left(\frac{k^2\pi^2 t}{a^2} + ik\pi\right) A_3 + (-1)^k \left(\frac{k^2\pi^2 t}{a^2} - ik\pi\right) A_4 \\
&\quad -2ik\pi A_5 + (-1)^k 2ik\pi A_6 \\
&= Re\left(I_k \exp\left(\frac{k^2\pi^2 t}{2a^2}\right)\right) \\
&= \frac{2k^2\pi^2 t}{C_k a^2} - (-1)^k \frac{2k^2\pi^2 t\,\mathrm{ch}(\delta a)}{C_k a^2} - 2\frac{(a^2\delta^2 - k^2\pi^2)k^2\pi^2 t}{C_k^2 a^2} \\
&\quad +(-1)^k \frac{2\,\mathrm{ch}(\delta a)(a^2\delta^2 - k^2\pi^2)k^2\pi^2 t - 4k^2\pi^2 a^3\delta\,\mathrm{sh}(\delta a)}{C_k^2 a^2} \\
&\quad + \frac{4k^4\pi^4 - 12k^2\pi^2 a^2\delta^2}{C_k^3} \\
&\quad +(-1)^k \frac{(12\delta^2 a^2 k^2\pi^2 - 4k^4\pi^4)\,\mathrm{ch}(\delta a)}{C_k^3}.
\end{aligned}$$

In particular :

$$\begin{aligned}
(I_k + I_{-k})\exp\left(\frac{k^2\pi^2 t}{2a^2}\right) &= \frac{k^2\pi^2}{C_k^2}(1 - (-1)^k\,\mathrm{ch}(\delta a)) \\
&\quad \times\left(\frac{4t(k^2\pi^2 + \delta^2 a^2)}{a^2} - 4\frac{(a^2\delta^2 - k^2\pi^2)t}{a^2}\right. \\
&\quad \left. + \frac{8k^2\pi^2 - 24a^2\delta^2}{C_k}\right) \\
&\quad -(-1)^k 8\frac{k^2\pi^2}{C_k^2} a\delta\,\mathrm{sh}(\delta a).
\end{aligned}$$

This achieves the proof of Proposition 4.4. □

References

[1] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[2] K. S. Chong, R. Cowan, and L. Holst. The ruin problem and cover times of asymmetric random walks and Brownian motions. *Adv. in Appl. Probab.*, 32(1) :177–192, 2000.

[3] R. Durrett. *Probability : theory and examples*. Duxbury Press, Belmont, CA, second edition, 1996.

[4] W. Feller. The asymptotic distribution of the range of sums of independent random variables. *Ann. Math. Statistics*, 22 :427–432, 1951.

[5] P. W. Glynn. On the range of a regenerative sequence. *Stochastic Process. Appl.*, 20(1) :105–113, 1985.

[6] P. Hsu and P. March. Brownian excursions from extremes. In *Séminaire de Probabilités, XXII*, volume 1321 of *Lecture Notes in Math.*, pages 502–507. Springer, Berlin, 1988.

[7] J.-P. Imhof. On the range of Brownain motion and its inverse process. *Ann. Prob.*, 13 :1011–1017, 1985.

[8] J.-P. Imhof. A construction of the Brownian path from BES^3 pieces. *Stochastic Process. Appl.*, 43(2) :345–353, 1992.

[9] Ph. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.

[10] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[11] P. Salminen and P. Vallois. On first range times of linear diffusions. *J. Theoret. Probab.*, to appear.

[12] Y. Siebenaler and P. Vallois. Hitting times of the range of a birth and death random walk. *Prépublication de l'Institut Élie Cartan de Nancy*, 1, 1999.

[13] P. Vallois. Diffusion arrêtée au premier instant où l'amplitude atteint un niveau donné. *Stochastics Stochastics Rep.*, 43(1-2) :93–115, 1993.

[14] P. Vallois. Decomposing the Brownian path via the range process. *Stoch. proc. and their app.*, 55 :211–226, 1995.

[15] P. Vallois. The range of a simple random walk on **Z**. *Adv. in Appl. Probab.*, 28(4) :1014–1033, 1996.

[16] P. Vallois and C. S. Tapiero. Range reliability in random walks. *Math. Methods Oper. Res.*, 45(3) :325–345, 1997. Stochastic models of reliability.

[17] D. Williams. Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc. (3)*, 28 :738–768, 1974.