New decompositions of 2-structures

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Abstract

We present a family of decompositions of 2-structures generalizing the modular decomposition, and $O(n^3)$ time algorithms to compute all these decompositions. These results can be applied to non-oriented, oriented and directed graphs. Bi-join decomposition of non-oriented graphs and of tournaments are two special cases of this family of decomposition. Two others special cases are generalisations of the bi-join decomposition on directed graphs.

1 Introduction

The well-known modular decomposition of graph has many applications in graph theory and algorithms. It is unique [8] and can be computed in linear time (*i.e.* in $O(n+m)$) on non-oriented graphs [11], on directed graphs [10], and in linear time (*i.e.* in $O(n^2)$) on 2-structures [9]. The bi-join decomposition is a generalisation of the modular decomposition on non-oriented graphs [13, 14] and on tournaments [2]. These two decompositions can be computed in linear time.

We present a family of decompositions of 2-structures which generalize the modular decomposition. We show that these decompositions are unique, and we present an algorithm to compute them in time $O(n^3)$ (for a fixed decomposition in the family). We apply these results to oriented and directed graphs. We give two new different decompositions for directed graphs which generalize the bi-join decomposition of non-oriented graphs and tournaments, and we give a new decomposition for oriented graphs. Bi-join decomposition of non oriented graphs and bi-join decomposition of tournament are also special cases of this family of decompositions.

After some preliminaries in section 2, we introduce in section 3 the G-joins and show that G-joins have the bipartitive property. In section 4 we define the G-join decomposition. For any fixed abelian group with some properties, there is a different G-join decomposition. In section 5 we give some special cases of decompositions on non-oriented, oriented and directed graphs. Finally, we present an $O(n^3)$ algorithm to compute the G-join decomposition in section 6, for any fixed abelian group.

2 Preliminaries

2.1 Graphs and 2-structures

A *directed graph* $G = (V, A)$ is a pair of a set of *vertices* V and a set of *arcs* $A \subseteq V \times V \setminus \{(u, u) : u \in V\}$. A *non-oriented graph* is a directed graph such that for all $(u, v) \in V^2$, with $u \neq v$, then

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$(u, v) \in A$ if and only if $(v, u) \in A$. An *oriented* graph is a directed graph such that for all (u, v), $(u, v) \in A \Rightarrow (v, u) \notin A$. A *tournament* is a oriented graph such that either $(u, v) \in A$ or $(v, u) \in A$.

Let $\mathcal{D}$ be a set. A 2-*structure on* $\mathcal{D}$ is a pair (V, e) such that $e : V \times V \rightarrow \mathcal{D}$. In this paper, every 2-structure is finite (*i.e.* V is finite). A 2-structure is *symmetric* if $e(u, v) = e(v, u)$ for all $u, v \in V$, $u \neq v$. Let σ be an involution on $\mathcal{D}$ (*i.e* a bijection such that $\sigma(\sigma(x)) = x$ for all $x \in \mathcal{D}$). A 2-structure (V, e) on $\mathcal{D}$ is σ-*symmetric* if $e(u, v) = \sigma(e(v, u))$ for all $u, v \in \mathcal{D}$, $u \neq v$.

A directed graph can be viewed as a 2-structure on $\mathbb{Z}_2$, a non oriented graph as a symmetric 2-structure on $\mathbb{Z}_2$, and a tournament as a σ-symmetric 2-structure on $\mathbb{Z}_2$ with $\sigma = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. There is a way to transform a 2-structure on $\mathcal{D}$ into a σ-symmetric 2-structure on $\mathcal{D} \times \mathcal{D}$: take $e'(u, v) = (e(u, v), e(v, u))$ and $\sigma((i, j)) = (j, i)$ for all $i, j \in \mathcal{D}$. For example a directed graph can also be viewed as a σ-symmetric 2-structure on $\mathbb{Z}_2 \times \mathbb{Z}_2$, with $\sigma = \left[\begin{smallmatrix} (0,0) & (0,1) & (1,0) & (1,1) \\ (0,0) & (1,0) & (0,1) & (1,1) \end{smallmatrix}\right]$.

2.2 Bipartitive families

A *bipartition* of a set V is a partition $\{X, Y\}$ of V such that $X \neq \emptyset$ and $Y \neq \emptyset$. We write sometimes $\{X, -\}$ instead of $\{X, V \setminus X\}$. Two bipartitions $\{X, Y\}$ and $\{X', Y'\}$ *overlap* (or $\{X, Y\}$ *overlaps* $\{X', Y'\}$) if $X \cap X'$, $X \cap Y'$, $Y \cap X'$ and $Y \cap Y'$ are non empty. A family $\mathcal{F}$ of bipartitions of V is *weakly bipartitive* if:

- for all $v \in V$, $\{\{v\}, V \setminus \{v\}\}$ is in $\mathcal{F}$, and
- for all $\{X, Y\}$ and $\{X', Y'\}$ in $\mathcal{F}$ such that $\{X, Y\}$ overlaps $\{X', Y'\}$, then $\{X \cap X', Y \cup Y'\}$, $\{X \cap Y', Y \cup X'\}$, $\{Y \cap X', X \cup Y'\}$ and $\{Y \cap Y', X \cup X'\}$ are in $\mathcal{F}$.

Moreover a weakly bipartitive family $\mathcal{F}$ is *bipartitive* if for all $\{X, Y\}$ and $\{X', Y'\}$ which overlap in $\mathcal{F}$, $\{X \Delta X', X \Delta Y'\}$ is in $\mathcal{F}$ (where $A \Delta B = (A \setminus B) \cup (B \setminus A)$). Bipartitive families are close to *partitive families* [3, 12] but deals with bipartitions of V instead of subsets of V.

A member of a bipartitive family is *strong* if it overlaps no other member in the family. A member $\{X, Y\}$ is *trivial* if $|X| = 1$ or $|Y| = 1$. Let $T = (V, E)$ be a tree. We denote by Leaves(T) the leaves of T. For $\beta \in V$, let $\{A^1_\beta, \ldots, A^{d(\beta)}_\beta\}$ be the connected components of $T - \beta$. Let $C^i_\beta = A^i_\beta \cap \text{Leaves}(T)$. For $e \in E$, let A^1_e and A^2_e be the connected components of $T - e$, and let $\{C^1_e, C^2_e\} = \{A^1_e \cap \text{Leaves}(T), A^2_e \cap \text{Leaves}(T)\}$. The following result can be found in [14] or in [6] using a different formalism. This result can also be easily showed from known results of weakly partitive families [3, 12].

Theorem 1. *[6, 14] Let $\mathcal{F}$ be a weakly bipartitive family $\mathcal{F}$ on V. Then there is a unique unrooted tree $T = (V_T, E_T)$, call the* representative tree, *such that* Leaves$(T) = V$, *and each internal node has at least 3 neighbors and is marked* `degenerate`, `linear` *or* `prime`, *such that:*

- *For all $e \in E_T$, $\{C^1_e, C^2_e\}$ is a strong member of $\mathcal{F}$ and there is no other strong members in $\mathcal{F}$.*
- *Let $\beta \in V_T$ be an internal node, and let k be the degree of β.*
  - *If β is* `degenerated`, *then for all $\emptyset \subsetneq I \subsetneq \{1, \ldots, k\}$, $\{\cup_{i \in I} C^i_\beta, -\}$ is in $\mathcal{F}$.*
  - *If β is* `linear`, *there is a ordering $C^1_\beta, \ldots, C^k_\beta$ such that for all $a, b \in \{1, k\}$ with $a \leq b$ and $(a, b) \neq (1, k)$, $\{\cup_{i \in \{a, \ldots, b\}} C^i_\beta, -\}$ is in $\mathcal{F}$.*
- *There is no other members in $\mathcal{F}$.*

Furthermore if $\mathcal{F}$ is bipartitive, then T has no `linear` node.

Decompositions based on bipartitive families have been studied in [6] under a formalism called *decomposition frame* with some properties. Some examples of this decomposition frame can be found in [4, 5]. Bipartitive families based decompositions are interesting since the bipartitivity imply an unique decomposition. Furthermore, this imply that a greedy algorithm to decompose the structure will always work: if we can find in polynomial time a decomposable bipartition in the structure, then we can decompose the whole structure in polynomial time.

2.3 Modular decomposition and bi-join decomposition

A *module* in a 2-structure $G = (V, e)$ is a non-empty $X \subseteq V$ such that for all $v \notin X$ and $u, u' \in X$, $e(v, u) = e(v, u')$ and $e(u, v) = e(u', v)$. The family of modules of a 2-structure is weakly partitive, and is partitive if the 2-structure is symmetric [7]. If a structure G has a non-trivial module X, then it can be decomposed into $G[X]$ and $G[V \setminus X \cup \{x\}]$, where $x \in X$. Note that the structure G can be easily reconstructed from $G[X]$ and $G[V \setminus X \cup \{x\}]$. The *modular decomposition* is defined by recursively decompose the structure by a non-trivial module. It can be represented by a tree, call the *modular decomposition tree*, which is exactly the representative tree of the family of modules.

A *bi-join* in a non-oriented graph $G = (V, E)$ is a bipartition $\{X, Y\}$ of V such that for all $v, v' \in X$, $\{N(v) \cap Y, Y \setminus N(v)\} = \{N(v') \cap Y, Y \setminus N(v')\}$. A *bi-join* in a tournament $G = (V, A)$ is a bipartition $\{X, Y\}$ of V such that for all $v, v' \in X$, $\{N^+(v) \cap Y, Y \setminus N^+(v)\} = \{N^+(v') \cap Y, Y \setminus N^+(v')\}$.

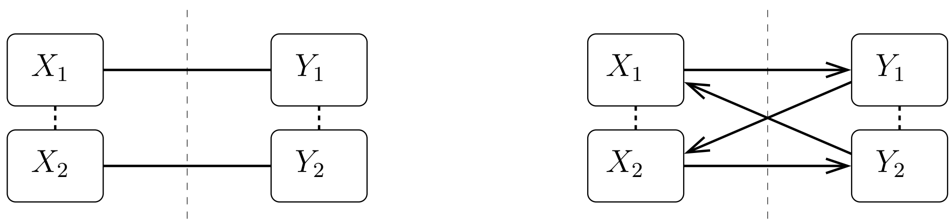

Figure 1: A bi-join in a non-oriented graph and a tournament. $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$.

If $X \subsetneq V$ is a module of G, then it is a bi-join of G. The family of bi-joins of a undirected graph is bipartitive [13], and the family of bi-joins of a tournament is weakly bipartitive [2]. If a graph has a non-trivial bi-join, then it can be decomposed into two graphs, and the *bi-join* decomposition is the recursive decomposition by a strong non-trivial bi-join. Since the family of bi-join is bipartitive, the bi-join decomposition tree is unique (and is isomorphic to the representative tree).

2.4 Abelian group

We recall axioms of an abelian group $(\mathcal{D}, \dot{+})$.

Neutral element: There is an element $\dot{0}$ in $\mathcal{D}$ such that for all a in $\mathcal{D}$, $\dot{0} \dot{+} a = a \dot{+} \dot{0} = a$.

Inverse element: For each a in $\mathcal{D}$ there is an element a^{-1} in $\mathcal{D}$ such that $a \dot{+} a^{-1} = a^{-1} \dot{+} a = \dot{0}$, where $\dot{0}$ is the neutral element. (We will wrote $\dot{-}a$ for a^{-1}.)

Associativity: For all a, b and c in $\mathcal{D}$, $(a \dot{+} b) \dot{+} c = a \dot{+} (b \dot{+} c)$.

Commutativity: For all a and b in $\mathcal{D}$, $a \dot{+} b = b \dot{+} a$.

3 G-joins

3.1 Definition

Throughout this section, we fix an abelian group $(\mathcal{D}, \dot{+})$. Let (V, e) be a 2-structure on $\mathcal{D}$. A pair (X, Y) with $X \neq \emptyset$, $Y \neq \emptyset$ and $V = X \uplus Y$ (*i.e.* $X \cup Y = V$ and $X \cap Y = \emptyset$) is a *G-join* if there is pairwise disjoin X_i and Y_i (for $i \in \mathcal{D}$) such that $X = \biguplus_{i\in\mathcal{D}} X_i$, $Y = \biguplus_{j\in\mathcal{D}} Y_j$, and for all $(i, j) \in \mathcal{D}^2$ and $(u, v) \in (X_i, Y_j)$, $e(u, v) = i \dot{+} j$. We start with some easy observations.

Proposition 2. *If (X, Y) is a G-join of G and $V' \subseteq V$ such that $V' \cap X \neq \emptyset$ and $V' \cap Y \neq \emptyset$, then $(X \cap V', Y \cap V')$ is a G-join of $G[V']$.*

Proposition 3. *If M is a module of (V, e), then $(M, V \setminus M)$ and $(V \setminus M, M)$ are G-joins of (V, e).*

Proposition 4. *For every pairwise different $a, b, c, d \in V$ such that there is a G-join $\{X, Y\}$ with $\{a, c\} \subseteq X$ and $\{b, d\} \subseteq Y$, then $e(c, d) = e(c, b) \dot{+} e(a, d) \dot{-} e(a, b)$.*

Lemma 5. *Let (V, e) be a 2-structure on $\mathcal{D}$. Let (X, Y) and (X', Y') be two G-joins of (V, e) such that $X \cap X' \neq \emptyset$ and $Y \cap Y' \neq \emptyset$. Then $(X \cap X', Y \cup Y')$ is a G-join of (V, e).*

Proof. Let X_i and Y_i (for $i \in \mathcal{D}$) such that $X = \biguplus_{i\in\mathcal{D}} X_i$, $Y = \biguplus_{j\in\mathcal{D}} Y_j$, and for all $(u, v) \in (X_i, Y_j)$, $e(u, v) = i \dot{+} j$. Similarly let X'_i and Y'_i ($i \in \mathcal{D}$) such that $X' = \biguplus_{i\in\mathcal{D}} X'_i$, $Y' = \biguplus_{j\in\mathcal{D}} Y'_j$, and for all $(u, v) \in (X'_i, Y'_j)$, $e(u, v) = i \dot{+} j$.

Since $Y \cap Y'$ is non-empty, let $v \in Y \cap Y'$, and let $j, j' \in \mathcal{D}$ such that $v \in Y_j \cap Y'_{j'}$. Suppose that $w \in X \cap X'$, and let $i, i' \in \mathcal{D}$ such that $w \in X_i \cap X'_{i'}$. Then $e(w, v) = i \dot{+} j = i' \dot{+} j'$. Thus $X \cap X' = \biguplus_{i\in\mathcal{D}} X_i \cap X'_{i\dot{+}j\dot{-}j'}$. Moreover, for all $u \in Y \cap Y'$, $e(w, u) = i \dot{+} k = i' \dot{+} k'$ (with $u \in Y_k \cap Y'_{k'}$), thus $k' = i \dot{-} i' \dot{+} k = j' \dot{-} j \dot{+} k$, and $Y \cap Y' = \biguplus_{k\in\mathcal{D}} Y_k \cap Y'_{k\dot{+}j'\dot{-}j}$.

For all $k \in \mathcal{D}$, let $X''_k = X_k \cap X'_{k\dot{+}j\dot{-}j'}$, and let $Y''_k = Y_k \cup Y'_{k\dot{+}j'\dot{+}j^{-1}}$. $X \cap X' = \biguplus_{i\in\mathcal{D}} X''_i$ and $Y \cup Y' = \biguplus_{k\in\mathcal{D}} Y''_k$. For all $u \in X''_i$ and $v \in Y''_k$, $e(u, v) = i \dot{+} k$. Thus $(X \cap X', Y \cup Y')$ is a G-join. □

3.2 G-joins in σ-symmetric 2-structures

A function f is a *isomorphism* for $(\mathcal{D}, \dot{+})$ if $f(a \dot{+} b) = f(a) \dot{+} f(b)$ for all $(a, b) \in \mathcal{D}^2$. From now σ will denote an involution on $\mathcal{D}$ such that the function $f : a \to \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism for $(\mathcal{D}, \dot{+})$ (where $\dot{0}$ is the neutral element).

Lemma 6. *Let (V, e) is a σ-symmetric 2-structure, and let X and Y such that $V = X \uplus Y$. Then (X, Y) is a G-join if and only if (Y, X) is a G-join.*

Proof. Let X_a and Y_a (for $a \in \mathcal{D}$) such that $X = \biguplus_{a\in\mathcal{D}} X_a$, $Y = \biguplus_{a\in\mathcal{D}} Y_a$, and for all $(u, v) \in (X_a, Y_b)$, $e(u, v) = a \dot{+} b$. Let $X'_a = X_{\sigma(a)}$ and $Y'_a = Y'_{\sigma(a)\dot{-}\sigma(\dot{0})}$, for all $a \in \mathcal{D}$. Since σ is a bijection, $X = \biguplus_{a\in\mathcal{D}} X'_a$ and $Y = \biguplus_{a\in\mathcal{D}} Y'_a$. Moreover, for all $u \in X'_a$ and $v \in Y'_b$, $e(u, v) = \sigma(a) \dot{+} \sigma(b) \dot{-} \sigma(\dot{0}) = f(a) \dot{+} f(b) \dot{+} \sigma(\dot{0}) = f(a \dot{+} b) \dot{+} \sigma(\dot{0}) = \sigma(a \dot{+} b)$, and $e(v, u) = a \dot{+} b$. Thus (Y, X) is a G-join. □

We say that $\{X, Y\}$ is a G-join of (V, e) if (X, Y) is a G-join of (V, e). Lemmas 5 and 6 show that if $\{X, Y\}$ and $\{X', Y'\}$ are two G-joins such that $\{X, Y\}$ overlaps $\{X', Y'\}$, then $\{X \cap X', Y \cup Y'\}$ is a G-join. Therefore we have:

Corollary 7. *The family of G-joins of a σ-symmetric 2-structure is weakly bipartitive.*

3.3 G-joins in symmetric 2-structures

Lemma 8. *Let (V,e) be a symmetric 2-structure. Let $\{X,Y\}$ and $\{X',Y'\}$ be two G-joins of (V,e) such that $\{X,Y\}$ overlaps $\{X',Y'\}$. Then $\{X\Delta X', X\Delta Y'\}$ is a G-join of (V,e).*

Proof. Let $v \in Y \cap Y'$, $w \in X \cap Y'$, and let $(j,j',l,l') \in \mathcal{D}^4$ such that $v \in Y_j \cap Y'_{j'}$ and $w \in X_l \cap Y'_{l'}$. As we show in proof of Lemma 5, $X \cap X' = \biguplus_{k \in \mathcal{D}} X_k \cap X'_{k \dot{+} j \dot{-} j'}$ and $Y \cap Y' = \biguplus_{k \in \mathcal{D}} Y_k \cap Y'_{k \dot{+} j' \dot{-} j}$. Using similar argument, $Y \cap X' = \biguplus_{k \in \mathcal{D}} Y_k \cap X'_{k \dot{+} l \dot{-} l'}$ and $X \cap Y' = \biguplus_{k \in \mathcal{D}} X_k \cap Y'_{k \dot{+} l' \dot{-} l}$.

Let $X''_k = (X_k \cap X'_{k \dot{+} j \dot{-} j'}) \cup (Y_{k \dot{+} j \dot{-} j' \dot{+} l' \dot{-} l} \cap Y'_{k \dot{+} l' \dot{-} l})$ and $Y''_k = (Y_k \cap X'_{k \dot{+} l \dot{-} l'}) \cup (X_{k \dot{+} j' \dot{-} j \dot{+} l \dot{-} l'} \cap Y'_{k \dot{+} j' \dot{-} j})$. For all $u \in X''_k$ and $v \in Y''_l$, $e(u,v) = k \dot{+} l$. Thus $\{\biguplus_{k \in \mathcal{D}} X''_k, \biguplus_{k \in \mathcal{D}} Y''_k\}$ is a G-join. □

With Lemma 5, we obtain:

Corollary 9. *The family of G-joins of a symmetric 2-structure is bipartitive.*

4 G-join decomposition

In this section, we fix an abelian group $(\mathcal{D}, \dot{+})$ and an involution σ such that $f : a \to \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism for $(\mathcal{D}, \dot{+})$. For most part, our terminology follows terminology used in [4, 6].

4.1 Simple decomposition

A G-join $\{X,Y\}$ is *trivial* if $|X| = 1$ or $|Y| = 1$. Since every singleton is a module, every bipartition $\{X,Y\}$ with $|X| = 1$ or $|Y| = 1$ is a G-join.

Let $G = (V,e)$ be a σ-symmetric 2 structure and $\{X,Y\}$ be a non-trivial G-join. Let $x \in X$ and $y \in Y$. A *simple decomposition* of (V,e) by the G-join (X,Y) is the decomposition into $G_1 = (X \cup \{y\}, e|_{X \cup \{y\}})$ and $G_2 = (Y \cup \{x\}, e|_{Y \cup \{x\}})$ with an additional *marker triplet* (x,y,α), where $\alpha = e(x,y)$ ($e|_X$ represents the function e induced by $X \times X$). We write $G \to (G_1, G_2, (x,y,\alpha))$. Note that this decomposition is not unique for a fixed $\{X,Y\}$.

The *simple composition* of (V_1, e_1), (V_2, e_2) and the marker triplet (x,y,α), with $V_1 \cap V_2 = \{x,y\}$, is the 2-structure $(V_1 \cup V_2, e)$ where $e(a,b) = e_1(a,b)$ for all $a,b \in V_1 \setminus \{y\}$, $e(a,b) = e_2(a,b)$ for all $a,b \in V_2 \setminus \{x\}$, and $e(a,b) = e_1(a,y) \dot{-} \alpha \dot{+} e_2(x,b)$ for all $a \in V_1 \setminus \{y\}$ and $b \in V_2 \setminus \{x\}$. By Proposition 4, if $((V_1,e_1),(V_2,e_2),(x,y,\alpha))$ is a simple decomposition of (V,e), then the simple composition of (V_1,e_1), (V_2,e_2) and (x,y,α) is (V,e).

Lemma 10. *Let $\{X,Y\}$ be a G-join of G, and $(G_1, G_2, (x,y,\alpha))$ be the simple decomposition of G by (X,Y). Let $\{X',Y'\}$ be a bipartition of V with $Y' \subsetneq Y$. Then $\{X',Y'\}$ is a G-join of G if and only if $\{\{x\} \cup Y \setminus Y', Y'\}$ is a G-join of G_2.*

Proof. If $\{X',Y'\}$ is a G-join of G then by Proposition 2 $\{\{x\} \cup Y \setminus Y', Y'\}$ is a G-join of G_2. Now suppose that $\{\{x\} \cup Y \setminus Y', Y'\}$ is a G-join of G_2. Let X'_a and Y'_a (for $a \in \mathcal{D}$) such that $(\{x\} \cup Y \setminus Y', Y') = (\biguplus_{a \in \mathcal{D}} X'_a, \biguplus_{a \in \mathcal{D}} Y'_a)$ and $e(u,v) = a \dot{+} b$ for all $u \in X'_a$ and $v \in Y'_b$. Since $\{X,Y\}$ is a G-join of G, let X_a and Y_a such that $(X,Y) = (\biguplus_{a \in \mathcal{D}} X_a, \biguplus_{a \in \mathcal{D}} Y_a)$ and $e(u,v) = a \dot{+} b$ for all $u \in X_a$ and $v \in Y_b$. Let $c,d \in \mathcal{D}$ such that $x \in X'_c$ and $y \in Y_d$. Let $X''_a = (X'_a \setminus \{x\}) \cup X_{a \dot{-} d \dot{-} c \dot{+} \alpha}$ and $Y''_a = Y'_a$. Let $u \in X''_a$ and $v \in Y''_b$. If $u \in X'_a$ then $e(u,v) = a \dot{+} b$. Otherwise $u \in X_{a \dot{-} d \dot{-} c \dot{+} \alpha}$ and by definition of simple decomposition, $e(u,v) = e(u,y) \dot{-} \alpha \dot{+} e(x,v) = a \dot{+} b$ since $e(u,y) = a \dot{-} d \dot{-} c \dot{+} \alpha \dot{+} d$ and $e(x,v) = c \dot{+} b$. Then $\{\biguplus_{a \in \mathcal{D}} X''_a, \biguplus_{a \in \mathcal{D}} Y''_a\} = \{X',Y'\}$ is a G-join of G. □

A G-join $\{X, Y\}$ is *strong* if it is a strong member of the bipartitive family of G-joins of G (*i.e.* there is no G-join $\{X', Y'\}$ such that $\{X, Y\}$ overlaps $\{X', Y'\}$). A simple decomposition is strong if it is induced by a strong G-join. The following Corollary follows from previous Lemma.

Corollary 11. *Let $\{X, Y\}$ be a G-join of G, and $(G_1, G_2, (x, y, \alpha))$ be the simple decomposition of G by (X, Y). Let $\{X', Y'\}$ be a bipartition of V with $Y' \cap Y$. Then $\{X', Y'\}$ is a strong G-join of G if and only if $\{\{x\} \cup Y \setminus Y', Y'\}$ is a strong G-join of G_2.*

4.2 G-join decompositions

A 2-structure is *prime* if all its G-joins are trivial. A 2-structure is *degenerated* if every bipartition is a G-join. A 2-structure $G = (V, E)$ is *linear* if there is a ordering $v_1, \ldots, v_n$ of the vertices such that for all $i, j \in \{1, \ldots, n\}$ with $i \leq j$ and $(i, j) \neq (1, n)$, $\{\{v_i, \ldots v_j\}, -\}$ is a G-join of G, and G has no others G-join. Every 2-structure with at most 3 vertices is degenerated, linear and prime, and every 2-structure with at least 4 vertices is either prime, degenerated, linear or none of these three cases. The following Lemma comes immediately from the bipartitivity of G-joins.

Lemma 12. *Let G be a 2-structure. G has no strong non-trivial G-join if and only if G is either prime, degenerated or linear*

Proof. If G has no strong non-trivial G-join, then representative tree of G has only one internal node β. Then G is prime, degenerated or linear if β is `prime`, `degenerate` or `linear`, respectively. □

The following Lemma gives a characterisation of degenerated graphs. Its straightforward inductive proof is given in appendix.

Lemma 13. *Suppose $\sigma(\dot{0}) = \dot{0}$. A σ-symmetric 2-structure with at least 4 vertices is degenerated if and only if there is an $\alpha \in \mathcal{D}$ such that $\sigma(\alpha) = \alpha$, and a function $f : V \to \mathcal{D}$ such that for all $u, v \in V$, $u \neq v$, $e(u, v) = \alpha \dotplus f(u) \dotplus \sigma(f(v))$.*

Let G be a 2-structure. *G-join decompositions* of G are defined recursively: $(\{G\}, \emptyset)$ is a G-join decomposition of G and if $(\mathcal{D}, M)$ is a G-join decomposition of G, $H \in \mathcal{D}$, and H_1, H_2 is a simple decomposition of H with marker triplet (u, v, α), then $((\mathcal{D} \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\})$ is a G-join decomposition of G. In this case we say that $(\mathcal{D}', M') = ((\mathcal{D} \setminus \{H\}) \cup \{H_1, H_2\}, M \cup \{(u, v, \alpha)\})$ is a simple decomposition of $(\mathcal{D}, M)$, and we write $(\mathcal{D}, M) \to (\mathcal{D}', M')$.

A G-join decomposition $(\mathcal{D}, M)$ is *minimal* if every 2-structure in $\mathcal{D}$ is prime. A G-join decomposition $(\mathcal{D}, M)$ is *good* if no $H \in \mathcal{D}$ has a strong non-trivial G-join. A G-join decomposition $(\mathcal{D}, M)$ is *standard* if it can be obtained from $(\{G\}, \emptyset)$ by a sequence of simple strong decompositions, and no $H \in \mathcal{D}$ has a strong non-trivial G-join. Note that minimal decompositions and standard decompositions are goods. The proof of the following lemma is similar to the proof given in [6].

Lemma 14. *Let $(\mathcal{D}, M)$ be a good decomposition of G. If there is no good decomposition $(\mathcal{D}', M')$ such that $(\mathcal{D}', M') \to (\mathcal{D}, M)$, then $(\mathcal{D}, M)$ is a standard decomposition of G.*

Proof. If $(\mathcal{D}, M)$ is not a standard decomposition then there is a simple decomposition in the sequence of decompositions which is not strong. Let $(\mathcal{D}_1, M_1) \to (\mathcal{D}_2, M_2) = ((\mathcal{D}_1 \setminus \{H\}) \cup \{H_1, H_2\}, M_1 \cup \{m\})$ be the last non-strong decomposition in the sequence. All the decompositions after $(\mathcal{D}_1, M_1) \to (\mathcal{D}_2, M_2)$ are strong and correspond to unique strong G-joins of G. We construct the decomposition $(\mathcal{D}', M')$ from $(\mathcal{D}_1, M_1)$ after simple decompositions of these strong G-joins. $(\mathcal{D}', M')$ is good since there is a simple decomposition for every strong G-join in G and $(\mathcal{D}', M') \rightarrow (\mathcal{D}, M)$ by the simple decomposition of the G-join corresponding to $(\mathcal{D}_1, M_1) \rightarrow (\mathcal{D}_2, M_2)$. □

The previous Lemma tells us that a standard decomposition can be obtained from a minimal decomposition by a sequence of simple compositions. This will be used in the decomposition algorithm presented in a next section.

A decomposition $(\mathcal{D}, M)$ of G induces a unrooted tree of vertex set $\mathcal{D}$ and H_1 is adjacent to H_2 if there is a $(x, y, \alpha) \in M$ such that x is a vertex of H_1 and y is a vertex of H_2. The decomposition tree of a standard decomposition is isomorphic to the representative tree of the weakly bipartitive family of G-joins and thus is unique. We call it the *standard decomposition tree*.

5 Specials cases of G-join decomposition

5.1 Bi-join decomposition

The bi-join decomposition [13, 14] is a special case with $(\mathcal{D}, \dot{+}) = (\mathbb{Z}_2, +)$. Lemma 13 says that degenerated graphs are disjoint union of two cliques if $\alpha = 1$ and complete bipartite graphs if $\alpha = 0$. This decomposition has no `linear` node since the structure is symmetric. The bi-join decomposition of tournaments [2] is the decomposition, with $(\mathcal{D}, \dot{+}) = (\mathbb{Z}_2, +)$ and $\sigma = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

5.2 Decomposition of oriented graphs

A directed graph G can be viewed as an σ-symmetric 2-structure on the set $\{(0,0), (1,0), (0,1)\}$, with $\sigma((i,j)) = (j,i)$ for $(i,j) \in \{(0,0), (1,0), (0,1)\}$. There is one abelian group on $\mathcal{D}$ such that $a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism. This abelian group, isomorphic to $(\mathbb{Z}_3, +)$, is given in figure 2.

5.3 Decompositions of directed graphs

A directed graph G is a 2-structure on $\mathbb{Z}_2$, and can be viewed as a σ-symmetric 2-structure (V, e) on $\mathbb{Z}_2 \times \mathbb{Z}_2$, with $\sigma((i,j)) = (j,i)$. There are two abelian groups such that $a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism. The first one (isomorphic to $(\mathbb{Z}_2^2, +)$) is given in figure 3, and the second (isomorphic to $(\mathbb{Z}_4, +)$) is given in figure 4. These two decompositions are generalizations of the bi-join decomposition on both non-oriented graphs and on tournament. They are mutually exclusive, that is there is a graph prime for the first one and completely decomposable for the other one, and *vice versa*.

6 Decomposition algorithm

From now, we fix an abelian group $(\mathcal{D}, \dot{+})$ and an involution σ such that $f : a \rightarrow \sigma(a) \dot{-} \sigma(\dot{0})$ is an isomorphism for $(\mathcal{D}, \dot{+})$.

6.1 Find a non trivial G-join

We give in this section a $O(n^2)$ algorithm for the following problem: given a 2-structure $G = (V, e)$ and $u, v \in V$, output a non trivial G-join $\{X, Y\}$ such that $u \in X$ and $v \in Y$, or output “No” if there is no such partition.

$\dot{+}$	(0,0)	(1,0)	(0,1)
(0,0)	(0,0)	(1,0)	(0,1)
(1,0)	(1,0)	(0,1)	(0,0)
(0,1)	(0,1)	(0,0)	(1,0)

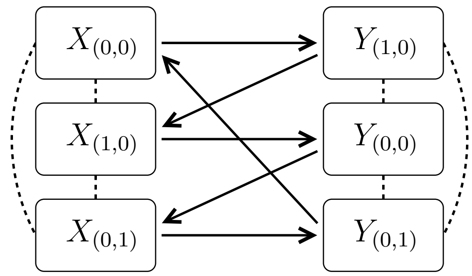

Figure 2: Decomposition for oriented graphs. (Dashed edge signify that two vertex can be adjacent or not.)

$\dot{+}$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

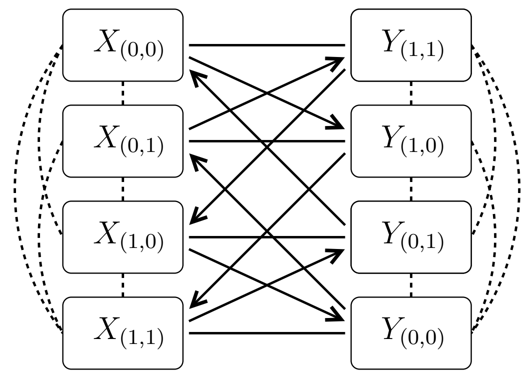

Figure 3: Decomposition for directed graphs (first).

$\dot{+}$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

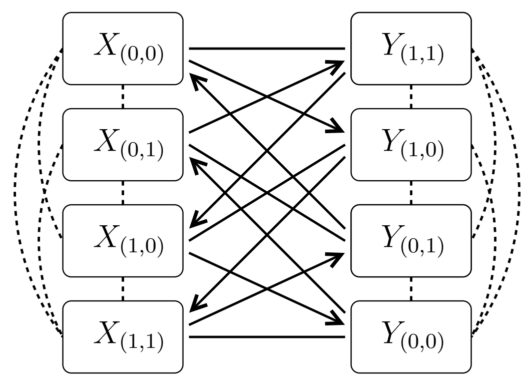

Figure 4: Decomposition for directed graphs (second).

A directed graph $G = (V, A)$ is *strongly connected* if for every $u, v \in V$ there is a path from u to v (*i.e.* there is a sequence $u_0 = u, u_1, \dots u_k = v$ such that for all $i \in \{0, \dots k-1\}$, $(u_i, u_{i+1}) \in A$). A *strongly connected component* is a maximal subset $W \subseteq V$ such that $G[W]$ is strongly connected. The strongly connected components form a partition of the vertices of G, and can be found in linear time [1]. Moreover, there is always a strongly connected component W such that there is no arcs from W to $V \setminus W$, since the incidence graph of strongly connected components is acyclic.

We transform our problem into a 2-SAT problem. We suppose w.l.o.g. that $u \in X_{\hat{0}}$ and thus $v \in Y_{e(u,v)}$. If a vertex $w \notin \{u, v\}$ is in X then it is in $X_{e(w,v)\dot{-}e(u,v)}$, and if w is in Y, it is in $Y_{e(u,v)}$. Let the 2-SAT problem with variable set $V \setminus \{u, v\}$, and $w \Rightarrow t$ if $e(w,v)\dot{-}e(u,v)\dot{+}e(u,t) \neq e(w,t)$. A variable w is true means that $w \in X$. Then there is a non trivial G-join if and only is there is a non trivial solution for the 2-SAT problem. Let $G_f = (V \setminus \{u, v\}, E_f)$ with $E_f = \{(w,t) : w, t \in V \setminus \{u, v\}$ and $e(w,v)\dot{-}e(u,v)\dot{+}e(u,t) \neq e(w,t)\}$. The 2-SAT problem has a non trivial solution if and only if the graph G_f is not strongly connected. In this case $\{X \cup \{u\}, V \setminus (X \cup \{u\})\}$ is a non trivial G-join of G. All these operations can be done in time $O(n^2)$. (Algorithms in pseudo-code are given in appendix.)

6.2 Compute a minimal G-join decomposition

If a 2-structure is not prime, then a G-join can be found in $O(n^3)$ time using the previous algorithm for a fixed $u \in V$ and for all $v \neq u$. So a naive algorithm to compute a minimal decomposition take $O(n^4)$ time. We can reach $O(n^3)$ by the following way. We remember the set $\mathcal{P}$ of subsets of V such that there is no non-trivial G-join which overlaps U for all $U \in \mathcal{P}$. $\mathcal{P}$ is a partition of V, and at each call of the sub-routine, either it succeed and we decompose the 2-structure, either it fails and we merge two sets in $\mathcal{P}$. So a minimal decomposition can be obtained with $O(n)$ call to the algorithm of section 6.1, and can computed in $O(n^3)$.

6.3 Compute a standard G-join decomposition

Lemma 14 says that a standard decomposition of G can be computed from a minimal decomposition, after some re-compositions. We show that we can test in time $O(n^2)$ if a composition of two 2-structures degenerated or linear is degenerated or linear.

Let $G_1 = (V_1, e_1)$ and $G_2 = (V_2, e_2)$ and a marker triplet (x, y, α), such that G_1 and G_2 have no strong non-trivial G-join. If G has no strong non-trivial G-join, then by Lemma 12, G is either degenerated or linear (since it cannot be prime). If G_1 or G_2 is not degenerated, then G must be linear. Moreover if G_1 and G_2 are linear, let $v_1, \dots, v_k$ be a linear ordering of the vertex of G_1, and let $v'_1, \dots, v'_{k'}$ be a linear ordering of G_2. W.l.o.g. $v_1 = y$ and $v'_1 = x$. Then if G is linear, $v_2, \dots, v_k, v'_2, \dots, v'_{k'}$ or $v_2, \dots, v_k, v'_{k'}, \dots, v'_2$ must be a linear ordering of G, and so either $\{\{v_2, v'_2\}, -\}$ or $\{v_2, v'_{k'}\}, -\}$ must be a G-join of G.

Let G_1 and G_2 be two 2-structures without strong non-trivial G-join. We want to known if the composition G of G_1 and G_2 with the marker triplet (x, y, α) is degenerated or linear (and to know a ordering of G if it is linear). Case 1: $|V_1| = |V_2| = 3$. All bipartitions of G can be tested in constant time, so the type of G (and a ordering if G is linear) can be computed in $O(1)$. Case 2: $|V_1| \neq 3$ or $|V_2| \neq 3$. Then G_1 or G_2 is non degenerated, or non linear. If G_1 and G_2 are degenerated, then G is degenerated if and only if $\{\{x, y\}, -\}$ is a G-join of G. If G_1 and G_2 are linear, with ordering $\{v_1 = y, \dots, v_k\}$ and $\{v'_1 = x, \dots, v'_{k'}\}$, then G is linear if $\{\{v_2, v'_2\}, -\}$ or $\{\{v_2, v'_{k'}\}, -\}$ is a G-join of G. In this case, $v_2, \dots, v_k, v'_2, \dots, v'_{k'}$ or $v_2, \dots, v_k, v'_{k'}, \dots, v'_2$ is ordering of G. In others cases, G

is neither degenerate nor linear. Moreover, to test if a bipartition is a G-join of a 2-structure can be done in $O(n^2)$.

There is at most $O(n)$ re-compositions (at most one for each edge in the decomposition tree). To summarize, we obtain:

Theorem 15. *A standard G-join decomposition can be computed in time $O(n^3)$.*

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7 Appendix

7.1 Proof of lemma 13

Let (V, e) be a 2-structure such that there is a $f : V \to \mathcal{D}$ and a α with $\sigma(\alpha) = \alpha$ and $e(u, v) = \alpha \dot{+} f(u) \dot{+} \sigma(f(v))$ for all $u, v \in V$, $u \neq v$. It is easy to see that (V, e) is σ-symmetric since $\sigma(\dot{0}) = \dot{0}$ and thus σ is a isomorphism for $(\mathcal{D}, \dot{+})$. Let $\{X, Y\}$ be a bipartition of V. Let $X_a = \{v \in X : a = f(v)\}$ and $Y_a = \{v \in Y : a = \alpha \dot{+} \sigma(f(v))\}$. For all $u \in X_a$ and $b \in Y_b$, $a \dot{+} b = f(u) \dot{+} \alpha \dot{+} \sigma(f(v)) = e(u, v)$, thus $\{X, Y\}$ is a G-join.

On the other hand, let (V, e) be a degenerated 2-structure such that $|V| \geq 4$.

Claim 1. For every pairwise different $a, b, c, d \in V$, $e(c, d) = e(c, b) \dot{+} e(a, d) \dot{-} e(a, b)$.

Proof. Since (V, e) is degenerated, $\{\{a, c\}, -\}$ is a G-join. By Proposition 4 we have the equality. □

Claim 2. For every pairwise different $a, b, c \in V$:

$$e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) = e(b, a) \dot{+} e(c, b) \dot{+} e(a, c).$$

Proof. Let $d \in V \setminus \{a, b, c\}$. Applying Claim 1, we get:

$$\begin{aligned} e(d, a) \dot{-} e(d, b) &= e(c, a) \dot{-} e(c, b) \\ e(d, b) \dot{-} e(d, c) &= e(a, b) \dot{-} e(a, c) \\ e(d, c) \dot{-} e(d, a) &= e(b, c) \dot{-} e(b, a). \end{aligned}$$

Thus

$$\begin{aligned} e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) &= e(b, a) \dot{+} e(c, b) \dot{+} e(a, c) \\ &= \sigma(e(a, b)) \dot{+} \sigma(e(b, c)) \dot{+} \sigma(e(c, a)). \end{aligned}$$

□

Case 1: $|V| = 4$. W.l.o.g $V = \{a, b, c, d\}$. Let:

$$\begin{aligned} \alpha &= e(a, b) \dot{+} e(b, c) \dot{+} e(c, a) \\ f(a) &= \dot{-} e(b, c) \\ f(b) &= \dot{-} e(a, c) \\ f(c) &= \dot{-} e(b, a) \dot{-} e(a, c) \dot{+} e(c, a) \\ f(d) &= e(d, a) \dot{+} e(c, b) \dot{-} \alpha \end{aligned}$$

From Claim 2, $\sigma(\alpha) = \alpha$. We get:

$$\begin{aligned}
f(a) \dotplus \sigma(f(b)) \dotplus \alpha &= \dot{-}e(b,c) \dot{-} e(c,a) \dotplus e(a,b) \dotplus e(b,c) \dotplus e(c,a) \\
&= e(a,b) \\
f(a) \dotplus \sigma(f(c)) \dotplus \alpha &= \dot{-}e(b,c) \dot{-} e(a,b) \dot{-} e(c,a) \dotplus e(a,c) \dotplus e(a,b) \dotplus e(b,c) \dotplus e(c,a) \\
&= e(a,c) \\
f(b) \dotplus \sigma(f(c)) \dotplus \alpha &= \dot{-}e(a,c) \dot{-} e(a,b) \dot{-} e(c,a) \dotplus e(a,c) \dotplus e(a,b) \dotplus e(b,c) \dotplus e(c,a) \\
&= e(b,c) \\
f(a) \dotplus \sigma(f(d)) \dotplus \alpha &= \dot{-}e(b,c) \dotplus e(a,d) \dotplus e(b,c) \dot{-} \alpha \dotplus \alpha \\
&= e(a,d) \\
f(b) \dotplus \sigma(f(d)) \dotplus \alpha &= \dot{-}e(a,c) \dotplus e(a,d) \dotplus e(b,c) \dot{-} \alpha \dotplus \alpha \\
&= e(b,d) \quad \text{(by Claim 1)} \\
f(c) \dotplus \sigma(f(d)) \dotplus \alpha &= \dot{-}e(b,a) \dot{-} e(a,c) \dotplus e(c,a) \dotplus e(a,d) \dotplus e(b,c) \dot{-} \alpha \dotplus \alpha \\
&= e(a,d) \dotplus e(c,b) \dot{-} e(a,b) \quad \text{(by Claim 2)} \\
&= e(c,d) \quad \text{(by Claim 1.)}
\end{aligned}$$

Thus f and α have the required property.

Case 2: $|V| > 4$. Let $v \in V$. $(V \setminus \{v\}, e)$ is degenerated and thus there is a $f' : V \setminus \{v\} \to \mathcal{D}$ and an $\alpha \in \mathcal{D}$ such that for all $u, v \in V \setminus \{v\}$, $e(u,v) = f'(u) \dotplus \sigma(f'(v)) \dotplus \alpha$. Let $u \neq v$, and let f such that $f(w) = f'(w)$ if $x \in V \setminus \{v\}$ and $f(v) = e(v,u) \dot{-} \sigma(f'(u)) \dot{-} \alpha$.

$$\begin{aligned}
f(u) \dotplus \sigma(f(v)) \dotplus \alpha &= f'(u) \dotplus e(u,v) \dot{-} \sigma(\sigma(f'(u))) \dot{-} \alpha \dotplus \alpha \\
&= e(u,v)
\end{aligned}$$

Let $w \in V \setminus \{u, v\}$ and $x \in V \setminus \{u, v, w\}$.

$$\begin{aligned}
f(w) \dotplus \sigma(f(v)) \dotplus \alpha &= f'(w) \dotplus e(u,v) \dot{-} \sigma(\sigma(f'(u))) \dot{-} \alpha \dotplus \alpha \\
&= f'(w) \dot{-} f'(u) \dotplus e(u,x) \dotplus e(w,v) \dot{-} e(w,x) \quad \text{(by Claim 1)} \\
&= e(w,v) \dotplus f'(w) \dot{-} f'(u) \dotplus f'(u) \dotplus \sigma(f'(x)) \dotplus \alpha \dot{-} f'(w) \dot{-} \sigma(f'(x)) \dot{-} \alpha \\
&= e(w,v).
\end{aligned}$$

Thus f and α have the required property.

7.2 Algorithm to find a non trivial G-join

Function FINDGJOIN($G=(V,e),u,v$)
Input: a 2-structure $G=(V,e)$ and $u,v \in V$, $u \neq v$
Output: a non trivial G-join $\{X,Y\}$ of G such that $u \in X$ and $v \in Y$,
or "No" is there is no such G-join
begin
$f_1(u) := \dot{0}$
$f_2(v) := e(u,v)$
For every $w \in V \setminus \{u,v\}$
$f_1(w) := e(w,v) \dot{-} e(u,v)$
$f_2(w) := e(u,w)$
$E_f := \{(w,t) : w,t \in V \setminus \{u,v\}$ and $f_1(w) \dot{+} f_2(t) \neq e(w,t)\}$
$G_f := (V \setminus \{u,v\}, E_f)$
if G_f is strongly connected
output "No"
Otherwise
Let W be a strongly connected component of G_f
such that there is no arc in G_f from W to $V \setminus W \setminus \{u,v\}$
output $\{\{u\} \cup W, V \setminus \{u\} \setminus W\}$
end {FINDGJOIN}

7.3 Algorithm to compute a minimal G-join decomposition

Function DECOMPOSEP($G,\mathcal{P}$)
Input: a 2-structure $G=(V,e)$ and a partition $\mathcal{P}$ of V
Output: a minimal G-join decomposition G
begin
If $|\mathcal{P}| = 1$ **then**
return $(\{G\}, \emptyset)$
Let $A, B \in \mathcal{P}$, $a \in A$ and $b \in B$
If FINDGJOIN(G,a,b) returns "no" **then**
$P := \{A \cup B\} \cup (\mathcal{P} \setminus \{A,B\})$
return DECOMPOSEP($G,\mathcal{P}$)
Let $\{X,Y\}$ be the G-join returned by FINDGJOIN
Decompose G into G_1 and G_2 by the G-join $\{X,Y\}$ with marker triplet (x,y,α)
$\mathcal{P}_1 := \{P \in \mathcal{P} : P \subseteq X\}$
$\mathcal{P}_2 := \{P \in \mathcal{P} : P \subseteq Y\}$
$(\mathcal{D}_1, M_1) :=$ DECOMPOSEP($G_1,\mathcal{P}_1$)
$(\mathcal{D}_2, M_2) :=$ DECOMPOSEP($G_2,\mathcal{P}_2$)
return $(\mathcal{D}_1 \cup \mathcal{D}_2, M_1 \cup M_2 \cup (x,y,\alpha))$
end {DECOMPOSEP}

Function DECOMPOSE(G)
Input: a 2-structure $G = (V, e)$
Output: a minimal G-join decomposition G
begin
  $\mathcal{P} := \{\{v\} : v \in V\}$
  return DECOMPOSEP($G, \mathcal{P}$)
end {DECOMPOSE}

7.4 Algorithm to compute a standard G-join decomposition

Function DECOMPOSESTANDARD(($\mathcal{D}, M$))
Input: a minimal G-join decomposition
Output: a standard G-join decomposition
begin
  for all $H \in \mathcal{D}$
    if H has exactly 3 vertices **then**
      mark H `degenerated` and `linear`, and set an arbitrary linear ordering for H
  for all $(x, y, \alpha) \in M$
    let $H_1 \in \mathcal{D}$ having vertex x, and **let** $H_2 \in \mathcal{D}$ having vertex y
    compute the composition H of H_1 and H_2
    if H is degenerated or linear **then**
      $(\mathcal{D}, M) := (\mathcal{D} \setminus \{H_1, H_2\} \cup \{H\}, M \setminus \{(x, y, \alpha)\}))$
      mark H `degenerated` or `linear`, and set the linear ordering of H if H is `linear`
  return $(\mathcal{D}, M)$
end {DECOMPOSESTANDARD}