

A characterization of the solutions of steady Hamilton-Jacobi equations*

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Abstract

In this note, we propose to revisit the approximate stationary Hamilton-Jacobi equations and analyse the corresponding solutions following certain properties of the hamiltonian. This enables us to give a characterization of the zero set of the limiting solution. We also remark that the analysis can be applied for evolution equations with a time periodic source term.

1 Introduction

In this note, we will be concerned with the viscosity solutions u_α of the approximate Hamilton-Jacobi type equations

$$\alpha u + H(x, Du) = 0, \quad x \in \mathbb{R}^N \quad (1)$$

for $\alpha > 0$ and with the characterization of these solutions and also of the limit of the sequence $(u_\alpha)_\alpha$ as α goes to zero. In the above, the hamiltonian H is a continuous function of its variables and Du denotes the gradient of u .

Now one knows that for $\alpha > 0$, equation (1) admits a unique viscosity solution under some suitable hypotheses on the hamiltonian H . We need notably a boundedness property wrt the space variable. For this sake, we will suppose that

$$H \text{ is periodic in } x. \quad (2)$$

We will also want some regularity of the solutions. This will be achieved by the coercivity condition

$$\lim_{|p| \rightarrow \infty} H(x, p) = \infty, \text{ uniformly in } x \in \mathbb{R}^N. \quad (3)$$

The above is a weak regularity assumption in the sense that no convexity assumption of H is called for. Under the hypotheses (2) and (3), for every $\alpha > 0$, equation (1) admits a unique viscosity solution $u_\alpha \in W^{1,\infty}(\mathbb{R}^N)$. Moreover u_α is periodic. We refer the reader to [5], and [1] for a review of the definition and properties of viscosity solutions

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for Hamilton-Jacobi equations.

The properties of u_α will naturally depend on those imposed on the hamiltonian H . We want to inquire about this dependence. For this, we start by assuming that the function $x \mapsto H(x, 0)$ has a sign. As a consequence, we have

LEMMA 1. Suppose $H(x, 0) \leq 0$ for all $x \in \mathbb{R}^N$. Then (i) $u_\alpha \geq 0$ for all $\alpha > 0$ and (ii) $(u_\alpha)_\alpha$ is increasing when α goes to zero.

We thus need to bound (u_α) if we want to say anything about its convergence. Notice also that, as we naturally expect the limit to be a solution of

$$H(x, Du) = 0, \quad x \in \mathbb{R}^N \quad (4)$$

we are brought to inquire about the well-posedness of (4) in the class of bounded continuous solutions. The latter solvability is not to be taken for granted. In fact, one can show that if the so-called ergodic equation

$$H(x, Du) = \lambda, \quad x \in \mathbb{R}^N \quad (5)$$

is solvable in the class of bounded solutions, then it is so for a unique ergodic constant $\lambda \in \mathbb{R}$. See for example the paper of Lions, Papanicolaou and Varadhan [6] where the above equation appears as the cell problem in a homogenization process. This unique λ , usually denoted by the letter c , is also called the critical value and the corresponding solutions, critical solutions, see for *e.g.* [3] and [4]. In fact the hypotheses (2) and (3) guarantee the solvability of (5) for some $\lambda \in \mathbb{R}$ and thus we just demand that the critical value be equal to zero in our case. Before going further, let us give sufficient conditions for this to be true.

LEMMA 2. Let H satisfy (2) and (3) with $H(x, 0) \leq 0$, $x \in \mathbb{R}^N$. Assume that there exists a point $x^* \in \mathbb{R}^N$ such that $H(x^*, p) \geq 0$ for all $p \in \mathbb{R}^N$. Then

(5) is solvable iff $\lambda = 0$.

A typical example will be the standard hamiltonian in classical mechanics which is given by

$$H(x, p) = |p|^2 - V(x)$$

with V periodic in x and $\min V = 0$. It is not difficult to see that H verifies the assumptions of the above lemma so that $|Du|^2 - V(x) = \lambda$ is solvable only for $\lambda = 0$. We thus recover the result observed in [6] where the above example is treated rather in details. Let us also point out that the critical value c can be rather explicitly characterized under some convexity assumptions on H . One can see for example the paper by Fathi-Siconolfi [4], where quasiconvex hamiltonians are considered and the one by Roquejoffre in a 1-D context [8], where it is notably shown that in some cases, which includes a strict convexity assumption of H wrt p , we have

$$c = \max_x \min_p H(x, p).$$

We now state

LEMMA 3. Let $H(x, 0) \leq 0$ and (4) be solvable. Then

$$u_\alpha \xrightarrow{\alpha \rightarrow 0} u_0 = \sup_{\alpha > 0} u_\alpha \quad \text{in } C^0(\mathbb{R}^N)$$

with u_0 a periodic solution of (4).

Now we want to make more precise the characterization of the limit u_0 . Define for this sake the set

$$Z = \{x \in \mathbb{R}^N : H(x, 0) = 0\}. \quad (6)$$

It is easy to verify that, due to the periodicity condition, Z is a nonempty set. We then propose

LEMMA 4. Under the hypotheses of Lemma 3, u_0 attains its minimum on Z and vanishes at the minima points.

In fact an analogous property applies for u_α . More precisely if x_0 is a minimum point of u_0 then it is a minimum point of u_α for all $\alpha > 0$ and $u_\alpha(x_0) = 0$. But of course, at this stage we cannot say that the minima points of u_0 and those of the functions u_α coincide, *i.e.* a point x_1 can be a minimum point of a function u_α but without being a minimum point of u_0 .

Also Lemma 4 tells us that if x_0 is a minimum point of u_0 then $x_0 \in Z$ and $u_0(x_0) = 0$. But does the zero set of u_0 coincide with Z ? We now give our main result which states that under an additional condition on H , the zero set of u_0 coincides with the zero set of u_α for all $\alpha > 0$ and is equal to the set Z .

THEOREM. Let H satisfy (2) and (3) with $H(x, 0) \leq 0$, $x \in \mathbb{R}^N$. Assume also that $H(x, p) \geq 0$ for all $x \in Z$ and $p \in \mathbb{R}^N$. Then the sequence of solutions u_α of (1) converges to a periodic solution u_0 of (4) whose zero set is exactly equal to Z . Moreover Z coincides also with the zero set of u_α for all $\alpha > 0$.

Note that the assumptions of the above theorem ensure the solvability of (4) via the Lemma 2. Let us point out that the above type of characterization is known for time asymptotic limits of Cauchy problems. See for *e.g.* [7] where the zero set of the limiting solution was shown to coincide with the maxima points of the initial condition. The analogy seems to stop here because in the case at present, there is of course no initial condition. What is important in both cases is some monotonicity character of the solutions, *wrt* the time variable for the Cauchy problem and *wrt* α for the steady case as stated in Lemma 1.

We proceed in the next section by giving an application of the above result and notably show how the previous analysis can be profitably used to characterize the asymptotic limit in the case of evolution equations having a time periodic source term. And finally the last part will be devoted to the proofs of the lemmas and of the theorem.

2 The time periodic case

We consider therefore the following Hamilton-Jacobi type equation

$$\alpha u + \partial_t u + H(x, Du) = f(t), \quad (x, t) \in \mathbb{R}^N \times \mathbb{R} \quad (7)$$

where f is a continuous T -periodic function in t . One knows that under assumptions (2) and (3), equation (7) admits a unique T -periodic viscosity solution $u_\alpha(x, t)$ in $W^{1,\infty}(\mathbb{R}^N \times \mathbb{R})$, see for example [2]. We enquire about the limit when α goes to zero. To simplify the presentation and without loss of generality, we will suppose that the mean of f over one period is zero.

We proceed by a decoupling of the equation (7). Consider for this sake

$$\alpha\phi + H(x, D\phi) = 0, \quad x \in \mathbb{R}^N \quad (8)$$

and

$$\alpha v + v_t = f(t), \quad t \in \mathbb{R}. \quad (9)$$

The *ode* (9) admits also a unique periodic solution v_α for every $\alpha > 0$ and which has the explicit formulation

$$v_\alpha(t) = \frac{e^{-\alpha t}}{1 - e^{-\alpha T}} \int_0^T e^{-\alpha(T-s)} f(s) ds + \int_0^t e^{-\alpha(t-s)} f(s) ds.$$

It is then not difficult to verify that

$$v_\alpha(t) \xrightarrow{\alpha \rightarrow 0} F(t) + C_* \quad \text{for all } t \in \mathbb{R}$$

where

$$F(t) = \int_0^t f(s) ds \quad \text{and the constant } C_* = \frac{1}{T} \int_0^T s f(s) ds.$$

Now if we denote by $\phi_\alpha(x)$ the solution of (8), then it is immediate to verify that $u_\alpha(x, t) = \phi_\alpha(x) + v_\alpha(t)$ is the solution of (7). Therefore the study of the limit, when α goes to zero, of the time periodic solution u_α of (7) is reduced to that of the solution ϕ_α of the stationary equation (9). In fact, the limiting solution, which is time periodic via F , can thus be completely explicit as we have

$$u_\alpha(x, t) \xrightarrow{\alpha \rightarrow 0} \phi_0(x) + F(t) + C_* \quad \text{uniformly in } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

with ϕ_0 as previously characterized following the assumptions on the hamiltonian H .

3 Proof of the Theorem

We start by proving the preliminary lemmas.

3.1 Proof of Lemma 1

Observe that as $H(x, 0) \leq 0$, 0 is a subsolution of (1). By comparison results, we then have $u_\alpha \geq 0$. Next take $0 < \alpha \leq \beta$ and notice that u_α is a supersolution of

$$\beta u(x) + H(x, Du) = 0, \quad x \in \mathbb{R}^N,$$

since we know that $(\beta - \alpha)u_\alpha \geq 0$. Again by comparison results, we have $u_\alpha \geq u_\beta$.

3.2 Proof of Lemma 3

We now suppose that (4) is solvable and let ϕ be a continuous bounded solution. Then observe that $\phi + \|\phi\|_\infty$ is a supersolution of (1) so that we have

$$0 \leq u_\alpha \leq \phi + \|\phi\|_\infty$$

i.e. $(u_\alpha)_\alpha$ is bounded. Considering the monotonicity of the sequence $(u_\alpha)_\alpha$, we then have

$$u_\alpha(x) \xrightarrow{\alpha \rightarrow 0} u_0(x) = \sup_{\alpha > 0} u_\alpha(x), \quad x \in \mathbb{R}^N.$$

In fact, due to the coercivity condition (3), $(u_\alpha)_\alpha$ is uniformly bounded in $W^{1,\infty}(\mathbb{R}^N)$ so that the above convergence is uniform, at least locally. The stability result of viscosity solutions then yields that u_0 is a viscosity solution of (4). The periodicity of u_0 results from that of each u_α and we are done with the proof of Lemma 3.

3.3 Proof of Lemma 4

To prove the lemma, first observe that the set $Z = \{x \in \mathbb{R}^N : H(x, 0) = 0\}$ is non empty. Indeed at a minimum point x_0 of u_0 (its existence is guaranteed by the periodicity of u_0) we have $H(x_0, 0) \geq 0$ which together with the assumption $H(x, 0) \leq 0$ leads to $H(x_0, 0) = 0$, *i.e.* $x_0 \in Z$. Note at this stage that H just bounded (instead of being periodic) would not have sufficed. It remains to show that $u_0(x_0) = 0$. We already know that $u_\alpha \geq 0$ for all $\alpha > 0$ and thus $u_0(x_0) \geq 0$. Now one can easily verify that $u_0 - u_0(x_0)$ is a supersolution of (1). By comparison result, we then have $u_0 - u_0(x_0) \geq u_\alpha$ for all $\alpha > 0$. A passage to the limit then leads to $u_0(x_0) \leq 0$ and thus $u_0(x_0) = 0$. As pointed out before, since for all α we have $0 \leq u_\alpha \leq u_0$, x_0 is also a minimum point for u_α for all $\alpha > 0$ and $u_\alpha(x_0) = 0$.

3.4 Proof of the theorem

For the time being we know that if u_0 vanishes, it does so at a point of Z . Proving the theorem comes therefore to verify that if x_0 is a point of Z then $u_0(x_0) = 0$ *i.e.* the zero set of u_0 coincides exactly with Z . Let then $x_0 \in Z$ and let us show that $u_\alpha(x_0) = 0, \forall \alpha > 0$. We know that $u_\alpha(x_0) \geq 0$. Suppose $u_\alpha(x_0) > 0$, for some $\alpha > 0$. As u_α is bounded and lipschitz continuous, it is almost everywhere differentiable and therefore one can find a sequence $(x_n)_n$ such that

$$x_n \xrightarrow{n \rightarrow \infty} x_0 \quad \text{and} \quad Du_\alpha(x_n) \quad \text{exists} \quad \forall n.$$

Set $p_n^\alpha = Du_\alpha(x_n)$. Then for all n , there holds

$$\alpha u_\alpha(x_n) + H(x_n, p_n^\alpha) = 0 \tag{10}$$

Due to the boundedness of u_α and the coercivity condition (3), the sequence $(p_n^\alpha)_n$ is bounded so that it converges (up to a subsequence) to some $p^\alpha \in \mathbb{R}^N$. A passage to the limit in (10) then leads to

$$\alpha u_\alpha(x_0) + H(x_0, p^\alpha) = 0.$$

Now this is not possible because $u_\alpha(x_0) > 0$ and by assumption $H(x_0, p^\alpha) \geq 0$. We thus have $u_\alpha(x_0) = 0$ and so will be $u_0(x_0)$ by passing to the limit in α .

Now it remains to prove the Lemma 2 and we will be done.

3.5 Proof of Lemma 2

We know by [6] that there exists a unique λ for which 5 is solvable. Therefore, to prove the lemma, it suffices to prove that $\lambda = 0$. Consider for this sake, for some $\alpha > 0$, the unique solution u_α of the equation

$$\alpha u_\alpha + H(x, Du_\alpha) = \lambda, \quad x \in \mathbb{R}^N. \quad (11)$$

Let x_0 be a global minimum point of u_α . By testing (11) with 0, one has $\alpha u_\alpha(x_0) \geq \lambda$ and thus

$$\alpha u_\alpha(x) \geq \lambda, \quad \forall x \in \mathbb{R}^N. \quad (12)$$

Consider now the point x^* such that $H(x^*, p) \geq 0$ for all p . As previously, we can find a sequence $(x_n)_n$ which has x^* as limit and such that

$$\alpha u_\alpha(x_n) + H(x_n, Du_\alpha(x_n)) = \lambda$$

holds for all n . And thus, at the limit n goes to infinity, we recover

$$\alpha u_\alpha(x^*) + H(x^*, p^\alpha) = \lambda,$$

which leads to $\alpha u_\alpha(x^*) \leq \lambda$, which together with (12) gives $\alpha u_\alpha(x^*) = \lambda$. Now as α was taken arbitrarily and $(u_\alpha)_\alpha$ bounded, by letting α go to zero, we obtain $\lambda = 0$.

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