

# Finite Projective Spaces, Geometric Spreads of Lines and Multi-Qubits

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## Abstract

Given a  $(2N - 1)$ -dimensional projective space over  $\text{GF}(2)$ ,  $\text{PG}(2N - 1, 2)$ , and its geometric spread of lines, there exists a remarkable mapping of this space onto  $\text{PG}(N - 1, 4)$  where the lines of the spread correspond to the points and subspaces spanned by pairs of lines to the lines of  $\text{PG}(N - 1, 4)$ . Under such mapping, a non-degenerate quadric surface of the former space has for its image a non-singular Hermitian variety in the latter space, this quadric being *hyperbolic* or *elliptic* in dependence on  $N$  being *even* or *odd*, respectively. We employ this property to show that generalized Pauli groups of  $N$ -qubits also form two distinct families according to the parity of  $N$  and to put the role of symmetric Pauli operators into a new perspective. The  $N = 4$  case is taken to illustrate the issue, due to its link with the so-called black-hole/qubit correspondence.

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Multiple qubit states play a key role in various fields of quantum information theory like quantum computing, coding and quantum error-correction (see, e. g., [1]). Recently, and rather surprisingly, they have also been recognized to be of great relevance for getting insights into the nature of entropy formulas of a certain class of stringy black hole solutions (see, e. g., [2]). It is, therefore, important to deepen our understanding of these fundamental building blocks of quantum world. In the present note we do so through the geometry of their associated generalized Pauli groups.

Let  $\text{PG}(d, q)$  be a  $d$ -dimensional projective space over  $\text{GF}(q)$ ,  $q$  being a power of a prime.<sup>1</sup> A  $t$ -spread  $\mathcal{S}$  of  $\text{PG}(d, q)$  is a set of  $t$ -dimensional subspaces of  $\text{PG}(d, q)$  which partitions its point-set [4]. If the elements of  $\mathcal{S}$  in a subspace  $V$  form a  $t$ -spread on  $V$ , one says that  $\mathcal{S}$  induces a  $t$ -spread on  $V$ . A  $t$ -spread  $\mathcal{S}$  is called *geometric* (or normal) if it induces a  $t$ -spread on each  $(2t + 1)$ -dimensional subspaces of  $\text{PG}(d, q)$  spanned by a pair of its elements. It is a well-known fact that  $\text{PG}(d, q)$  possesses a  $t$ -spread iff  $(t + 1)|(d + 1)$ ; moreover, this condition is also sufficient for  $\text{PG}(d, q)$  to have a geometric  $t$ -spread. B. Segre showed [5] that a geometric  $t$ -spread of  $\text{PG}(N(t + 1) - 1, q)$ ,  $N \geq 2$ , gives rise to a projective space  $\text{PG}(N - 1, q^{t+1})$  as follows: the points of this space are the elements of  $\mathcal{S}$  and its lines are the  $(2t + 1)$ -dimensional subspaces spanned by any two distinct elements of  $\mathcal{S}$ , with incidence inherited from  $\text{PG}(N(t + 1) - 1, q)$ . For a particular case of  $t = 1$  (i. e., a spread of lines), Dye [6] demonstrated that a *hyperbolic* or an *elliptic* quadric of  $\text{PG}(2N - 1, q)$  has an induced (geometric) spread of lines if and only if  $N$  is, respectively, *even* or *odd*, in which case it is mapped onto a non-singular Hermitian variety  $\text{H}(N - 1, q^2)$  of  $\text{PG}(N - 1, q^2)$ . We shall now show that this property has for  $q = 2$  a very interesting physical implication.

It is already a firmly established fact [7, 8, 9, 10] that the commutation relations between the elements of the generalized Pauli group of  $N$ -qubits,  $N \geq 2$ , can be completely reformulated in the geometrical language of symplectic polar space of rank  $N$  and order two,  $\text{W}(2N - 1, 2)$ ; the generalized Pauli operators (discarding the identity) answer to the

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<sup>1</sup>For the standard mathematical nomenclature and notation employed in what follows, see, e. g., [3].

points of  $W(2N - 1, 2)$ , a maximally commuting subset has its representative in a maximal totally isotropic subspace of  $W(2N - 1, 2)$  and commuting translates into collinear. One of the most natural representations of  $W(2N - 1, 2)$  is that in terms of the points and the set of totally isotropic subspaces of  $PG(2N - 1, 2)$  endowed with a symplectic polarity. Employing this representation, it has been found in [8] that in the real case the *symmetric* elements/operators of the  $N$ -qubit Pauli group *always* lie on a *hyperbolic* quadric in the ambient space  $PG(2N - 1, 2)$ . Combining this fact with Dye's result, we arrive at our main observation: *it is only for  $N$  even when all symmetric generalized Pauli operators of  $W(2N - 1, 2)$  can be mapped to the points of an Hermitian variety of the space  $PG(N - 1, 4)$  associated through a geometric spread of lines with the ambient space  $PG(2N - 1, 2)$* . Hence, in this regard, when it comes to generalized Pauli groups 'even-numbered' multi-qubits are found to stand on a slightly different footing than 'odd-numbered' ones.

We shall finish this communication by briefly mentioning an especially interesting even case,  $N = 4$ . Here, a hyperbolic quadric  $Q^+(7, 2)$  of  $PG(7, 2)$  formed by the symmetric operators is well known for its puzzling triality swapping points and two systems of generators and has for its spread-induced image an Hermitian surface  $H(3, 4)$  of  $PG(3, 4)$  (see, e.g., [11]). This Hermitian surface is, in turn, nothing but the generalized quadrangle  $GQ(4, 2)$  in disguise (see, e.g., [12]), the dual of which —  $GQ(2, 4)$  — was found to play a prominent role in the so-called black-hole-qubit correspondence, by fully encoding the  $E_{6(6)}$  symmetric entropy formula describing black holes and black strings in  $D = 5$  [13]. Our finding thus, *inter alia*, not only opens up an unexpected window through which also four-qubit Pauli group, like its lower rank cousins, could find its way into some black hole entropy formula(s), but also puts the role of symmetric operators into a new perspective. It is also important to keep in mind this remarkable *three-to-one* correspondence, i.e., that it is always a triple of (collinear) operators of the ambient space  $PG(7, 2)$  which comprises a single point of  $PG(3, 4)$ .

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