Singular solutions of some nonlinear parabolic equations with spatially inhomogeneous absorption*

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Abstract

We study the limit behaviour of solutions of $\partial_t u - \Delta u + h(|x|)\,|u|^{p-1}\,u = 0$ in $\mathbb{R}^N \times (0,T)$ with initial data $k\delta_0$ when $k \to \infty$, where h is a positive nondecreasing function and $p > 1$. If $h(r) = r^\beta$, $\beta > N(p-1) - 2$, we prove that the limit function u_∞ is an explicit very singular solution, while such a solution does not exist if $\beta \leq N(p-1)-2$. If $\liminf_{r\to 0} r^2 \ln(1/h(r)) > 0$, u_∞ has a persistent singularity at $(0,t)$ $(t \geq 0)$. If $\int_0^{r_0} r\ln(1/h(r))\,dr < \infty$, u_∞ has a pointwise singularity localized at $(0,0)$.

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1 Introduction

Consider

$$\partial_t u - \Delta u + h(x)\,|u|^{p-1}\,u = 0 \quad \text{in } Q_T := \mathbb{R}^N \times (0,T), \tag{1.1}$$

with $p > 1$ and h is a nonnegative measurable function defined in $\mathbb{R}^N$. It is well known that if

$$\iint_{Q_T} h(x) E^p(x,t)\,dx\,dt < \infty, \tag{1.2}$$

where $E(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ is the heat kernel, then, for any $k > 0$ there exists a unique solution $u = u_k$ to (1.1) satisfying initial condition

$$u(.,0) = k\delta_0 \tag{1.3}$$

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in the sense of measures in $\mathbb{R}^N$. Furthermore the mapping $k \mapsto u_k$ is increasing. If it assumed that h is positive essentially locally bounded from above and from below in $\mathbb{R}^N \setminus \{0\}$, then the set $\{u_k\}$ is also bounded in the $C^1_{loc}(\overline{Q}_T \setminus \{0 \times (0,\infty)\})$-topology. Thus there exist $u_\infty := \lim_{k\to\infty} u_k$ and u_∞ is a solution of (1.1) in $Q_T \setminus \{0 \times (0,\infty)\}$. Furthermore u_∞ is continuous in $\overline{Q}_T \setminus \{0 \times [0,\infty)\}$ and vanishes on $\mathbb{R}^N \setminus \{0\} \times \{0\}$. Only two situations can occur:

(i) Either $u_\infty(0,t)$ is finite for every $t > 0$ and u_∞ is a solution of (1.1) in Q_T. Such a solution which has a pointwise singularity at $(0,0)$ is called a *very singular solution* (abr. V.S.S.)

(ii) Or $u_\infty(0,t) = \infty$ for every $t > 0$ and u_∞ is a solution of (1.1) in $Q_T \setminus \{0 \times (0,\infty)\}$ only. Such a solution with a persistent singularity is called a *razor blade* (abr. R. B.).

In the well-known article [4], Brezis, Peletier and Terman proved in 1985 that u_∞ is a V.S.S., if $h(x) \equiv 1$. Furthermore they showed that $u_\infty(x,t) = t^{-1/(p-1)} f(x/\sqrt{t})$ for $(x,t) \in Q_T$ where f is the unique positive (and radial) solution of the problem

$$\left\{\begin{array}{l} -\Delta f - \dfrac{1}{2}\eta.\nabla f - \dfrac{1}{p-1} f + |f|^{p-1} f = 0 \quad \text{in } \mathbb{R}^N \\ \qquad \lim_{|\eta|\to\infty} |\eta|^{2/(p-1)} f(\eta) = 0. \end{array}\right. \tag{1.4}$$

Their proof of existence and uniqueness relied on shooting method in ordinary differential equations (abr. O.D.E.). The already mentioned self-similar very singular solutions of the problem (1.4) was discovered independently in [6] too. Later on, a new proof of existence, has been given by Escobedo and Kavian [8] by a variational method in a weighted Sobolev space. More precisely they proved that the following functional

$$v \mapsto J(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 - \frac{1}{p-1} v^2 + \frac{2}{p+1}|v|^{p+1} \right) K(\eta) d\eta \tag{1.5}$$

achieves a nontrivial minimum in $H^1_K(\mathbb{R}^N)$, where $K(\eta) = e^{|\eta|^2/4}$.

In this article we first study equation (1.1) when $h(x) = |x|^\beta$ $(\beta \in \mathbb{R})$. Looking for self-similar solutions under the form $u(x,t) = t^{-(2+\beta)/2(p-1)} f(x/\sqrt{t})$, we are led to

$$\left\{\begin{array}{l} -\Delta f - \dfrac{1}{2}\eta.\nabla f - \dfrac{2+\beta}{2(p-1)} f + |\eta|^\beta |f|^{p-1} f = 0 \quad \text{in } \mathbb{R}^N \\ f \in H^1_{loc}(\mathbb{R}^N) \cap L^{p+1}_{loc}(\mathbb{R}^N; |\eta|^\beta d\eta) \cap C^2(\mathbb{R}^N \setminus \{0\}) \\ \qquad \lim_{|\eta|\to\infty} |\eta|^{(2+\beta)/(p-1)} f(\eta) = 0, \end{array}\right. \tag{1.6}$$

and the associated functional

$$v \mapsto J(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 - \frac{2+\beta}{2(p-1)} v^2 + \frac{2}{p+1}|\eta|^\beta |v|^{p+1} \right) K(\eta) d\eta. \tag{1.7}$$

We prove the following

Theorem A *I- Assume $\beta \leq N(p-1) - 2$; then there exists no nonzero solution to (1.6).*

II- Assume $\beta > N(p-1) - 2$; then there exists a unique positive solution f^* to (1.6).

One of the key arguments in the study of isolated singularities of (1.1) is the following a priori estimate

$$|u(x,t)| \leq \frac{\tilde{c}}{(t + |x|^2)^{(2+\beta)/2(p-1)}} \quad \forall (x,t) \in Q_T \tag{1.8}$$

valid for any $p > 1$ and $\beta > -2$. The remarkable aspect of this proof is that it is based upon the auxiliary construction of the maximal solution of (1.1) under a selfsimilar form. Next we give two proofs of II, one based upon scaling transformations and asymptotic analysis of O.D.E., combining ideas from [4], [5] and [10], and the second based on variational methods, extending some ideas from [8] and valid in a more general context. As a consequence we prove

Theorem B *Assume $\beta > N(p-1)-2$, then $u_\infty(x,t) = t^{-(2+\beta)/2(p-1)} f^*(x/\sqrt{t})$.*

It must be noticed that, if $\beta \leq N(p-1)-2$, u_k does not exist, and more precisely, the isolated singularities of solutions of (1.1) are removable.

Next we consider the case of more degenerate potentials $h(x)$:

$$\frac{h(x)}{|x|^\alpha} \to 0 \quad \text{as} \quad |x| \to 0 \;\; \forall \alpha > 0. \tag{1.9}$$

In the set of such potentials we find the borderline which separates the above mentioned two possibilities (i) — (V.S.S.) and (ii) — (R.B). Remark that in the case of flat potentials like (1.9), the corresponding solution $u_\infty(x,t)$ does not have self-similar structure and we have to find some alternative techniques for the study of the structure of u_∞. The main results of the paper are the following two statements.

Theorem C (sufficient condition for V.S.S. solution) *Assume that the function h is continuous and positive in $\mathbb{R}^N \setminus \{0\}$ and verifies the following flatness condition*

$$|x|^2 \ln\left(\frac{1}{h(x)}\right) \leq \omega(|x|) \Leftrightarrow h(x) \geq e^{-\omega(|x|)/|x|^2} \quad \forall x \in \mathbb{R}^N, \tag{1.10}$$

where the function $\omega \geq 0$ is nondecreasing, satisfies the following Dini-like condition

$$\int_0^1 \frac{\omega(s)ds}{s} < \infty, \tag{1.11}$$

and the additional technical condition

$$s\omega'(s) \leq (2-\alpha_0)\omega(s) \quad \text{near } 0, \tag{1.12}$$

for some $\alpha_0 \in (0,2)$. Then $u_\infty(x,t) < \infty$ for any $(x,t) \in Q_T$. Furthermore there exists positive constants C_i $(i = 1,2,3)$, depending only on N, α_0 and p, such that

$$\int_{\mathbb{R}^N} u_\infty^2(x,t)\,dx \leq C_1 t \exp\left[C_2\left(\Phi^{-1}(C_3 t)\right)^{-2}\right] \quad \forall t > 0, \tag{1.13}$$

where Φ^{-1} is the inverse function of

$$\Phi(\tau) := \int_0^\tau \frac{\omega(s)}{s}\,ds.$$

Notice that (1.11)-(1.12) is satisfied if $h(x) \geq Ce^{-|x|^{\theta-2}}$ for some $\theta > 0$.

Theorem D (sufficient condition for R.B. solution) *Assume h is continuous and positive in $\mathbb{R}^N \setminus \{0\}$ and satisfies*

$$\liminf_{x\to 0} |x|^2 \ln\left(\frac{1}{h(x)}\right) > 0 \Leftrightarrow \exists \omega_0 = \text{const} > 0 : h(x) \leq \exp\left(-\frac{\omega_0}{|x|^2}\right). \tag{1.14}$$

Then $u_\infty(0,t) = \infty$ for any $t > 0$, and $t \mapsto u_\infty(x,t)$ is increasing. If we denote $U(x) = \lim_{t\to\infty} u_\infty(x,t)$, then U is the minimal large solution of

$$-\Delta u + h(x)u^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.15}$$

i.e. the smallest solution of (1.15) which satisfies

$$\int_{B_\epsilon} u(x)dx = \infty \quad \forall \epsilon > 0. \tag{1.16}$$

Theorem C is proved by some new version of local energy method. A similar variant of this method was used in [1] for the study of extinction properties of solutions of nonstationary diffusion-absorption equations.

Theorem D is obtained by constructing local appropriate sub-solutions. The monotonicity and the limit property of u_∞ are characteristic of razor blades solutions [16].

A natural question which remains unsolved is to characterize u_∞ if the potential $h(x)$ satisfies

$$h(x) \approx \exp\left(-\frac{\omega(|x|)}{|x|^2}\right),$$

where $\omega(s) \to 0$ as $s \to 0$ and

$$\int_0^1 \frac{\omega(s)ds}{s} = \infty.$$

This article is the natural continuation of [12], [14] where (1.1) is replaced by

$$\partial_t u - \Delta u + h(t)\,|u|^{p-1}\,u = 0 \quad \text{ in } Q_T. \tag{1.17}$$

In equation (1.17), the function $h \in C([0,T])$ is positive in $(0,T]$ and vanishes only at $t = 0$. In the particular case $h(t) = t^\beta$ $(\beta > 0)$, u_k exists if and only if $1 < p < 1 + 2(1+\beta)/N$, and u_∞ is an explicit very singular solution. If $h(t) \geq e^{-\omega(t)/t}$ where ω is positive, nondecreasing and satisfies

$$\int_0^1 \frac{\sqrt{\omega(s)}ds}{s} = \infty,$$

then u_∞ has a pointwise singularity at $(0,0)$. If the degeneracy of h is stronger, namely

$$\liminf_{t\to 0} t \ln h(t) > -\infty,$$

it is proved that the singularity of u_k propagates along the axis $t = 0$; at end, u_∞ is nothing else than the (explicit) maximal solution $\Psi(t)$ of the O.D.E.

$$\Psi' + h(t)\Psi^p = 0 \quad \text{ in } (0,\infty). \tag{1.18}$$

A very general and probably difficult **open problem** generalizing (1.1) and (1.17) is to study the propagation phenomenon of singularities starting from $(0,0)$ when (1.1) is replaced by

$$\partial_t u - \Delta u + h(x,t)\,|u|^{p-1}\,u = 0 \quad \text{ in } Q_T, \tag{1.19}$$

where $h \in C(\overline{Q}_T)$ is nonnegative and vanishes only on a curve $\Gamma \subset \overline{Q}_T$ starting from $(0,0)$. It is expected that two types of phenomena should occur:

(i) either u_∞ has a pointwise singularity at $(0,0)$,

(ii) or u_∞ is singular along Γ or a connected part of Γ containing $(0,0)$.

It is natural to conjecture that the order of degeneracy should be measured in terms of the parabolic distance to Γ and of the slope of Γ in the space $\mathbb{R}^N \times \mathbb{R}$. This could serve as a starting model for nonlinear heat propagation in inhomogeneous fissured media.

Our paper is organized as follows: 1 Introduction - 2 The power case - 3 Pointwise singularities - 4 Existence of razor blades.

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2 The power case

In this section we assume that $h(x) = |x|^\beta$ with $\beta \in \mathbb{R}$, and the equation under consideration is the following

$$\partial_t u - \Delta u + |x|^\beta |u|^{p-1} u = 0 \quad \text{ in } Q_T := \mathbb{R}^N \times (0,T), \tag{2.1}$$

with $p > 1$. By a solution we mean a function $u \in C^{2,1}(Q_T)$. Let $E(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ be the heat kernel in Q_T and $\mathbb{E}[\phi]$ the heat potential of a function (or measure) ϕ defined by

$$\mathbb{E}[\phi](x,t) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/4t} \phi(y)\, dy. \tag{2.2}$$

If there holds

$$\iint_{Q_T} E^p(x,t)|x|^\beta dx\, dt < \infty, \tag{2.3}$$

it is easy to prove (see [12, Prop 1.2], and [18, Th 6.12]), that for any $k \in \mathbb{R}$, there exists a unique function $u = u_k \in L^1(B_R \times (0,T)) \cap L^p(B_R \times (0,T); |x|^\beta dx)$ such that

$$\iint_{Q_T} \left(-u\partial_t\zeta - u\Delta\zeta + |x|^\beta |u|^{p-1} u\zeta\right) dx\, dt = k\zeta(0,0), \tag{2.4}$$

for any $\zeta \in C_0^{2,1}(\mathbb{R}^N \times [0,T))$. By the maximum principle $k \mapsto u_k$ is increasing. Next, it is straightforward that (2.3) is fulfilled as soon as

$$\beta > \max\{N(p-1) - 2; -N\}. \tag{2.5}$$

2.1 The *a priori* estimate and the maximal solution

In order to prove an *a priori* estimate, we introduce the auxiliary N dimensional equation in the variable $\eta = x/\sqrt{t}$

$$-\Delta f - \frac{1}{2}\eta.\nabla f - \gamma f + |\eta|^\beta |f|^{p-1} f = 0, \tag{2.6}$$

where $\gamma = (2+\beta)/2(p-1)$.

Proposition 2.1 *Let $a > 0$ and $\beta \in \mathbb{R}$; then there exists a unique nonnegative function $F_a \in H^1_{loc}(B_a) \cap L^{p+1}_{loc}(B_a; |\eta|^\beta d\eta)$ solution of (2.6) and satisfying*

$$\lim_{|\eta| \to a} F_a(\eta) = \infty. \tag{2.7}$$

Furthermore $a \mapsto F_a$ is decreasing.

Proof. Set $K(\eta) = e^{|\eta|^2/4}$. Then (2.6) becomes

$$-K^{-1} div(K\nabla f) - \gamma f + |\eta|^\beta |f|^{p-1} f = 0. \tag{2.8}$$

Step 1- Boundary behaviour. First we claim that

$$\lim_{|\eta| \to a} (a - |\eta|)^{2/(p-1)} F_a(\eta) = \left(\frac{2(p+1)}{a^{p\beta}(p-1)^2}\right)^{1/(p-1)}. \tag{2.9}$$

Actually, if $0 < b < |\eta| < a$, u satisfies

$$-K^{-1} div(K\nabla F_a) - \gamma F_a + CF_a^p \leq 0$$

with $C = \min\{a^\beta, b^\beta\}$. We perform a standard variant of the two-sides estimate method used in [17] : we set $\Gamma := B_\rho \setminus B_b$ with $b < \rho < a$, $\alpha = (\rho - b)/2$ and denote by z the solution of

$$\left\{ \begin{array}{l} z'' - Cz^p = 0 \quad \text{in } (-\alpha, \alpha) \\ z(-\alpha) = z(\alpha) = \infty. \end{array} \right. \tag{2.10}$$

Then z is an even function and is computed by the formula

$$\int_{z(t)}^{\infty} \frac{ds}{\sqrt{s^{p+1} - z(0)^{p+1}}} = \sqrt{\frac{2C}{p+1}}(\alpha - t) \quad \forall t \in [0, \alpha). \tag{2.11}$$

Notice also that $\lim_{\alpha \to 0} z(t) = \infty$, uniformly on $(-\alpha, \alpha)$ and

$$\lim_{t \to \alpha} (t - \alpha)^{2/(p-1)} z(t) = \left(\frac{2(p+1)}{C^p (p-1)^2} \right)^{1/(p-1)}. \tag{2.12}$$

We set $Z(\eta) = z(|\eta| - (\rho + b)/2)$ and we look for a super-solution in Γ under the form $w = MZ(\eta)$ $(M > 1)$. Then

$$-K^{-1} div(K \nabla w) - \gamma w + C w^p = M \left((M^{p-1} - 1) C z^p - \left(\frac{N-1}{|\eta|} + \frac{|\eta|}{2} \right) z' - \gamma z \right).$$

Since

$$z'(t) = \sqrt{\frac{2C}{p+1}} \sqrt{z^{p+1}(t) - z(0)^{p+1}} < C^* z^{(p+1)/2}(t), \quad \text{with } C^* = \sqrt{\frac{2C}{p+1}},$$

we derive

$$-K^{-1} div(K \nabla w) - \gamma w + C w^p \geq M \left((M^{p-1} - 1) C z^p - \left(\frac{N-1}{b} + \frac{a}{2} \right) C^* z^{(p+1)/2} - \gamma z \right) \tag{2.13}$$

on $\{\eta : (\rho - b)/2 \, |\eta| < \rho\}$; and the same inequality holds true on $\{\eta : \rho < |\eta| < (\rho - b)/2\}$, up to interverting a and b. For any $M > 1$, we can choose $b > 0$ such that for any $b < \rho < a$, the right-hand side of (2.13) is positive and maximum principle applies in $B_\rho \setminus B_b$. Thus $MZ \geq F_a$ in Γ. Furthermore, the previous comparison still holds if we take $\rho = a$, which implies $\alpha = (a - b)/2$. Therefore, using the explicit value of C

$$\limsup_{|\eta| \to a} (a - |\eta|)^{2/(p-1)} F_a(\eta) \leq M \left(\frac{2(p+1)}{\min\{a^{p\beta}, b^{p\beta}\}(p-1)^2} \right)^{1/(p-1)}. \tag{2.14}$$

Because $M > 1$ and $0 < b < a$ are arbitrary, we derive

$$\limsup_{|\eta| \to a} (a - |\eta|)^{2/(p-1)} F_a(\eta) \leq \left(\frac{2(p+1)}{a^{p\beta}(p-1)^2} \right)^{1/(p-1)}. \tag{2.15}$$

For the estimate from below we notice that u satisfies

$$-K^{-1} div(K \nabla F_a) - \gamma F_a + \tilde{C} F_a^p \geq 0$$

in $\{\eta : b < |\eta| < a\}$, with $\tilde{C} = \max\{a^\beta, b^\beta\}$. Taking now $\alpha = a - b$, we denote by $\tilde{z}$ the positive solution of

$$\left\{ \begin{array}{l} \tilde{z}'' + \gamma \tilde{z} - \tilde{C} \tilde{z}^p = 0 \quad \text{in } (0, \alpha) \\ \tilde{z}(0) = 0 \\ \tilde{z}(\alpha) = \infty. \end{array} \right. \tag{2.16}$$

Then $\tilde{z}$ is computed by the formula

$$\int_{\tilde{z}(t)}^{\infty} \frac{ds}{\sqrt{\tilde{z}'^2(0) - \gamma s^2 + 2\tilde{C}s^{p+1}/(p+1)}} = \alpha - t \quad \forall t \in [0, \alpha), \tag{2.17}$$

and formula (2.12) is valid provided C be replaced by $\tilde{C}$. We fix $A \in \partial B_a$ with coordinates $(a, 0, ..., 0)$, and look for a subsolution under the form $\tilde{w}(\eta) = M\tilde{z}(\eta_1 - b)$ with $0 < M < 1$. Then

$$-K^{-1}div(K\nabla\tilde{w}) - \gamma\tilde{w} + \tilde{C}\tilde{w}^p = \tilde{M}\left((\tilde{M}^{p-1} - 1)\tilde{z}^p - \frac{\eta_1}{2}\tilde{w}'\right) \leq 0,$$

since $\tilde{w}' \geq 0$. Applying again the maximum principle, we derive $\tilde{w}(\eta) \leq F_a$ in $B_a \cap \{\eta : b < \eta_1 < a\}$. But clearly the direction η_1 is arbitrary and can be replaced by any radial direction. Thus

$$\liminf_{|\eta|\to a} (a - |\eta|)^{2/(p-1)} F_a(\eta) \geq \tilde{M}\left(\frac{2(p+1)}{\max\{a^{p\beta}, b^{p\beta}\}(p-1)^2}\right)^{1/(p-1)}. \tag{2.18}$$

In turn, (2.18) implies

$$\liminf_{|\eta|\to a} (a - |\eta|)^{2/(p-1)} F_a(\eta) \geq \left(\frac{2(p+1)}{a^{p\beta}(p-1)^2}\right)^{1/(p-1)}, \tag{2.19}$$

and (2.9) follows from (2.15) and (2.19).

Step 2- Uniqueness. If F' is another nonnegative solution of (2.6) satisfying the same boundary blow-up conditions, then for any $\epsilon > 0$, $F'_\epsilon = (1+\epsilon)F'$ is a super solution. Thus, for $\delta > 0$,

$$\begin{aligned}\iint_{B_a} &\left(-\frac{div(K\nabla F_a)}{F_a + \delta} + \frac{div(K\nabla F'_\epsilon)}{F'_\epsilon + \delta} + |\eta|^\beta \left(\frac{F_a^p}{F_a + \delta} - \frac{F'^p_\epsilon}{F'_\epsilon + \delta}\right) K\right) ((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+ d\eta \\ &\leq \gamma \iint_{B_a} \left(\frac{F_a}{F_a + \delta} - \frac{F'_\epsilon}{F'_\epsilon + \delta}\right) ((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+ K d\eta.\end{aligned}$$

By monotonicity

$$\left(\frac{F_a^p}{F_a + \delta} - \frac{F'^p_\epsilon}{F'_\epsilon + \delta}\right) ((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+ \geq 0,$$

and

$$0 \leq \left(\frac{F_a}{F_a + \delta} - \frac{F'_\epsilon}{F'_\epsilon + \delta}\right) ((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+ \leq ((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+.$$

By Lebesgue's theorem, since (2.9) implies that $((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+$ has compact support in B_a,

$$\lim_{\delta\to 0} \iint_{B_a} \left(\frac{F_a}{F_a + \delta} - \frac{F'_\epsilon}{F'_\epsilon + \delta}\right) ((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+ K d\eta = 0.$$

Using Green formula, we obtain

$$\begin{aligned}\iint_{B_a} &\left(-\frac{div(K\nabla F_a)}{F_a + \delta} + \frac{div(K\nabla F'_\epsilon)}{F'_\epsilon + \delta}\right) ((F_a + \delta)^2 - (F'_\epsilon + \delta)^2)_+ K d\eta \\ &= \iint_{F_a \geq F'_\epsilon} \left(\left|\nabla F_a - \frac{F_a + \delta}{F'_\epsilon + \delta}\nabla F'_\epsilon\right|^2 + \left|\nabla F'_\epsilon - \frac{F'_\epsilon + \delta}{F_a + \delta}\nabla F_a\right|^2\right) K d\eta \geq 0.\end{aligned}$$

Letting $\delta \to 0$, we derive, by Fatou's theorem,

$$\iint_{F_a \geq F'_\epsilon} \left(F_a^{p-1} - F_\epsilon'^{p-1}\right)(F_a^2 - F_\epsilon'^2) K d\eta \leq 0.$$

Thus $F_a \leq F'_\epsilon$. Since ϵ is arbitrary, $F_a \leq F'$. The reverse inequality is the same. The monotonicity of $a \mapsto F_a$ is proved in a similar way, by the previous form of maximum principle.

Step 3- Existence with finite boundary value. We shall first prove the existence of a positive solution w_k of (2.6) with boundary value equal to $k > 0$ for small value of a, and we shall let $k \to \infty$ in order to obtain one solution satisfying (2.7). We denote by J_a the functional defined over $H_0^1(B_a) \cap L^{p+1}(B_a; |\eta|^\beta d\eta)$ by

$$J_a(w) = \frac{1}{2}\int_{B_a} \left(|\nabla w|^2 - \gamma w^2 + \frac{1}{p+1}|\eta|^\beta |w|^{p+1}\right) K(\eta) d\eta.$$

Let $k > 0$ and $\kappa \in C^1(\overline{B}_a)$ with $0 \leq \kappa(\eta) \leq k$, $supp(\kappa) \subset \overline{B}_a \setminus B_{a/2}$, $\kappa(\eta) \equiv k$ on $\overline{B}_a \setminus B_{2a/3}$. If $v \in H_0^1(B_a) \cap L^{p+1}(B_a; |\eta|^\beta d\eta)$ and $w := v + \kappa$, then

$$J_a(w) = J_a(v + \kappa) \geq J_a(v) + J_a(\kappa) + \int_{B_a} \left(\nabla v.\nabla\kappa - \gamma v\kappa - |\eta|^\beta |v|^p \kappa\right) K(\eta) d\eta.$$

Since $\gamma \leq \lambda_a$, it follows from Cauchy-Schwarz and Hölder-Young inequalities that

$$J_a(w) \geq (1 - \epsilon^2) J_a(v) - \frac{p^p}{\epsilon^{2p}} J_a(\kappa)$$

for $0 < \epsilon < 1$. Because $\lim_{a\to 0} \lambda_a = \infty$, there exists $a_0 \in (0, \infty]$ such that, for any $0 < a < a_0$, $J_a(v)$ is bounded from below on $H_0^1(B_a) \cap L^{p+1}(B_a; |\eta|^\beta d\eta)$. Thus there exists a minimizer w_k such that $w_k = v + \kappa$ with v in the above space; w_k is a solution of (2.6) and $w_k|_{\partial B_a} = k$. Furthermore w_k is positive. Notice that if $\gamma \leq 0$, $a_0 = \infty$, in which case there exists a solution w_k for any $k > 0$ and any $a > 0$. The uniqueness of $w_k > 0$, is a consequence of the monotonicity of the mapping $k \mapsto w_k$ that we prove by a similar argument as in Step 2: if $k < k'$, there holds

$$\iint_{w_k > w_{k'}} \left(w_k^{p-1} - w_{k'}^{p-1})(w_k^2 - w_{k'}^2)\right) |\eta|^\beta K d\eta \leq 0,$$

which implies $w_k < \tilde{w}_k$. Uniqueness and radiality follows immediately, thus w_k solves the differential equation

$$\left\{\begin{array}{l} -w'' - \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) w' - \gamma w + r^\beta w^p = 0 \quad \text{on } (0, a) \\ w(a) = k \quad \text{and } w \in H^1_{rad}(B_a) \cap L^{p+1}_{rad}(B_a; |\eta|^\beta d\eta). \end{array}\right. \tag{2.20}$$

Next we shall assume $\gamma > 0$, equivalently $\beta > -2$. If w_k is a positive solution of (2.20) and $\lambda > 1$ (resp. $\lambda < 1$) λw_k is a super-solution (resp. a sub-solution) larger (resp. smaller) than w_k. Note that $\beta > -2$ implies $w_k(0) > 0$ while $\beta > -1$ implies also $w'_k(0) = 0$. Thus, by [13], there exists a solution $w_{\lambda k}$ with boundary data λk, and this solution is positive because $w_k \leq w_{\lambda k} \leq \lambda w_k$ (resp. $\lambda w_k \leq w_{\lambda k} \leq w_k$). Consequently, the set $\mathcal{A}$ of the positive $\tilde{a}$ such that there exists a positive solution of (2.20) on $(0, a)$ for any $a < \tilde{a}$ is not empty and independent of k. Furthermore, if for some $\tilde{a} > 0$ and some $k_0 > 0$, there exists some positive w_{k_0} solution of (2.20) on $0, \tilde{a})$, then for any $0 < a < \tilde{a}$ and any $k > 0$, there exists a positive solution w_k of (2.20). Since $r \mapsto \max\{k, (\gamma_+ a^{-\beta})^{1/(p-1)}\}$ is a super-solution, there holds

$$w_k(r) \leq \max\{k, (\gamma_+ a^{-\beta})^{1/(p-1)}\} \quad \forall r \in [0, a]. \tag{2.21}$$

Let us assume that $a^* = \sup \mathcal{A} < \infty$. Because of (2.21) and local regularity of solutions of elliptic equations, for any $\epsilon, \epsilon' > 0$, $w_k'(a)$ is bounded uniformly with respect if $\epsilon \leq a < a^* - \epsilon'$. But since (2.20) implies

$$a^{N-1} e^{a^2/4} w_k'(a) = \epsilon^{N-1} e^{\epsilon^2/4} w_k'(\epsilon) + \int_\epsilon^a (r^\beta w_k^p - \gamma w_k) r^{N-1} e^{r^2/4} dr,$$

$w_k'(a)$ is actually uniformly bounded on $[\epsilon, a^*)$. It follows from the local existence and uniqueness theorem that there exists $\delta > 0$, independent of $a < a^*$ such that there exists a unique solution z defined on $[a, a+\delta]$ to

$$\left\{ \begin{array}{l} -z'' - \left(\dfrac{N-1}{r} + \dfrac{r}{2} \right) z' - \gamma z + r^\beta z^p = 0 \quad \text{on } (0,a) \\ z(a) = k, \; z'(a) = w_k'(a), \end{array} \right. \tag{2.22}$$

and δ and $k > 0$ can be chosen such that $z > 0$ in $[a, a+\delta]$. This leads to the existence of a positive solution to (2.20) on $[0, a+\delta]$. If $a^* - a < \delta$, which contradicts the maximality of a^*. Therefore $a^* = \infty$.

Step 4- End of the proof. We have already seen that $k \mapsto w_k$ is increasing. By Step 1, we know that, for any $a > 0$, and some $b < a$, there holds

$$w_k(|\eta|) \leq C(a - |\eta|)^{-1/(p-1)} \text{on } B_a \setminus B_b. \tag{2.23}$$

In particular

$$w_k(b) \leq C^* = C^*(a, b, p, N)$$

Next

$$w_k(r) \leq \max\{C^*, (\gamma_+ b^{-\beta})^{1/(p-1)}\} \quad \forall r \in [0, b]. \tag{2.24}$$

Combining (2.23) and (2.24) implies that w_k is locally uniformly bounded on $[0, a)$. Since $k \mapsto w_k$ is increasing, the existence of $F_a := w_\infty = \lim_{k \to \infty} w_k$ follows. The fact that $a \mapsto F_a$ decreases is a consequence of the fact that $F_{a'}$ is finite on ∂B_a for any $a < a'$. $\square$

Remark. In the sequel we set $F_\infty = \lim_{a \to \infty} F_a$. Then F_∞ is a nondecreasing, nonnegative solution of (2.6). Using asymptotic analysis, is is easy to prove that there holds:

(i) if $\beta \neq 0$

$$F_\infty(\eta) = \left(\frac{1}{p-1} \right)^{1/(p-1)} |\eta|^{-\beta/(p-1)} (1 + \circ(1)) \quad \text{as } |\eta| \to \infty; \tag{2.25}$$

(ii) if $\beta = 0$,

$$F_\infty(\eta) \equiv \left(\frac{1}{p-1} \right)^{1/(p-1)}. \tag{2.26}$$

Furthermore, if $\beta > -2$, it follows by the strict maximum principle that $F_a(0) = \min\{F_a(\eta) : |\eta| < a\} > 0$. This observation plays a fundamental role for obtaining estimate from above.

Proposition 2.2 *Assume $p > 1$ and $\beta > -2$. Then any solution u of (2.1) in Q_T which verifies*

$$\lim_{t \to 0} u(x,t) = 0 \quad \forall x \neq 0, \tag{2.27}$$

satisfies

$$|u(x,t)| \leq \min \left\{ c^* \, |x|^{-(2+\beta)/(p-1)} \, ; t^{-(2+\beta)/2(p-1)} F_\infty(x/\sqrt{t}) \right\} \quad \forall (x,t) \in Q_T \setminus \{0\}, \tag{2.28}$$

where $c^* = c^*(N, p, \beta)$.

Proof. Let $\epsilon > 0$ and $a > 0$ and $\mathcal{P}_{a,\epsilon} = \{(x,t) : t > \epsilon, |x|/\sqrt{t-\epsilon} < a\}$. By the previous remark $\min F_a > 0$, thus the function $W(x,t) = (t-\epsilon)^{-(2+\beta)/2(p-1)} F_a(|x|/\sqrt{t-\epsilon})$, which is a solution of (2.1) in $\mathcal{P}_{a,\epsilon}$ tends to infinity on the boundary on $\mathcal{P}_{a,\epsilon}$; since u is finite in $Q_T \cap \mathcal{P}_{a,\epsilon}$, W dominates u in this domain. Letting successively $\epsilon \to 0$ and $a \to \infty$ yields to $u \leq F_\infty$. The estimate from below is similar. Next we consider $x \in \mathbb{R}^N \setminus \{0\}$, then $v = |u|$ satisfies (by Kato's inequality)

$$\partial_t v - \Delta v + C(x) v^p \leq 0 \quad \text{in } B_{|x|/2}(x) \times (0,T),$$

where $C(x) = \max\{(|x|/2)^\beta ; (3|x|/2)^\beta\}$. It is easy to construct a function under the form $w(y) = \Lambda \left(|x|^2 - 4|x-y|^2\right)^{-2/(p-1)}$ which satisfies

$$\left\{\begin{array}{l} -\Delta w + C(x) w^p = 0 \quad \text{in } B_{|x|/2}(x) \\ \lim_{|x-y| \to |x|/2} w = \infty, \end{array}\right.$$

with $\Lambda = \Lambda(x) = c^* \, |x|^{(2-\beta)/(p-1)}$, $c^* = c^*(N,p,\beta) > 0$. Using (2.27), it follows from Lebesgue's theorem that $u(y,t) \leq w(y)$ in $B_{|x|/2}(x) \times [0,T)$, thus $u(x,t) \leq w(x) = c^* |x|^{-(2+\beta)/(p-1)}$. Estimate from below is similar. $\square$

The construction of the first part of the proof of Proposition 2.2 (estimate in $\mathcal{P}_{a,\epsilon}$) shows that, without condition (2.27), equation (2.1) admits a maximal solution u_M.

Proposition 2.3 *Assume $p > 1$ and $\beta > -2$. Then any solution u to (2.1) satisfies*

$$|u(x,t)| \leq u_M(x,t) := t^{-(2+\beta)/2(p-1)} F_\infty(x/\sqrt{t}) \quad \forall (x,t) \in Q_T \setminus \{0\}. \tag{2.29}$$

As a variant of (2.28), we have the following Keller-Osserman type parabolic estimate which extends the classical one due to Brezis and Friedman in the case $\beta = 0$ (see [3]).

Proposition 2.4 *Under the assumptions of Proposition 2.2 there holds*

$$|u(x,t)| \leq \frac{\tilde{c}}{(|x|^2 + t)^{(2+\beta)/2(p-1)}} \quad \forall (x,t) \in Q_T \setminus \{0\}, \tag{2.30}$$

with $\tilde{c} = \tilde{c}(N,p,\beta)$.

Proof. Assume $|x|^2 \leq t$, then

$$\begin{aligned} \frac{1}{(|x|^2 + t)^{(2+\beta)/2(p-1)}} &\geq 2^{-(2+\beta)/2(p-1)} t^{-(2+\beta)/2(p-1)} \\ &\geq \frac{2^{-(2+\beta)/2(p-1)}}{\min\{F_\infty(\eta) : |\eta| \leq 1\}} t^{-(2+\beta)/2(p-1)} F_\infty(x/\sqrt{t}). \end{aligned} \tag{2.31}$$

Assume $|x|^2 \geq t$, then

$$\frac{1}{(|x|^2 + t)^{(2+\beta)/2(p-1)}} \geq 2^{-(2+\beta)/(p-1)} |x|^{-(2+\beta)/(p-1)}. \tag{2.32}$$

Combining (2.31) and (2.32) gives (2.30). $\square$

2.2 Isolated singularities and the very singular solution

Theorem 2.5 *Assume $p > 1$ and $-2 < \beta \leq N(p-1) - 2$. Then any solution u to (2.1) which satisfies (2.27) is identically* 0.

Proof. If $-(2+\beta)/(p-1) + N - 1 > -1$, equivalently $\beta < N(p-1) - 2$, the function $x \mapsto |x|^{-(2+\beta)/(p-1)}$ is locally integrable in $\mathbb{R}^N$, thus $u(.,t) \to 0$ in $L^1_{loc}(\mathbb{R}^N)$ as $t \to 0$. For $\epsilon > 0$ there exists $R = R(\epsilon)$ such that $u(x,t) \leq \epsilon$ for any $|x| \geq R$ and $t > 0$. Thus

$$u(x, t+\tau) \leq \epsilon + \mathbb{E}[u\chi_{B_R} u(.,\tau)](x,t) \quad \forall t > 0, \tau > 0 \,\text{and}\; x \in \mathbb{R}^N, \tag{2.33}$$

where $\mathbb{E}[\phi]$ denotes the heat potential of the measure ϕ (see (2.2)). Letting successively $\tau \to 0$ and $\epsilon \to 0$, yields to $u \leq 0$. In the same way $u \geq 0$. In the case $\beta = N(p-1) - 2$ estimate (2.30) reads

$$|u(x,t)| \leq \frac{\tilde{c}}{(|x|^2 + t)^{N/2}}.$$

From this estimate, the proof of [3, Th 2, Steps 5, 6] applies and we recall briefly the steps

(i) By choosing positive test functions ϕ_n which vanish in $\mathcal{V}_n = \{(x,t) : |x|^2 + t \leq n^{-1}\}$ and are constant on $\mathcal{V}'_n = \{(x,t) : |x|^2 + t \geq 2n^{-1}\}$, we first prove that, for any $\rho > 0$,

$$\iint_{B_\rho \times (0,T)} \left(|u(x,t)| + |x|^\beta |u|^p\right) dxdt < \infty. \tag{2.34}$$

Thus, using the same test function, we derive that the identity

$$\iint_{Q_T} \left(-u\partial_t \zeta - u\Delta\zeta + |x|^\beta \, |u|^{p-1} \, u\zeta\right) dx\, dt = 0, \tag{2.35}$$

holds for any $\zeta \in C_0^{2,1}(\mathbb{R}^N \times [0,T))$. The uniqueness yields to $u = 0$. $\square$

Proof of Theorem A- case I. In the case $-2 < \beta \leq N(p-1) - 2$, the result is a consequence of Theorem 2.5. Next we assume $\beta \leq -2$. If f is a solution of (1.6), it satisfies

$$f(\eta) = \circ(|\eta|^{-(2+\beta)/(p-1)}) \quad \text{as } |\eta| \to \infty.$$

If $\beta = -2$, the equation becomes

$$-\Delta f - \frac{1}{2}\eta.\nabla f + |\eta|^{-2} |f|^{p-1} f = 0,$$

and $f(\eta) \to 0$ at infinity. Since any positive constant is a supersolution, $f \leq 0$. Similarly $f \geq 0$.
If $\beta < -2$, for $\epsilon > 0$ the function $\eta \mapsto \epsilon|\eta|^{-(2+\beta)/(p-1)} = \psi(\eta)$ belongs to $W^{1,1}_{loc}(\mathbb{R}^N)$ since $\beta < -2$ and satisfies

$$\begin{aligned} -\Delta\psi - \frac{1}{2}\eta.\nabla\psi - \frac{2+\beta}{2(p-1)}\psi + |\eta|^\beta |\psi|^{p-1}\psi \\ = \epsilon r^{-(2+\beta)/(p-1)-2} \left(\left(\frac{2+\beta}{p-1}\right)\left(\frac{2+\beta}{p-1} + 2 - N\right) + \epsilon^{p-1}\right). \end{aligned}$$

Therefore, either if $N \geq 2$ or $N = 1$ and $\beta \leq -(p+1)$, ψ is a super-solution of (1.6) for any $\epsilon > 0$. The conclusion follows as above.

Finally we treat the case $N = 1$ and $-(p+1) < \beta < -2$ where there exists a particular solution of

$$f'' + \frac{r}{2} f' + \frac{2+\beta}{2(p-1)} f - r^\beta |f|^{p-1} f = 0 \quad \text{on } \mathbb{R}_+,$$

under the form $f_1(r) = A_{\beta,p} r^{-(2+\beta)/(p-1)}$. Furthermore, if $f \geq 0$ (which can be always assumed by the maximum principle), it is a subsolution of the linear equation

$$\phi'' + \frac{r}{2}\phi' + \frac{2+\beta}{2(p-1)}\phi = 0$$

Noticing that this equation has a solution ϕ_1 which has the same behaviour at infinity than the explicit solution of (1.4), namely

$$\phi_1(r) = cr^{-(2+\beta)/(p-1)}(1 + \circ(1)),$$

by standard methods (see e.g. [10, Prop A1]), the second solution ϕ_2 behaves in the following way

$$\phi_2(r) = cr^{(2+\beta)/(p-1)-1} e^{-r^2/4}(1 + \circ(1)) \quad \text{as } r \to \infty.$$

Consequently, by the maximum principle, any solution f of (1.4) on $\mathbb{R}$ such that $f(r) = \circ(\phi_1(r))$ at infinity, verifies

$$|f(r)| \leq C|r|^{(2+\beta)/(p-1)-1} e^{-r^2/4} \quad \text{for } |r| \geq 1. \tag{2.36}$$

Using the equation, we obtain that

$$f'(r) = e^{r^2/4}\int_r^\infty \left(s^\beta |f(s)|^{p-1} f(s) - \frac{2+\beta}{p-1} f(s)\right) ds,$$

thus

$$|f'(r)| \leq Cr^{(2+\beta)/(p-1)-2} e^{-r^2/4} \quad \text{for } |r| \geq 1. \tag{2.37}$$

Since $f \in H^1_{loc}(\mathbb{R})$, we derive that for any $n \in \mathbb{N}_*$,

$$\int_{-n}^{n} \left(f'^2 - \frac{2+\beta}{p-1} f^2\right) e^{r^2/4} dr \leq e^{n^2/4}\left(f(n)f'(n) - f(-n)f'(-n)\right).$$

Because of (2.36) and (2.36), this last term tends to 0 as $n \to \infty$. Therefore

$$\int_{-\infty}^{\infty} \left(f'^2 - \frac{2+\beta}{p-1} f^2\right) e^{r^2/4} dr = 0 \Longrightarrow f = 0,$$

which end the proof. $\square$

Remark. The method of proof used in the case $N = 1$ and $-p-1 < \beta < -2$ is actually valid in any dimension, for any $\beta \leq -2$. But it relies strongly on the fact that $f \in H^1_{loc}(\mathbb{R}^N)$, while the other methods use only $f \in W^{1,1}_{loc}(\mathbb{R}^N)$.

Proposition 2.6 *Assume* $\beta > \max\{N(p-1)-2; -N\}$ *. Then for any* $k > 0$ *there exists a unique solution* u_k *of (2.1) with initial data* $k\delta_0$*. Furthermore* $k \mapsto u_k$ *is increasing and* $u_\infty := \lim_{k\to\infty} u_k$ *satisfies* $u_\infty(x,t) = t^{-(2+\beta)/2(p-1)} f_\infty(x/\sqrt{t})$*, where* f_∞ *is positive, radially symmetric and satisfies*

$$\left\{\begin{array}{l} -\Delta f_\infty - \frac{1}{2}\eta.\nabla f_\infty - \gamma f_\infty + |\eta|^\beta f_\infty^p = 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|\eta|\to\infty} |\eta|^{(2+\beta)/(p-1)} f_\infty(\eta) = 0. \end{array}\right. \tag{2.38}$$

Proof. The existence of u_k and the monotonicity of $k \mapsto u_k$ has already been seen. By the uniform continuity of the u_k in any compact subset of $\bar{Q}_T \setminus \{(0,0)\}$, the function u_∞ satisfies

$$\lim_{t\to 0} u_\infty(x,t) = 0 \quad \forall x \neq 0. \tag{2.39}$$

For $\ell > 0$ and u is defined in Q_∞, we set

$$T_\ell[u](x,t) := \ell^{(2+\beta)/2(p-1)}u(\sqrt{\ell}x, \ell t). \tag{2.40}$$

If u satisfies equation (2.1) in Q_∞, $T_\ell[u]$ satisfies it too. Because of uniqueness

$$T_\ell[u_k] = u_{\ell^{(2+\beta)/2(p-1)-N/2}k}. \tag{2.41}$$

Using the continuity of $u \mapsto T_\ell[u]$ and the definition of u_∞, we can let $k \to \infty$ in (2.41) and derive (by taking $\ell t = 1$ and replacing t by ℓ),

$$T_\ell[u_\infty] = u_\infty \Longrightarrow u_\infty(x,t) = t^{-(2+\beta)/2(p-1)}u_\infty(x/\sqrt{t}, 1). \tag{2.42}$$

Setting $f_\infty(\eta) = u_\infty(x/\sqrt{t}, 1)$ with $\eta = x/\sqrt{t}$, it is straightforward that f_∞ satisfies (2.38) (using in particular 2.39). Furthermore f_∞ is radial and positive as the u_k are. $\square$

Lemma 2.7 *The function f_∞ satisfies*

$$f_\infty(\eta) = c|\eta|^{2\gamma-N}e^{-|\eta|^2/4}\left(1 + \circ(|\eta|^{-2})\right) \quad \text{as } |\eta| \to \infty, \tag{2.43}$$

for some $c = c_{N,p,\beta} > 0$. Furthermore

$$f'_\infty(\eta) = -\frac{c}{2}c|\eta|^{2\gamma+1-N}e^{-|\eta|^2/4}\left(1 + \circ(|\eta|^{-2})\right) \quad \text{as } |\eta| \to \infty. \tag{2.44}$$

Proof. Set $r = |\eta|$ and denote $f_\infty(\eta) = f_\infty(r)$. Then f_∞ satisfies,

$$f''_\infty + \left(\frac{N-1}{r} + \frac{r}{2}\right)f'_\infty + \gamma f_\infty - r^\beta |f_\infty|^{p-1}f_\infty = 0 \quad \text{on } (0,\infty), \tag{2.45}$$

and $\lim_{r\to\infty} r^{2\gamma}f_\infty(r) = 0$. We consider the auxiliary equation

$$f'' + \left(\frac{N-1}{r} + \frac{r}{2}\right)f' + \gamma f = 0 \quad \text{on } (0,\infty). \tag{2.46}$$

By [10, Prop A1], (2.46) admits two linearly independent solutions defined on $(0,\infty)$, y_1 and y_2 such that

$$y_1(r) = r^{-2\gamma}(1 + \circ(1)) \quad \text{and} \quad y_2(r) = r^{2\gamma-N}e^{-r^2/4}(1 + \circ(1)), \tag{2.47}$$

as $r \to \infty$. Next we choose $R > 0$ large enough so that the maximaum principle applies for equation (2.46) on $[R,\infty)$ and the y_j are positive on the same interval. For $\delta > 0$, $Y_\delta = \delta y_1 + f_\infty(R)y_2/y_2(R)$ is a supersolution for (2.45). Furthermore $f_\infty(r) = \circ(Y_\delta)$ at infinity. Letting $\delta \to 0$ yields to

$$f_\infty(r) \le \frac{f_\infty(R)}{y_2(R)}y_2(r) \quad \forall r \ge R. \tag{2.48}$$

Using (2.47) we derive

$$0 \le f_\infty(\eta) \le C|\eta|^{2\gamma-N}e^{-|\eta|^2/4} \quad \forall\, |\eta| \ge 1.$$

Plugging this estimate into (2.45), we derive (2.43) from standard perturbation theory for second order linear differential equation [2, p. 132-133]. Finally, (2.44) follows directly from (2.43) and (2.45). $\square$

An alternative proof of the existence of f_∞ is linked to calculus of variations. In the case $\beta = 0$, this was performed by Escobedo and Kavian [8]. This construction is based upon the study of the following functional

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 - \gamma v^2 + \frac{2}{p+1}|\eta|^\beta |v|^{p+1}\right)K(\eta)d\eta, \tag{2.49}$$

defined over the functions in $H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$.

Proposition 2.8 *Assume $p > 1$ and $\beta > N(p-1) - 2$. Then there exists a positive function $\tilde{f}_\infty \in H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$ satisfying*

$$-\Delta \tilde{f}_\infty - \frac{1}{2}\eta.\nabla \tilde{f}_\infty - \gamma \tilde{f}_\infty + |\eta|^\beta \tilde{f}^p_\infty = 0 \quad in\ \mathbb{R}^N. \tag{2.50}$$

We recall that the eigenvalues of $-K^{-1}div(K\nabla.)$ are the $\lambda_k = (N+k)/2$, with $k \in \mathbb{N}$ and the eigenspaces H_k are generated by $D^\alpha \phi$ where $\phi(\eta) = K^{-1}(\eta) = e^{-|\eta|^2/4}$ and $|\alpha| = k$. It is straightforward to check that J is C^1. In order to apply Ekeland Lemma, we have just to prove that J is bounded from below in $H^1_K(\mathbb{R}^N)$. As we shall see it later on, the proof is easy when $\beta < N(p-1)/2$, and more difficult when $\beta \geq N(p-1)/2$.

Lemma 2.9 *For any $v \in H^1_K(\mathbb{R}^N)$, there holds*

$$\frac{1}{4}\int_{\mathbb{R}^N} \left(2N + |\eta|^2\right) v^2 K(\eta) d\eta \leq \int_{\mathbb{R}^N} |\nabla v|^2 K(\eta) d\eta.$$

Proof. We borrow the proof to Escobedo and Kavian. Put $w = v\sqrt{K}$. Then

$$\sqrt{K}\nabla v = \nabla w - \frac{w}{2}\eta.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla v|^2 K(\eta) d\eta = \int_{\mathbb{R}^N} \left(|\nabla w|^2 - w\nabla w.\eta + \frac{1}{4} w^2 \left|\eta\right|^2\right) d\eta.$$

Because

$$-\int_{\mathbb{R}^N} w\nabla w.\eta d\eta = \frac{N}{2}\int_{\mathbb{R}^N} w^2 d\eta,$$

there holds

$$\int_{\mathbb{R}^N} |\nabla v|^2 K(\eta) d\eta = \int_{\mathbb{R}^N} \left(|\nabla w|^2 + \frac{N}{2} w^2 + \frac{1}{4} w^2 \left|\eta\right|^2\right) d\eta.$$

This implies the formula. □

Lemma 2.10 *Let $p > 1$ and $\beta < N(p-1)/2$. For any $\epsilon > 0$ there exists $C = C(\epsilon, p) > 0$ and $R = R(\epsilon, p) > 0$ such that*

$$\int_{\mathbb{R}^N} v^2 K(\eta) d\eta \leq \epsilon \int_{\mathbb{R}^N} |\nabla v|^2 K(\eta) d\eta + C \left(\int_{\mathbb{R}^N} |v|^{p+1} |\eta|^\beta K(\eta)\right)^{2/p+1}.$$

Proof. For $R > 0$ there holds

$$\int_{|\eta|\leq R} v^2 K(\eta) d\eta \leq \left(\int_{|\eta|\leq R} |v|^{p+1} |\eta|^\beta K(\eta) d\eta\right)^{2/(p+1)} \left(\int_{|\eta|\leq R} |\eta|^{-2\beta/(p-1)} K(\eta) d\eta\right)^{(p-1)/(p+1)}.$$

Since $\beta < N(p-1)/2 \Longleftrightarrow N > 2\beta/(p-1)$, we obtain

$$\left(\int_{|\eta|\leq R} |\eta|^{-2\beta/(p-1)} K(\eta) d\eta\right)^{(p-1)/(p+1)} = C(R, N, p).$$

By Lemma 2.9

$$\int_{|\eta|\geq R} v^2 K(\eta) d\eta \leq \frac{4}{R^2}\int_{\mathbb{R}^N} |\nabla v|^2 K(\eta) d\eta.$$

The estimate follows by taking $\epsilon = 4R^{-2}$. $\square$

It follows from the previous Lemmas that J is bounded from below in the space $H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$ whenever $N(p-1)/2 - 2 < \beta < N(p-1)/2$. Next we consider the case $\beta > 0$ and we shall restrict the study to radial functions.

Lemma 2.11 *Assume $\beta > 0$. The functional J is bounded from below on the set*

$$X = \left\{ v \in H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N) : v \geq 0, v \textit{ radial and decreasing} \right\}.$$

Proof. For $0 < \delta < R$, we write $J(v) = J_{\delta,R}(v) + J'_{\delta,R}(v) + J''_{\delta,R}(v)$ where

$$J_{\delta,R}(v) = \frac{1}{2}\int_{|\eta|\leq\delta} \left(|\nabla v|^2 - \gamma v^2 + \frac{2}{p+1} |\eta|^\beta |v|^{p+1} \right) K(\eta) d\eta,$$

$$J'_{\delta,R}(v) = \frac{1}{2}\int_{\delta<|\eta|<R} \left(|\nabla v|^2 - \gamma v^2 + \frac{2}{p+1} |\eta|^\beta |v|^{p+1} \right) K(\eta) d\eta,$$

and

$$J''_{\delta,R}(v) = \frac{1}{2}\int_{|\eta|>R} \left(|\nabla v|^2 - \gamma v^2 + \frac{2}{p+1} |\eta|^\beta |v|^{p+1} \right) K(\eta) d\eta.$$

Using Lemma 2.10, we fix R large enough so that $J''_{\delta,R}$ is bounded from below in $H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$. By Hölder's inequality $J'_{\delta,R}$ is bounded from below, thus we are left with $J_{\delta,R}$. We assume that v is positive, radial, nonincreasing and $v(\delta) = c = \min\{v(x) : |x| \leq \delta\}$. Then

$$|v|^{p+1} = v^{p+1} = (v - c + c)^{p+1} \geq (v-c)^{p+1} + c^{p+1} \text{ and } v^2 \leq 2(v-c)^2 + 2c^2,$$

$$J_{\delta,R} \geq \frac{1}{2}\int_{|\eta|\leq\delta} \left(|\nabla(v-c)|^2 - 2\gamma(v-c)^2 + \frac{2}{p+1} |\eta|^\beta |v-c|^{p+1} \right) K(\eta) d\eta + L(c),$$

where

$$L(c) = \frac{c^{p+1}}{p+1}\int_{|\eta|\leq\delta} |\eta|^\beta K(\eta) d\eta - \gamma c^2 \int_{|\eta|\leq\delta} K(\eta) d\eta.$$

Clearly $L(c) \geq M$ for some M independent of c. Therefore we are reduced to study the functional $J_{\delta,R}$ defined by

$$J_{\delta,R}(w) = \frac{1}{2}\int_{|\eta|\leq\delta} \left(|\nabla w|^2 - 2\gamma w^2 + \frac{2}{p+1} |\eta|^\beta |w|^{p+1} \right) K(\eta) d\eta$$

over $H^1_{0,K}(B_\delta) \cap L^{p+1}_{|\eta|^\beta K}(B_\delta)$. Here we can fix $\delta > 0$ small enough so that the first eigenvalue of $-K^{-1} div(K\nabla .)$ is larger than 2γ, thus $J_{\delta,R}(v)$ is bounded from below in the class of radially symmetric nonincreasing, nonnegative functions v, and so is J. $\square$

Lemma 2.12 *Let v be a radially symmetric function in $H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$. Then there exists a radially symmetric decreasing function $\tilde{v} \in H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$ such that $J(\tilde{v}) \leq J(v)$.*

Proof. We define the two curves

$$C_1 = \left\{ (s,x) \in \mathbb{R}_+ \times \mathbb{R}_+ : -2^{-1}\gamma x^2 + (p+1)^{-1} s^\beta x^{p+1} = 0 \right\} = \left\{ x = \left(2^{-1}(p+1)\gamma s^{-\beta}\right)^{1/(p-1)} \right\},$$

and

$$C_2 = \left\{(s,x) \in \mathbb{R}_+ \times \mathbb{R}_+ : -\gamma x + s^\beta x^p = 0\right\} = \left\{x = \left(\gamma s^{-\beta}\right)^{1/(p-1)}\right\}.$$

For fixed $s > 0$ the function $x \mapsto -2^{-1}\gamma x^2 + (p+1)^{-1} s^\beta x^{p+1}$ vanishes at $x = 0$. It has the following properties:

(i) it is decreasing for $0 < x < \left(\gamma s^{-\beta}\right)^{1/(p-1)}$,

(ii) it achieves a minimum at $x_s = \left(\gamma s^{-\beta}\right)^{1/(p-1)}$,

(iii) and it is increasing for $x > \left(\gamma s^{-\beta}\right)^{1/(p-1)}$ with infinite limit. Furthermore it vanishes at $\tilde{x}_s = \left(2^{-1}(p+1)\gamma s^{-\beta}\right)^{1/(p-1)}$.

Let v be a radially symmetric positive function. By approximation of radial elements in $H^1_{0,K}(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$, we can assume that v is C^2 with nondegenerate isolated extrema. We can also assume that the graph of v has at most a countable of intersections with C_2, $a_1 < a_2 < a_3 ... < a_k < ...$, that the set of points $\{a_k\}$ is discrete, that all the intersections are transverse and that, for every $j \geq 0$,

$$v(s) < \left(\gamma s^{-\beta}\right)^{1/(p-1)} \quad \text{on } (a_{2j}, a_{2j+1}),$$

where $a_0 = 0$, and

$$v(s) > \left(\gamma s^{-\beta}\right)^{1/(p-1)} \quad \text{on } (a_{2j+1}, a_{2j+2+1}).$$

The modifications of the function v is performed by local modification on each interval (a_k, a_{k+1}):

Step 1- The construction of $\tilde{v}$ on (a_{2j}, a_{2j+1}) is as follows. Let $\alpha_1 < \alpha_2 < ...$ be the sequence of local extrema of v, with $v(\alpha_{2i+1})$ local minimum and $v(\alpha_{2i+2})$ local maximum. By extension, since $v'(a_{2j+1}) > -\beta/(p-1)\gamma^{1/(p-1)} a_{2j+1}^{-(\beta+p-1)/(p-1)}$, $v(a_{2j+1})$ is a local maximum of v on (a_{2j}, a_{2j+1}).
If $\max\{(\alpha_{2i+1}) : i \geq 1\} \leq v(a_{2j+1})$, then $\tilde{v} = \max\{v, v(a_{2j+1})\}$.
If $\max\{v(\alpha_{2i+1}) : i \geq 1\} > v(a_{2j+1})$, we define the increasing sequence $\{\alpha_{2i_d+1}\}$ by

$$v(\alpha_{2i_0+1}) = \max\{v(\alpha_{2i+1}) : i \geq 1\},$$

$$v(\alpha_{2i_1+1}) \max\{v(\alpha_{2i+1}) : i > i_0\},$$

and by induction,

$$v(\alpha_{2i_d+1}) \max\{v(\alpha_{2i+1}) : i > i_{d-1}\}.$$

Thus we can assume that the local maxima of v are less than $v(a_{2j+1})$ on the last interval $(\alpha_{2i_d+1}, a_{2j+1})$. Next we define the function $\tilde{v}$ by $\tilde{v} = \max\{v, v(\alpha_{2i_0+1}\}$ on $(a_{2j}, \alpha_{2i_0+1})$, $\tilde{v} = \max\{v, v(\alpha_{2i_1+1}\}$ on $(\alpha_{2i_0+1}, \alpha_{2i_1+1})$. By induction, $\tilde{v} = \max\{v, v(\alpha_{2i_{d-1}+1}\}$ on $(\alpha_{2i_{d-1}+1}, \alpha_{2i_d+1})$. Finally $\tilde{v} = \max\{v, v(a_{2j+1})\}$ on the last interval $(\alpha_{2i_d+1}, a_{2j+1})$. The function $\tilde{v}$ is Lipschitz continuous, nonincreasing and, because $v(s) \leq \tilde{v}(s) \leq \left(\gamma s^{-\beta}\right)^{1/(p-1)}$, there holds

$$\begin{aligned}\int_{a_{2j} \leq |\eta| \leq a_{2j+1}} &\left(|\nabla \tilde{v}|^2 - \gamma \tilde{v}^2 + \frac{2}{p+1} |\eta|^\beta |\tilde{v}|^{p+1}\right) K(\eta) d\eta \\ &\leq \int_{a_{2j} \leq |\eta| \leq a_{2j+1}} \left(|\nabla v|^2 - \gamma v^2 + \frac{2}{p+1} |\eta|^\beta |v|^{p+1}\right) K(\eta) d\eta.\end{aligned} \tag{2.51}$$

Step 2- The construction of $\tilde{v}$ on (a_{2j+1}, a_{2j+2}) follows the same principle. Let $\beta_1 < \beta_2 < ... < \beta_d$ be the sequence of local minima of v on this interval. Furthermore $v(a_{2j+1})$ is the minimum of v on (a_{2j+1}, a_{2j+2}) and $v'(a_{2j+2}) < -\beta/(p-1)\gamma^{1/(p-1)} a_{2j+2}^{-(\beta+p-1)/(p-1)}$.

On (a_{2j+1},β_1) we set $\tilde{v}=\min\{v,v(a_{2j+1})\}$. On (β_1,β_2), $\tilde{v}=\min\{v,\tilde{v}(\beta_1)\}$. By induction $\tilde{v}=\min\{v,\tilde{v}(\beta_i)\}$ on (β_i,β_{i+1}). On the last interval (β_d,b_{2j+2}), $\tilde{v}=\min\{v,\tilde{v}(\beta_d)\}$. Because $\tilde{v}\leq v$ on this interval and $x\mapsto -2^{-1}\gamma x^2+(p+1)^{-1}s^\beta x^{p+1}$ is increasing above the curve C_2, we obtain similarly

$$\begin{aligned}\int_{a_{2j+1}\leq|\eta|\leq a_{2j+2}}&\left(|\nabla\tilde{v}|^2-\gamma\tilde{v}^2+\frac{2}{p+1}|\eta|^\beta|\tilde{v}|^{p+1}\right)K(\eta)d\eta\\ &\leq\int_{a_{2j+1}\leq|\eta|\leq a_{2j+2}}\left(|\nabla v|^2-\gamma v^2+\frac{2}{p+1}|\eta|^\beta|v|^{p+1}\right)K(\eta)d\eta.\end{aligned}\tag{2.52}$$

By construction $\tilde{v}$ is nonincreasing. Combining (2.51) and (2.52), we obtain $J(\tilde{v})\leq J(\tilde{v})$. $\square$

Proof of Proposition 2.8. It follows from the previous lemmas that J is bounded from below on X and the function $\phi=K^{-1}$ belongs to X. Furthermore

$$J(t\phi)=\frac{(N-2\gamma)t^2}{4}\int K^{-1}(\eta)d\eta+\frac{|t|^{p+1}}{p+1}\int\phi^p(\eta)d\eta.$$

Since $\beta>N(p-1)-2\iff N-2\gamma<0$, the infimum m of J over radially symmetric functions is negative but finite and achieved by a decreasing function. Let $\{v_n\}\subset X$ a sequence such that $J(v_n)\downarrow m$. Then $\{v_n\}$ remains bounded in $H^1_K(\mathbb{R}^N)\cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$. Up to a subsequence we can assume that v_n converges weakly in $H^1_K(\mathbb{R}^N)$ and in $L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$ and strongly in $L^1_K(\mathbb{R}^N)$ to some function v. Moreover this convergence holds a.e., and, since $v_n\in X$ the same holds with v. Going to the limit in the functional yields to

$$J(v)\leq\liminf_{n\to\infty}J(v_n)=m;$$

thus v is a critical point. $\square$

The following uniqueness result holds.

Proposition 2.13 *Assume $p>1$ and $\beta>N(p-1)-2$. Then $f_\infty=\tilde{f}_\infty$. Furthermore f_∞ is the unique positive solution of (2.38).*

Proof. We first prove that $\tilde{f}_\infty$ is the unique positive radial solution of (2.50) belonging to $H^1_K(\mathbb{R}^N)\cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$. We denote $r=|\eta|$ and $\tilde{f}_\infty(\eta)=\tilde{f}_\infty(r)$. Let $\hat{f}$ be another solution in the same class. Thus there exists $\{r_n\}$ converging to ∞ such that $\hat{f}(r_n)\to 0$. For $\epsilon>0$, set $\tilde{f}_\epsilon=\tilde{f}_\infty+\epsilon$. For $n\geq n_0$, large enough, $w_+(r_n)=0$, thus, as in the proof of Proposition 2.1,

$$\begin{aligned}\iint_{B_{r_n}}\left(\left|\nabla\hat{f}-\frac{\hat{f}}{\tilde{f}_\epsilon}\nabla\tilde{f}_\epsilon\right|^2+\left|\nabla\tilde{f}_\epsilon-\frac{\tilde{f}_\epsilon}{\hat{f}}\nabla\hat{f}\right|^2\right)Kd\eta+\gamma\iint_{B_{r_n}}\frac{\epsilon}{\tilde{f}_\epsilon}(\hat{f}^2-\tilde{f}_\epsilon^2)_+Kd\eta&\\ +\iint_{B_{r_n}}|\eta|^\beta(\hat{f}^{p-1}-\tilde{f}_\epsilon^{p-1})(\hat{f}^2-\tilde{f}_\epsilon^2)_+Kd\eta\leq 0.&\end{aligned}$$

We let successively $r_n\to\infty$ with Fatou's lemma, and $\epsilon\to 0$ with Lebesgue's theorem, since $\epsilon/\tilde{f}_\epsilon\leq 1$ and $(\hat{f}^2-\tilde{f}_\epsilon^2)_+\leq\hat{f}^2+\tilde{f}_\infty^2\in L^1_K(\mathbb{R}^N)$. We get

$$\iint_{\mathbb{R}^N}\left(\left|\nabla\hat{f}-\frac{\hat{f}}{\tilde{f}_\infty}\nabla\tilde{f}_\infty\right|^2+\left|\nabla\tilde{f}_\infty-\frac{\tilde{f}_\infty}{\hat{f}}\nabla\hat{f}\right|^2+|\eta|^\beta(\hat{f}^{p-1}-\tilde{f}_\infty^{p-1})(\hat{f}^2-\tilde{f}_\infty^2)_+\right)Kd\eta\leq 0,$$

which implies $\hat{f} \leq \tilde{f}_\infty$. In the same way $\tilde{f}_\infty \leq \hat{f}$. By Lemma 2.7, $f_\infty \in H^1_K(\mathbb{R}^N) \cap L^{p+1}_{|\eta|^\beta K}(\mathbb{R}^N)$. Thus $f_\infty = \tilde{f}_\infty$. $\square$

We end this section with a classification result

Theorem 2.14 *Assume $p > 1$ and $\beta > N(p-1) - 2$ and let u be a positive solution of (2.1) which satisfies (2.27). Then,*

(i) either there exists $k \geq 0$ such that $u = u_k$,

(i) or $u = u_\infty$.

Proof. Because of (2.27), the initial trace $tr(u)$ of u is is a outer regular Borel measure concentrated at 0 (see [12]). Then either the initial trace is a Radon measure, say $k\delta_0$, and we get (i), or

$$\lim_{t \to 0} \int_{B_\epsilon} u(x,t)dx = \infty, \tag{2.53}$$

for every $\epsilon > 0$. This implies $u \geq u_\infty$ as in [11]. Notice that, in this article, this estimate is performed in the case $\beta = 0$, but the proof in the general case is the same. In order to prove that $u \leq u_\infty$, we consider, for $\epsilon > 0$, the minimal solution $v := v_\epsilon$ of

$$\left\{ \begin{array}{r} \partial_t v - \Delta v + |x|^\beta |v|^{p-1} v = 0 \quad \text{in } \; Q_T \\ tr(v) = \nu_{\bar{B}_\epsilon}, \end{array} \right. \tag{2.54}$$

where $\nu_{\bar{B}_\epsilon}$ is the outer regular Borel measure such that $\nu_{\bar{B}_\epsilon}(E) = 0$ for any Borel set $E \subset \mathbb{R}^N$ such that $E \cap \bar{B}_\epsilon = \emptyset$, and $\nu_{\bar{B}_\epsilon}(E) = \infty$ otherwhile. This solution is constructed as the limit, when $m \to \infty$ of the solution $v_{\epsilon,m}$ of (2.1) verifying $v_{\epsilon,m}(.,0) = m\chi_{\bar{B}_\epsilon}$. Clearly $u \leq v_\epsilon$. Furthermore, for any $\ell > 0$,

$$T_\ell[v_{\epsilon,m}] = v_{\epsilon/\sqrt{\ell}, m\ell^{(2+\beta)/2(p-1)}} \Longrightarrow T_\ell[v_\epsilon] = v_{\epsilon/\sqrt{\ell}} \Longrightarrow T_\ell[v_0] = v_0, \tag{2.55}$$

where $v_0 = \lim_{\epsilon \to 0} v_\epsilon$. This, and the fact that $\lim_{t\to 0} v_0(x,t) = 0$ for every $x \in \mathbb{R}^N \setminus \{0\}$, imply that $v_0(x,t) = t^{-(2+\beta)/2(p-1)} f_\infty(x/\sqrt{t}) = u_\infty(x,t)$. At the end, since $u \leq v_\epsilon \Longrightarrow u \leq v_0$, it follows $u \leq u_\infty$. $\square$

3 Existence of very singular solutions

In this section, we study the singular set of the solution u_∞, in the case of strongly degenerate potential (1.9), using some variant of the local energy estimate (abr. L.E.E.) method in the spirit of Saint-Venant's principle. The L.E.E. technique was first used for singular solutions of quasilinear parabolic equations in [15]. An adaption of this method to the study of conditions of removability of the point singularities of solutions of the quasilinear parabolic equations of diffusion-strong absorption type was presented in [9]. In [14] there was elaborated a variant of the L.E.E. method, which allowed to find sharp conditions on the time dependent absorption potential, guaranteing existence of very singular solutions of the Cauchy problem to diffusion-strong absorption type equation with point singularity set. Here we provide a new application of the L.E.E. method in describing the transformation of V.S.S solution into the R.B. solution in terms of the flatness of the absorption potential in the space variables.

We consider the sequence of the Cauchy problems

$$u_t - \Delta u + h(|x|)|u|^{p-1}u = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,T), \; p > 1, \tag{3.1}$$

$$u(x,0)=u_{0,k}(x)=M_k\ \exp(-2^{-1}\mu_0Nk)\delta_k(x), \tag{3.2}$$

where δ_k is a regularized Dirac measure: $\delta_k\in C(\mathbb{R}^N),\ \delta_k\rightharpoonup\delta$ weakly in the sense of measures as $k\to\infty$,

$$\operatorname{supp}\delta_k\subset\{x:|x|\le\exp(-\mu_0k)\}\quad\forall k\in\mathbb{N}, \tag{3.3}$$

where the constant $\mu_0>0$ will be defined later on, and

$$M_k=\exp\exp k\quad\forall k\in\mathbb{N}. \tag{3.4}$$

Without loss of generality we suppose that

$$\|\delta_k\|^2_{L_2(\mathbb{R}^N)}\le\exp(\mu_0Nk). \tag{3.5}$$

We write the potential h in the equation (3.1) under the form,

$$h(s)=\exp(-\omega(s)s^{-2})\quad\forall s\ge0, \tag{3.6}$$

where $\omega(s)\ge0$ is arbitrary nondecreasing function on $[0,\infty)$.

Theorem 3.1 *Let the function $\omega(s)$ defined in (3.6) satisfy additionally the following Dini-like condition*

$$\int_0^{d_1}\omega(s)s^{-1}ds\le d_2<\infty,\quad d_1=\text{const}>0, \tag{3.7}$$

and the following technical condition

$$\frac{s\omega'(s)}{\omega(s)}\le2-\alpha_0\quad\forall s\in(0,s_0),\ s_0>0,\ 0<\alpha_0=\text{const}<2. \tag{3.8}$$

Then the following a priori estimate of solutions u_k of the problem (3.1), (3.2), (3.5), holds uniformly with respect to $k\in\mathbb{N}$,

$$\int_{\mathbb{R}^N}|u_k(x,t)|^2dx\le C_1t\exp\left[C_2\left(\Phi^{-1}\left(\frac{t}{C_3}\right)\right)^{-2}\right], \tag{3.9}$$

where the constants $C_1>0,C_2>0,C_3>0$ do not depend on k. Here $\Phi^{-1}(s)$ is the inverse function to

$$s\mapsto\Phi(s):=\int_0^s\frac{\omega(r)}{r}d\tau.$$

Let us define the following families of domains

$$B(s):=\{x:|x|<s\},\quad\Omega(s):=\mathbb{R}^N\setminus B(s),$$

$$Q_{t_1}^{t_2}(s):=\Omega(s)\times(t_1,t_2),\quad\forall s>0,\quad\forall 0\le t_1<t_2\le T.$$

Let $u(x,t)\equiv u_k(x,t)$ be a solution of the problem (3.1), (3.2) under consideration. We introduce the energy functions

$$I(s,\tau):=\int_0^\tau\int_{\Omega(s)}\left(|\nabla_xu|^2+h(|x|)|u|^{p+1}\right)dx\,dt, \tag{3.10}$$

and

$$J(s,t)=\int_{\Omega(s)}|u(x,t)|^2dx,\quad E(s,t)=\int_{B(s)}|u(x,t)|^2dx. \tag{3.11}$$

Lemma 3.2 *The energy functions $J(s,t)$, $I(s,t)$ defined by (3.10), (3.11) corresponding to an arbitrary solution $u = u_k$ of problem (3.1), (3.2) satisfy the following a priori estimate*

$$J(s,t) + I(s,t) \le ctg(s) := ct \left(\int_0^s r^{-\frac{(N-1)(p-1)}{p+3}} h(r)^{\frac{2}{p+3}} dr \right)^{-\frac{p+3}{p-1}}, \quad \forall s \ge \exp(-\mu_0 k), \tag{3.12}$$

uniformly with respect to $k \in \mathbb{N}$.

By c, c_i we denote different positive constants, which depend on known parameters N, p, α_0, d_2 only, and their value may change from lines to lines.

Proof. Multiplying equation (3.1) by u and integrating in $Q_{t_1}^{t_2}(s)$, we obtain the following starting relation after standard computations,

$$2^{-1} \int_{\Omega(s)} |u(x,t_2)|^2 dx + \iint_{Q_{t_1}^{t_2}(s)} \left(|\nabla_x u|^2 + h(|x|)|u|^{p+1} \right) dx\, dt =$$
$$= 2^{-1} \int_{\Omega(s)} |u(x,t_1)|^2 dx + \int_{t_1}^{t_2} \int_{|x|=s} u \frac{\partial u}{\partial n}\, d\sigma\, dt := R_0 + R_1. \tag{3.13}$$

Let us estimate R_1 from above. Using Holder's and Young's inequalities we have

$$\left| \int_{|x|=s} u(x,t) \frac{\partial u}{\partial n}\, d\sigma \right| \le cs^{\frac{(N-1)(p-1)}{2(p+1)}} \left(\int_{|x|=s} |\nabla_x u|^2 d\sigma \right)^{1/2} \left(\int_{|x|=\tau} |u|^{p+1} d\sigma \right)^{\frac{1}{p+1}} \le$$
$$\le cs^{\frac{(N-1)(p-1)}{2(p+1)}} h(s)^{-\frac{1}{p-1}} \left(\int_{|x|=s} \left(|\nabla_x u|^2 + h(s)|u|^{p+1} \right) d\sigma \right)^{\frac{p+3}{2(p+1)}}.$$

Integrating in t, we get

$$\left| \int_0^\tau \int_{|x|=s} u \frac{\partial u}{\partial n}\, d\sigma\, dt \right| \le cs^{\frac{(N-1)(p-1)}{2(p+1)}} h(s)^{-\frac{1}{p-1}} \tau^{\frac{p-1}{2(p+1)}}$$
$$\times \left(\int_0^\tau \int_{|x|=s} \left(|\nabla_x u|^2 + h(s)|u|^{p+1} \right) d\sigma\, dt \right)^{\frac{p+3}{2(p+1)}}. \tag{3.14}$$

It is easy to see that

$$-\frac{d}{ds} I(s,\tau) = \int_0^\tau \int_{|x|=s} \left(|\nabla_x u|^2 + h(s)|u|^{p+1} \right) ds, \quad -\frac{d}{ds} J(s,t) \ge 0.$$

Because of the property (3.3) satisfied by $u_{0,k}$, and estimate (3.14), we derive the following inequality from relation (3.13) with $t_2 = t$, $t_1 = 0$, $s \ge \exp(-\mu_0 k)$,

$$J(s,t) + I(s,t) \le c\, t^{\frac{p-1}{2(p+1)}} h(s)^{-\frac{1}{p+1}} s^{\frac{(N-1)(p-1)}{2(p+1)}} \left(-\frac{d}{ds}(I(s,t) + J(s,t)) \right)^{\frac{p+3}{2(p+1)}}. \tag{3.15}$$

Solving this ordinary differential inequality (abr. O.D.I.) with respect to the function $I(s,t) + J(s,t)$, we deduce that estimate (3.12) holds for arbitrary $s \ge \exp(-\mu_0 k)$. □

Next, we define $s_k > 0$ by the relation

$$g(s_k) = M_k^{\varepsilon_0} = \exp(\varepsilon_0 \exp\, k), \tag{3.16}$$

where $0 < \varepsilon_0 < 1$ will be defined later on. Now we have to guarantee that

$$s_k \geq \exp(-\mu_0 k) := \overline{s}_k \quad \forall k > k_0(\varepsilon_0, \alpha_0, \nu_0, p). \tag{3.17}$$

Using [1, Lemma A1], it follows from the definitions (3.6)of function $h(.)$ and (3.12) of function $g(.)$, that the next estimate holds,

$$\left(\frac{2\alpha_0}{p+3}\right)^{\frac{p+3}{p-1}} g_1(s) \leq g(s) \leq \left(\frac{4}{p+3}\right)^{\frac{p+3}{p-1}} g_1(s), \tag{3.18}$$

where $g_1(s) = s^{N-1-\frac{3(p+3)}{p-1}} \omega(s)^{\frac{p+3}{p-1}} \exp\left(\frac{2}{(p-1)} \frac{\omega(s)}{s^2}\right)$, α_0 is constant from condition (3.8). The following simpler estimate follows from (3.18):

$$\exp\left(\frac{\omega(s)}{s^2}\frac{2}{(p-1)}(1-\nu_0)\right) \leq g(s) \leq \exp\left(\frac{\omega(s)}{s^2}\frac{2}{(p-1)}(1+\nu_0)\right), \tag{3.19}$$

for any $s \in (0, s_0)$, where $s_0 = s_0(\nu_0) \to 0$ as $\nu_0 \to 0$. As a consequence of definition (3.16) of s_k, and using (3.19), we get,

$$\frac{\omega(s_k)}{s_k^2}\frac{2(1-\nu_0)}{(p-1)} \leq \varepsilon_0 \exp k. \tag{3.20}$$

Integrating (3.8), we deduce that ω satisfies

$$\omega(s) \geq s^{2-\alpha_0} \quad \forall s > 0. \tag{3.21}$$

Combining (3.21) and (3.20) we derive:

$$s_k \geq \left(\frac{2(1-\nu_0)}{\varepsilon_0(p-1)}\right)^{\frac{1}{\alpha_0}} \exp\left(-\frac{k}{\alpha_0}\right). \tag{3.22}$$

Next we define μ_0 from (3.2) and set $\mu_0 = 2\alpha_0^{-1}$. It follows from (3.22) that (3.17) is satisfied for all $k > k_0 = k_0(\varepsilon_0, \alpha_0, \nu_0, p)$. As result we derive that estimate (3.12) obtained in Lemma 3.2 is valid for $s = s_k$, i.e.

$$J(s_k, t) + I(s_k, t) \leq ctg(s_k) \quad \forall k \geq k_0 = k_0(\varepsilon_0, \alpha_0, \nu_0, p). \tag{3.23}$$

In order to find estimates characterizing the behaviour of the energy function $E(s_k, t)$ with respect to the variable $t > 0$, we introduce the nonnegative cut-off function $\varphi_k \in C^1(\mathbb{R})$ defined by

$$\varphi_k(s) = 1 \text{ if } s < s_k, \quad \varphi_k(s) = 0 \text{ if } s \geq 2s_k, \ \varphi_k'(s) \leq cs_k^{-1}. \tag{3.24}$$

Multiplying (3.1) by $u_k\varphi_k^2(|x|)$ and integrating with respect to x, we get

$$2^{-1}\frac{d}{dt}\int_{\mathbb{R}^N} u^2(x,t)\varphi_k^2(|x|)dx + \int_{\mathbb{R}^N} |\nabla_x(u\varphi_k)|^2 dx + \int_{\mathbb{R}^N} h(|x|)\varphi_k^2|u|^{p+1}dx$$
$$\leq \int_{\mathbb{R}^N} u^2(x,t)|\nabla_x\varphi_k(|x|)|^2 dx := \mathbb{R}_1. \tag{3.25}$$

By (3.24) and (3.23), we obtain

$$\mathbb{R}_1 \leq c_1 s_k^{-2}\int_{s_k<|x|<2s_k} |u(x,t)|^2 dx \leq c_1 s_k^{-2} J(s_k, t) \leq c_2 s_k^{-2} tg(s_k). \tag{3.26}$$

Using (3.25), (3.26) and Poincaré's inequality we derive the following differential inequality,

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N} u^2(x,t)\varphi_k^2 dx\right) + d_0 s_k^{-2}\int_{B(2s_k)} u^2(x,t)\varphi_k^2 dx \le \overline{c} s_k^{-2} t g(s_k), \quad d_0 > 0. \tag{3.27}$$

If we set

$$\psi_k(t) := \int_{\mathbb{R}^N} |u_k(x,t)|^2 \varphi_k^2(|x|)dx,$$

it is straightforward that (3.27) implies that the following O.D.I. holds,

$$\psi_k'(t) + d_0 s_k^{-2}\psi_k(t) \le \overline{c} s_k^{-2} t g(s_k); \tag{3.28}$$

furthermore, we can rewrite (3.28) under the form

$$\psi_k'(t) + \frac{d_0}{2} s_k^{-2}\psi_k(t) + 2^{-1}\left(d_0 s_k^{-2}\psi_k(t) - 2\overline{c} s_k^{-2} t g(s_k)\right) \le 0. \tag{3.29}$$

Using the relations (3.2), (3.5) satisfied by $u_{k,0}$, we see that ψ_k verifies,

$$\psi_k(0) \le \int_{\mathbb{R}^N} |u_{k,0}(x)|^2 dx \le M_k. \tag{3.30}$$

At last, we define the t_k by

$$t_k = \gamma\omega(s_k) \tag{3.31}$$

where ω is the function in (3.6) and $\gamma > 0$ is a parameter which will be made precise in the next lemma.

Lemma 3.3 *There exists a constant $\gamma > 0$, which does not depend on k, such that any solution ψ_k of problem* (3.29), (3.30) *satisfies the following a priori estimate*

$$\psi_k(\overline{t}_k) \le 2d_0^{-1}\overline{c}\,\overline{t}_k g(s_k) \quad \forall_k > \overline{k}(\varepsilon_0, \nu_0), \tag{3.32}$$

for some $\overline{t}_k \le t_k$, where t_k is defined by (3.31).

Proof. Let us assume that (3.32) is not true, and for any $\gamma > 0$ there exist $k \ge k_0$ such that

$$\psi_k(t) > 2d_0^{-1}\overline{c} t g(s_k) \ \ \forall t : 0 < t < \gamma\omega(s_k) \equiv t_k. \tag{3.33}$$

This relation combined with (3.29) implies the following inequality,

$$\psi_k'(t) + \frac{d_0}{2} s_k^{-2}\psi_k(t) \le 0 \quad \forall t : 0 < t \le \gamma\omega(s_k).$$

Solving this O.D.I. and using (3.30), we get

$$\psi_k(t) \le \psi_k(0)\exp\left(-\frac{d_0 t}{2s_k^2}\right) \le M_k \exp\left(-\frac{d_0 t}{2s_k^2}\right) \quad \forall t \le \gamma\omega(s_k). \tag{3.34}$$

We derive easily the next estimate from (3.34) and (3.33)

$$M_k \exp\left(\frac{-d_0\gamma\omega(s_k)}{2s_k^2}\right) \ge 2d_0^{-1}\overline{c} g(s_k)\gamma\omega(s_k). \tag{3.35}$$

Using (3.16) and (3.4), we deduce from this last inequality,

$$(1-\varepsilon_0)\exp k \geq \frac{d_0\gamma\omega(s_k)}{2s_k^2} + \ln(2d_0^{-1}\overline{c}\gamma) - \ln(\omega(s_k)^{-1}). \tag{3.36}$$

Similarly to (3.20), it follows, from (3.19) and the definition (3.16) of s_k, that there holds

$$\frac{\omega(s_k)}{s_k^2}\frac{2(1+\nu_0)}{(p-1)} \geq \varepsilon_0 \exp k. \tag{3.37}$$

Using this estimate and (3.36), we derive

$$(1-\varepsilon_0)\exp k \geq \frac{d_0\gamma(p-1)\varepsilon_0}{4(1+\nu_0)}\exp k + \ln(d_0^{-1}2\overline{c}\gamma) - \ln(\omega(s_k))^{-1}. \tag{3.38}$$

Noticing that (3.21) implies

$$\ln(\omega(s_k))^{-1} \leq (2-\alpha_0)\ln(s_k^{-1}), \tag{3.39}$$

and (3.22) can be writen under the form

$$\ln(s_k^{-1}) \leq \frac{1}{\alpha_0}\ln\left(\frac{\varepsilon_0(p-1)}{2(1-\nu_0)}\right) + \frac{k}{\alpha_0}, \tag{3.40}$$

we deduce the following inequality from (3.39), (3.40) and (3.38),

$$\begin{aligned}(1-\varepsilon_0)\exp k \geq \frac{d_0\gamma(p-1)\varepsilon_0}{4(1+\nu_0)}\exp k + \ln(2d_0^{-1}\overline{c}\gamma) - (2-\alpha_0)\frac{k}{\alpha_0} \\ - \frac{(2-\alpha_0)}{\alpha_0}\ln\left(\frac{\varepsilon_0(p-1)}{2(1-\nu_0)}\right).\end{aligned} \tag{3.41}$$

If we define γ_0 by the equality

$$(1-\varepsilon_0) = \frac{d_0\gamma(p-1)\varepsilon_0}{8(1+\nu_0)} \Leftrightarrow \gamma = \frac{(1-\varepsilon_0)(1+\nu_0)8}{d_0(p-1)\varepsilon_0} := \gamma_0, \tag{3.42}$$

then inequality (3.41) yields to

$$\frac{(2-\alpha_0)}{\alpha_0}k \geq (1-\varepsilon_0)\exp k + \ln(2d_0^{-1}\overline{c}\gamma_0) - \frac{(2-\alpha_0)}{\alpha_0}\ln\left(\frac{\varepsilon_0(p-1)}{2(1-\nu_0)}\right).$$

It is clear that we can find $\overline{k} = \overline{k}(\varepsilon_0,\nu_0) < \infty$ such that the last inequality becomes impossible for $k \geq \overline{k}$, contradiction. Consequently, (3.33) does not hold for $\gamma = \gamma_0$ and estimate (3.32) is true with $\gamma = \gamma_0$. □

Proof of Theorem 3.1. Comparing definition (3.11) of $E(s,t)$ and definition of ψ_k, we easily see that

$$E(s_k,t) \leq \psi_k(t) \Rightarrow E(s_k,\overline{t}_k) \leq \psi_k(\overline{t}_k). \tag{3.43}$$

Therefore, using estimates (3.12), (3.32) and (3.43), we obtain

$$\int_{\mathbb{R}^N} |u_k(x,\overline{t}_k)|^2 dx = E(s_k,\overline{t}_k) + J(s_k,\overline{t}_k) \leq (d_0^{-1}\overline{c}+c)\overline{t}_k g(s_k). \tag{3.44}$$

Next we estimate the right-hand side of (3.44). Using (3.16), (3.31) and inequality (3.32), we get

$$\bar{t}_k g(s_k) \le \gamma_0 \omega(s_k) M_k^{\varepsilon_0} \le \gamma_0 \omega(s_0) \exp(\varepsilon_0 \exp k), \tag{3.45}$$

where γ_0 is defined by (3.42) and $s_0 > 0$ by (3.8). We obtain easily from (3.45)

$$(\bar{c} d_0^{-1} + c)\bar{t}_k g(s_k) \le \exp\left[\left(\varepsilon_0 + \frac{\ln(\gamma_0 \omega(s_0)(c + \bar{c} d_0^{-1}))}{\exp k}\right)\exp k\right]. \tag{3.46}$$

Let k_1 be the smallest integer such that

$$\ln\left(\gamma_0 \omega(s_0)(c + \bar{c} d_0^{-1})\right) \le \varepsilon_0 \exp k_1, \tag{3.47}$$

equivalently

$$k_1 = \left[\ln\left(\varepsilon_0^{-1} \ln\left(\gamma_0 \omega(s_0)(c + \bar{c} d_0^{-1})\right)\right)\right] + 1,$$

where $[a]$ denote integer part of a. Then it follows from (3.46)

$$(\bar{c} d_0^{-1} + c) t_k g(s_k) \le \exp(2\varepsilon_0 \exp k) \quad \forall k > k_1. \tag{3.48}$$

If we fix ε_0 such that

$$2\varepsilon_0 \le e^{-1}, \tag{3.49}$$

then the next estimate follows from (3.44) and (3.45)–(3.49)

$$\int_{\mathbb{R}^N} |u_k(x, \bar{t}_k)|^2 dx \le M_{k-1}, \tag{3.50}$$

for all $k \ge \max\{k_0, \bar{k}, k_1\}$, where k_0 is from (3.17), $\bar{k}$ – from (3.32), and k_1 from (3.47). Estimate (3.50) is the final step of the first round of computations. For the second round, we begin by definiting s_{k-1} analogously to s_k:

$$g(s_{k-1}) = M_{k-1}^{\varepsilon_0} = \exp(\varepsilon_0 \exp(k-1)). \tag{3.51}$$

From estimate (3.12), we obtain

$$J(s_{k-1}, t) + I(s_{k-1}, t) \le ctg(s_{k-1}), \tag{3.52}$$

since $s_{k-1} > s_k$. Analogously to φ_k, we define the function φ_{k-1} and set

$$\psi_{k-1}(t) := \int_{\mathbb{R}^N} |u_k(x,t)|^2 |\varphi_{k-1}(x)|^2 dx.$$

In the same way as (3.28), the following O.D.I. follows

$$\psi'_{k-1}(t) + d_0 s_{k-1}^{-2} \psi_{k-1}(t) \le \bar{c} s_{k-1}^{-2} t g(s_{k-1}) \quad \forall t > \bar{t}_k. \tag{3.53}$$

Using (3.50), we derive

$$\psi_{k-1}(\bar{t}_k) \le M_{k-1}, \quad \bar{t}_k \le t_k. \tag{3.54}$$

If we analyze the Cauchy problem (3.53)—(3.54) similarly as problem (3.28)—(3.30) was analyzed in Lemma 3.3, we obtain the following *a priori* estimate for $\psi_{k-1}(t)$,

$$\psi_{k-1}(\bar{t}_k + \bar{t}_{k-1}) \le 2 d_0^{-1} \bar{c} (\bar{t}_k + \bar{t}_{k-1}) g(s_{k-1}), \tag{3.55}$$

where $\overline{t}_{k-1} \leq t_{k-1} := \gamma_0\omega(s_{k-1})$ and γ_0 is defined in (3.42). It is clear that

$$E(s_{k-1}, t) \leq \psi_{k-1}(t) \quad \forall t \geq \overline{t}_k,$$

consequently

$$E(s_{k-1}, \overline{t}_k + \overline{t}_{k-1}) \leq \psi_{k-1}(\overline{t}_k + \overline{t}_{k-1}) \leq 2d_0^{-1}\overline{c}(\overline{t}_k + \overline{t}_{k-1})g(s_{k-1}). \tag{3.56}$$

From (3.52), we deduce

$$J(s_{k-1}, \overline{t}_k + \overline{t}_{k-1}) + I(s_{k-1}, \overline{t}_k + \overline{t}_{k-1}) \leq c(\overline{t}_k + \overline{t}_{k-1})g(s_{k-1}). \tag{3.57}$$

Summing estimates (3.56) and (3.57) we obtain

$$\int_{\mathbb{R}^N} |u_k(x, \overline{t}_k + \overline{t}_{k-1})|^2 dx \leq (\overline{c}d_0^{-1} + c)(\overline{t}_k + \overline{t}_{k-1})g(s_{k-1}), \tag{3.58}$$

and we use this last estimate for performing a similar third round of computations. Iterating this process j times, we deduce

$$\int_{\mathbb{R}^N} \left| u_k\left(x, \sum_{i=k}^{k-j} \overline{t}_i\right)\right|^2 dx \leq (\overline{c}d_0^{-1} + c)\left(\sum_{i=k}^{k-j} \overline{t}_i\right) g(s_{k-j}). \tag{3.59}$$

In particular, we can take $j = k - l$, where $l \in N$ satisfies

$$l \geq l_0 := \max\{k_0, \overline{k}, k_1\}. \tag{3.60}$$

Then we obtain:

$$\int_{\mathbb{R}^N} \left| u_k\left(x, \sum_{i=k}^{l} \overline{t}_i\right)\right|^2 dx \leq (\overline{c}d_0^{-1} + c)\left(\sum_{i=k}^{l} \overline{t}_i\right) g(s_l). \tag{3.61}$$

Next, we have to estimate from above the sum of the $\overline{t}_i$ for which there holds

$$\sum_{i=k}^{l} \overline{t}_i \leq \sum_{i=k}^{l} \gamma_0\omega(s_i), \tag{3.62}$$

where s_i is defined by $g(s_i) = M_i^{\varepsilon_0}$. By the same way as in (3.37), we obtain

$$s_i^2 \leq \frac{2(1+\nu_0)\omega(s_i)}{(p-1)\varepsilon_0}\exp(-i) \leq \frac{2(1+\nu_0)\omega(s_0)}{(p-1)\varepsilon_0}\exp(-i) \quad \forall i \geq l_0,$$

where l_0 is the integer appearing in (3.60), and from this inequality follows

$$s_i \leq \left(\frac{2(1+\nu_0)\omega(s_0)}{(p-1)\varepsilon_0}\right)^{1/2} \exp\left(-\frac{i}{2}\right) := C_1 \exp\left(-\frac{i}{2}\right). \tag{3.63}$$

Therefore, using the monotonicity of the function ω, we derive

$$\begin{aligned}\sum_{i=k}^{l} \omega(s_i) \leq \sum_{i=k}^{l} \omega\left(C_1 \exp\left(-\frac{i}{2}\right)\right) &\leq -\int_k^{l-1} \omega\left(C_1 \exp\left(-\frac{s}{2}\right)\right) ds \\ &\leq 2\int_{C_1\exp(-\frac{k}{2})}^{C_1\exp(-\frac{l-1}{2})} y^{-1}\omega(y)dy \\ &\leq 2\int_0^{C_1\exp(-\frac{l-1}{2})} y^{-1}\omega(y)dy \\ &:= 2\Phi\left(C_1\exp(-\frac{l-1}{2})\right).\end{aligned} \tag{3.64}$$

As a consequence of (3.62) and (3.64), we get

$$\sum_{i=k}^{l} \overline{t}_i \leq \sum_{i=\infty}^{l} t_i \leq 2\gamma_0 \Phi\left(C_1 \exp(-\frac{l-1}{2})\right) := T_l. \tag{3.65}$$

The Dini condition (3.7) implies that $T_l \to 0$ as $l \to \infty$. Next, we deduce from (3.61) that

$$\int_{\mathbb{R}^N} |u_k(x,T_l)|^2 dx \leq C_2 T_l g(s_l), \quad C_2 = \overline{c} d_0^{-1} + c \quad \forall k \geq l \geq l_0. \tag{3.66}$$

Using the fact that $s_l : g(s_l) = M_l^{\varepsilon_0}$ and (3.66), we derive

$$\int_{\mathbb{R}^N} |u_k(x,T_l)|^2 dx \leq C_2 T_l \exp(\varepsilon_0 \exp l). \tag{3.67}$$

Because (3.65) implies

$$\exp l = eC_1^2 \left(\Phi^{-1}\left(\frac{T_l}{2\gamma_0}\right)\right)^{-2}, \tag{3.68}$$

we get the following inequality by plugging this last relation into (3.67):

$$\int_{\mathbb{R}^N} |u_k(x,T_l)|^2 dx \leq C_2 T_l \exp\left[e \cdot \varepsilon_0 C_1^2 \left(\Phi^{-1}\left(\frac{T_l}{2\gamma_0}\right)\right)^{-2}\right] \quad \forall l \geq l_0.$$

At last, combining last estimate with (3.68), we obtain

$$\int_{\mathbb{R}^N} |u_k(x,t)|^2 dx \leq C_2 t \exp\left[e^2 \cdot \varepsilon_0 C_1^2 \left(\Phi^{-1}\left(\frac{t}{2\gamma_0}\right)\right)^{-2}\right] \quad \forall t > 0,$$

which ends the proof. □

Example 3.4 *Assume* $\omega(s) = s^{2-\alpha_0}$, $0 < \alpha_0 < 2$. *Then*

$$\Phi(s) = \int_0^s s^{1-\alpha_0} ds = \frac{s^{2-\alpha_0}}{2-\alpha_0} \Rightarrow \Phi^{-1}(s) = (2-\alpha)^{\frac{1}{2-\alpha_0}} s^{\frac{1}{2-\alpha_0}}.$$

Consequently, estimate (3.9) reads as follows,

$$\int_{\mathbb{R}^N} |u_k(x,t)|^2 dx \leq C_1 t \exp\left[C_2 \left(\frac{C_3}{2-\alpha_0}\right)^{\frac{2}{2-\alpha_0}} t^{-\frac{2}{2-\alpha_0}}\right] \quad \forall t > 0.$$

4 Razor blades

In this section we consider potential $h(|x|)$ of the form $e^{-\ell(x)}$ ($= e^{-\omega(|x|)/|x|^2}$ as in (3.6)) and equation (1.1) is written under the form

$$\partial_t u - \Delta u + e^{-\ell(|x|)} |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{4.1}$$

where $\ell \in C(\mathbb{R}^N)$ is positive, nonincreasing and $\lim_{r\to 0} \ell(r) = \infty$. Our main result is the following

Theorem 4.1 *Assume $p > 1$ and ℓ satisfies*

$$\liminf_{x\to 0} |x|^2 \ell(x) > 0. \tag{4.2}$$

Then the solution u_k of the problem (1.1), (1.3), exists for any $k > 0$ and $u_\infty := \lim_{k\to\infty}$ is a solution of (4.1) in $Q_\infty \setminus \{0\} \times \mathbb{R}^+$ with the following properties,

$$\lim_{t\to 0} u_\infty(x,t) = 0 \quad \forall x \neq 0 \quad \text{and} \quad \lim_{x\to 0} u_\infty u(x,t) = \infty \quad \forall t > 0. \tag{4.3}$$

Furthermore $t \mapsto u_\infty(x,t)$ is increasing and $\lim_{t\to\infty} u_\infty(x,t) = U(x)$ for every $x \neq 0$ where $U = \lim_{k\to\infty} U_k$ and U_k solves

$$-\Delta U_k + e^{-\ell(x)} U_k^p = k\delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{4.4}$$

Proof. By assumption (4.2), property (1.2) is fulfilled. Thus for $k > 0$ there exists $u := u_k$ solution of (4.1), (1.3). Moreover, for any $k > 0$ there exists a solution U_k of (4.4) (see [18]); the mapping $k \mapsto U_k$ is increasing and $U = \lim_{k\to\infty} U_k$ exists, because of Keller-Osserman estimate. U is the minimal solution of

$$-\Delta V + e^{-\ell(x)} V^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{4.5}$$

verifying

$$\int_{B_\epsilon} V(x)dx = \infty \quad \forall \epsilon > 0. \tag{4.6}$$

If we denote by $\bar{U}$ the maximal solution of (4.5), it is classical that $\bar{U} = \lim_{\epsilon\to 0} \bar{U}_\epsilon$ where

$$\left\{\begin{array}{l} -\Delta \bar{U}_\epsilon + e^{-\ell(x)} \bar{U}_\epsilon^p = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_\epsilon \\ \lim_{|x|\to\epsilon} \bar{U}_\epsilon(x) = \infty. \end{array}\right. \tag{4.7}$$

Since any u_k is bounded from above by $\bar{U}$, the local equicontinuity of the u_k in $\bar{Q}_T \setminus \{(0,0)\}$ implies that u_∞ satisfies $\lim_{t\to 0} u_\infty(x,t) = 0$ for all $x \neq 0$.

Step 1: Formation of the razor blade. <u>The Case 1: $1 < p < 1 + 2/N$</u>. For $\epsilon > 0$, $e^{-\ell(|x|)} \leq e^{-\ell(\epsilon)}$ for $|x| \leq \epsilon$. Therefore

$$\partial_t u - \Delta u + e^{-\ell(\epsilon)} |u|^{p-1} u \geq 0, \quad \text{in } B_\epsilon \times (0,\infty). \tag{4.8}$$

and $u \geq v_\epsilon$ in $B_\epsilon \times (0,T)$ where v_ϵ solves

$$\left\{\begin{array}{r} \partial_t v_\epsilon - \Delta v_\epsilon + e^{-\ell(\epsilon)} |v_\epsilon|^{p-1} v_\epsilon = 0 \quad \text{in } B_\epsilon \times (0,\infty) \\ v_\epsilon = 0 \quad \text{in } \partial B_\epsilon \times (0,\infty) \\ v_\epsilon(x,0) = \infty\delta_0 \quad \text{in } B_\epsilon, \end{array}\right. \tag{4.9}$$

where the initial condition is to be understood in the sense $\lim_{k\to\infty} k\delta_0$. We put

$$w_\epsilon(x,t) = \epsilon^{2/(p-1)} e^{-\ell(\epsilon)/(p-1)} v_\epsilon(\epsilon x, \epsilon^2 t).$$

Then $w_\epsilon = w$ is independent of ϵ and solves

$$\left\{\begin{array}{r} \partial_t w - \Delta w + |w|^{p-1} w = 0 \quad \text{in } B_1 \times (0,\infty) \\ w = 0 \quad \text{in } \partial B_1 \times (0,\infty) \\ w(x,0) = \infty\delta_0 \quad \text{in } B_1. \end{array}\right. \tag{4.10}$$

Therefore
$$u(0,1) \geq v_\epsilon(0,1) = \epsilon^{-2/(p-1)} e^{\ell(\epsilon)/(p-1)} w(0,\epsilon^{-2}). \tag{4.11}$$
The longtime behaviour is given in [7] where it is proved

$$\lim_{\tau\to\infty} e^{\lambda_1 \tau} w(0,\tau) = \kappa\phi_1(0).$$

In this formula ϕ_1 is the first eigenfunction of $-\Delta$ in $W_0^{1,2}(B_1)$, λ_1 the corresponding eigenvalue and $\kappa > 0$. Thus
$$u(0,1) \geq \delta\epsilon^{-2/(p-1)} e^{\ell(\epsilon)/(p-1)} e^{\lambda_1\epsilon^{-2}} \phi_1(0), \tag{4.12}$$
for some $\delta > 0$, if ϵ is small enough. If we assume

$$\lim_{\epsilon\to 0} \left(\frac{2}{p-1} \ln \epsilon^{-1} + \frac{\ell(\epsilon)}{p-1} - \lambda_1 \epsilon^{-2} \right) = \infty, \tag{4.13}$$

it implies
$$u(0,1) = \infty \Longrightarrow u(0,t) = \infty \quad \forall t > 0. \tag{4.14}$$
Moreover, the unit ball B_1 can be replaced by any ball B_R and λ_1 by $\lambda_R = R^{-2}\lambda_1$. Therefore the sufficient condition for a Razor blade is that it exists some $c > 0$ such that

$$\lim_{\epsilon\to 0} \left(\ell(\epsilon) - c\epsilon^{-2}\right) = \infty. \tag{4.15}$$

An equivalent condition is
$$\liminf_{\epsilon\to 0} \epsilon^2 \ell(\epsilon) > 0. \tag{4.16}$$
The general case. If $p > 1$ is arbitrary, we consider $\beta > 0$ such that $\beta > N(p-1) - 2$, and we write

$$e^{-\ell(x)} = |x|^\beta e^{-\ell(x) - \beta \ln|x|}.$$

For $R > 0$ small enough $x \mapsto \tilde{\ell}(x) := \ell(x) + \beta \ln|x|$ is positive, increasing and satisfies the same blow-up condition (4.2) as ℓ. Clearly u_k is bounded from below on $B_R \times (0,\infty)$ by the solution $\tilde{u} := \tilde{u}_k$ of
$$\left\{ \begin{array}{rl} \partial_t \tilde{u} - \Delta\tilde{u} + |x|^\beta e^{-\tilde{\ell}(x)} |\tilde{u}|^{p-1} \tilde{u} = 0 & \text{in } B_R \times (0,\infty) \\ \tilde{u} = 0 & \text{in } \partial B_R \times (0,\infty) \\ \tilde{u}(x,0) = k\delta_0 & \text{in } B_R. \end{array} \right. \tag{4.17}$$
Therefore, for $0 < \epsilon < R$, $\tilde{u}_\infty$ is bounded from below on $B_\epsilon \times (0,\infty)$ by the solution v_ϵ of

$$\left\{ \begin{array}{rl} \partial_t v_\epsilon - \Delta v_\epsilon + |x|^\beta e^{-\tilde{\ell}(\epsilon)} |v_\epsilon|^{p-1} v_\epsilon = 0 & \text{in } B_\epsilon \times (0,\infty) \\ v_\epsilon = 0 & \text{in } \partial B_\epsilon \times (0,\infty) \\ v_\epsilon(x,0) = \infty\delta_0 & \text{in } B_\epsilon. \end{array} \right. \tag{4.18}$$

If we set
$$w_\epsilon(x,t) = \epsilon^{(2+\beta)/(p-1)} e^{-\ell(\epsilon)/(p-1)} v_\epsilon(\epsilon x, \epsilon^2 t),$$
then $w_\epsilon = w$ is independent of ϵ and

$$\left\{ \begin{array}{rl} \partial_t w - \Delta w + |x|^\beta |w|^{p-1} w = 0 & \text{in } B_1 \times (0,\infty) \\ w = 0 & \text{in } \partial B_1 \times (0,\infty) \\ w(x,0) = \infty\delta_0 & \text{in } B_1. \end{array} \right. \tag{4.19}$$

By a straightforward adaptation of the result of [7], there still holds

$$\lim_{\tau\to\infty} e^{\lambda_1\tau} w(0,\tau) = \kappa\phi_1(0)$$

for some $\kappa > 0$. The remaining of the proof is the same as in case $1 < p < 1 + 2/N$.

Step 2: Asymptotic behaviour. A key observation is that, for any $\tau > 0$ and any $\epsilon_0 > 0$

$$\int_{\epsilon_0} u_\infty(x,\tau)dx = \infty. \tag{4.20}$$

We give the proof in the case $1 < p < 1 + 2/N$, the general case being similar. By step 1

$$\int_{B_\epsilon} u(x,\tau)dx \geq \int_{B_\epsilon} v_\epsilon(x,\tau)dx = \epsilon^{-2/(p-1)+N} e^{\ell(\epsilon)/(p-1)} \int_{B_1} w(y,\epsilon^{-2}\tau)dy. \tag{4.21}$$

If we fix τ and use [7], there exists ϵ_0 such that $w(y,\epsilon^{-2}\tau) \geq 2^{-1}\kappa e^{-\lambda_1\epsilon^{-2}\tau}\phi_1(y)$ for $\epsilon \leq \epsilon_0$ and $y_1 \in B_1$. Therefore

$$\int_{B_\epsilon} u(x,\tau)dx \geq c\epsilon^{-2/(p-1)+N} e^{\ell(\epsilon)/(p-1)-\lambda_1\epsilon^{-2}\tau}, \tag{4.22}$$

for some constant $c > 0$. If τ is small enough, the right-hand side of (4.22) tends to infinity as $\epsilon \to 0$, so does the left-hand side. This implies (4.20). For any $k > 0$ and any $\epsilon > 0$, there exists $m = m(\epsilon) > 0$ such that

$$\int_{B_\epsilon} \min\{u(x,\tau), m\}dx = k,$$

thus, if we set $\phi_m = \min\{u(x,\tau), m\}\chi_{B_\epsilon}$, then u is bounded from below on $\mathbb{R}^N \times (\tau,\infty)$ by the solution $v = v_{\epsilon,k}$ of

$$\left\{ \begin{array}{r} \partial_t v - \Delta v + e^{-\ell(x)}|v|^{p-1}v = 0 \quad \text{in } \mathbb{R}^N \times (\tau,\infty) \\ v(x,\tau) = \phi_m(x) \quad \text{in } \mathbb{R}^N. \end{array} \right. \tag{4.23}$$

When $\epsilon \to 0$, $\phi_m(.) \to k\delta_0$ weakly in $\mathfrak{M}(\mathbb{R}^N)$. By standard approximation property, $v(\epsilon,k) \to v_{0,k}$ which is a solution of

$$\left\{ \begin{array}{r} \partial_t v - \Delta v + e^{-\ell(x)}|v|^{p-1}v = 0 \quad \text{in } \mathbb{R}^N \times (\tau,\infty) \\ v(.,\tau) = k\delta_0 \quad \text{in } \mathbb{R}^N. \end{array} \right. \tag{4.24}$$

By uniqueness, $v_{0,k}(x,t) = u_k(x,t-\tau)$. Letting $k \to \infty$ yields to

$$u_\infty(x,t+\tau) \geq u_\infty(x,t) \quad \forall (x,t) \in Q_T. \tag{4.25}$$

This implies that $t \mapsto u_\infty(x,t)$ is increasing for every $x \in \mathbb{R}^N$. Because $u(x,t) \leq U(x)$, it is straightforward that $\lim_{x\to\infty} u(x,t) = \tilde{U}(x)$ exists in $\mathbb{R}^N \setminus \{0\}$.

Step 3: Identification of the limit. If $\zeta \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, there holds

$$\int_T^{T+1}\int_{\mathbb{R}^N} \left(-u(x,t)\Delta\zeta(x) + e^{-\ell(x)}u^p(x,t)\zeta(x)\right) dx\,dt = \int_{\mathbb{R}^N} \left(u(x,T) - u(x,T+1)\right)\zeta(x)dx.$$

By Lebesgue's theorem

$$\int_{\mathbb{R}^N} \left(-\tilde{U}(x)\Delta\zeta(x) + e^{-\ell(x)}\tilde{U}^p(x)\zeta(x)\right) dx = 0, \tag{4.26}$$

and, from (4.20),

$$\int_{\epsilon_0} \tilde{U}(x)dx = \infty, \tag{4.27}$$

for any $\epsilon_0 > 0$. Therefore $\tilde{U}$ is a solution of the stationary equation (4.4) in $\mathbb{R}^N \setminus \{0\}$ with a strong singularity at 0. For $k > 0$ and $\epsilon > 0$ there exists $k(\epsilon) > 0$ such that

$$\int_{B_\epsilon} U_{k(\epsilon)} dx = k.$$

Let $v := v_{k,\epsilon}$ be the solution of

$$\left\{ \begin{array}{r} \partial_t v - \Delta v + e^{-\ell(x)}|v|^{p-1}v = 0 \quad \text{in } Q_T, \\ v(.,0) = U_{k(\epsilon)}\chi_{B_\epsilon} \quad \text{in } \mathbb{R}^N. \end{array} \right. \tag{4.28}$$

Since $v_{k,\epsilon}(.,0) \leq U_{k(\epsilon)}(.)$, the maximum principle implies $v_{k,\epsilon} \leq U_{k(\epsilon)}$. If we let $\epsilon \to 0$, $v_{k,\epsilon}$ converges to the solution u_k with initial data $k\delta_0$. Furthermore $k(\epsilon) \to \infty$ as $\epsilon \to 0$. Therefore

$$u_k(x,t) \leq U(x) \quad \forall (x,t) \in Q_T. \tag{4.29}$$

Letting successively $k \to \infty$ and $t \to \infty$ implies

$$\tilde{U}(x) \leq U(x) \quad \forall x \in \mathbb{R}^N. \tag{4.30}$$

Since U is the minimal solution of (4.5) verifying (4.6), it follows that $U = \tilde{U}$. $\square$

References

[1] Belaud Y., Shishkov A. : Long-time extinction of solutions of some semilinear parabolic equations, **Journal Diff. Equ. 238**, 64-86 (2007).

[2] Bellman R. : **Stability Theory of Differential Equations**McGraw-Hill Book Company, Inc. 1953.

[3] Brezis H, Friedman A. : Nonlinear parabolic problems involving measures as initial conditions, **J. Math. Pures Appl. 62**, 73-97 (1983).

[4] Brezis H, Peletier L. A., Terman D. : A very singular solution of the heat equation with absorption, **Arch. Rat. Mech. Anal. 95**, 185-206 (1986).

[5] Brezis H., Oswald L. : Singular solutions for some semilinear elliptic equations, **Arch. Rat. Mech. Anal. 99**, 249-259 (1987)

[6] Galaktionov V., Kurdyumov S., Samarskii A. : On asymptotic "eigenfunctions" of the Cauchy problem for a nonlinear parabolic equation, **Math. USSR Sbornik 54**, 421-455 (1986).

[7] Gmira A., Véron L. : Asymptotic behaviour of the solution of a semilinear parabolic equation, **Mh. Math. 94**, 299-311 (1982).

[8] Escobedo M., Kavian O. : Variational problems related to self-similar solutions of semilinear heat equations, **Nonlinear Anal. 10**, 1103-1133 (1987).

[9] Galaktionov V., Shishkov A. : Higher order quasilinear parabolic equations with singular initial data, **Commun. in Contemp. Math. 8** (3), 331-354 (2006).

[10] Marcus M., Véron L. : Semilinear parabolic equations with measure boundary data and isolated singularities, **J. Anal. Math. 85** , 245-290 (2001).

[11] Marcus M., Véron L. : The initial trace of positive solutions of semilinear parabolic equations, **Comm. in P.D.E. 24**, 1445-1499 (1999).

[12] Marcus M., Véron L. : Initial trace of positive solutions to semilinear parabolic inequalities, **Advanced Nonlinear Studies 2**, 395-436 (2002).

[13] Ratto A., Rigoli M., Véron L. : Conformal deformation of hyperbolic space, **J. Funct. Anal. 121**, 15-77 (1994).

[14] Shishkov A., Véron L. : The balance between diffusion and absorption in semilinear parabolic equations, **Rend. Lincei Mat. Appl. 18**, 59-96 (2007).

[15] Shishkov A. : Propagation of perturbation on a singular Cauchy problem for degenerate quasilinear parabolic equations, **Sbornik: Mathematiks 187**(9), 1391-1410 (1996).

[16] Vazquez J. L., Véron L. : Different kinds of singular solutions of nonlinear parabolic equations, **Nonlinear Problems in Applied Mathematics**, Vol. in Honor of I. Stakgold, SIAM ed. 240-249 (1996).

[17] Véron L. : Semilinear elliptic equations with uniform blow-up on the boundary, **J. d'Analyse Math. 59**, 231-250 (1992).

[18] Véron L. : **Singularities of Second Order Quasilinear Equations**, Pitman Research Notes in Math. **353**, Addison Wesley Longman (1996).