SEMICLASSICAL SCATTERING AMPLITUDE AT THE MAXIMUM OF THE POTENTIAL

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Abstract. We study the scattering amplitude for Schrödinger operators at a critical energy level, which is a unique non-degenerate maximum of the potential. We do not assume that the maximum point is non-resonant and use results of [5] to analyze the contributions of the trapped trajectories. We prove a semiclassical expansion of the scattering amplitude and compute its leading term. We show that it has different orders of magnitude in specific regions of phase space. We also prove upper and lower bounds for the resolvent in this setting.

1. Introduction

We consider the semiclassical behavior of the scattering amplitude at energy $E > 0$ for Schrödinger operators

$$P(x, hD) = -\frac{h^2}{2}\Delta + V(x) \tag{1.1}$$

where V is a real valued C^∞ function on $\mathbb{R}^n$, which vanishes at infinity. We suppose that E is close to a critical energy level E_0 for P, which corresponds to a non-degenerate global maximum of the potential. Here, we address the case where this maximum is unique.

Let us recall that, if $V(x) = \mathcal{O}(\langle x\rangle^{-\rho})$ for some $\rho > (n+1)/2$, then for any $\omega \neq \theta \in \mathbb{S}^{n-1}$ and $E > 0$, the problem

$$\begin{cases} P(x, hD)u = Eu, \\ u(x, h) = e^{i\sqrt{2E}x\cdot\omega/h} + \mathcal{A}(\omega, \theta, E, h)\dfrac{e^{i\sqrt{2E}x\cdot\theta/h}}{|x|^{(n-1)/2}} + o(|x|^{(1-n)/2}) \text{ as } x \to +\infty,\ \dfrac{x}{|x|} = \theta, \end{cases}$$

has a unique solution in $L^2_{\text{loc}}(\mathbb{R}^n)$. The scattering amplitude at energy E for the incoming direction ω and the outgoing direction θ is the real number $\mathcal{A}(\omega, \theta, E, h)$.

For potentials that are not decaying that fast at infinity, the scattering amplitude cannot be so easily defined through a stationary approach: If $V(x) = \mathcal{O}(\langle x\rangle^{-\rho})$ for some $\rho > 1$,

Date: April 2, 2008.

2000 *Mathematics Subject Classification.* 81U20,35P25,35B38,35C20.

Key words and phrases. Scattering amplitude, critical energy, Schrödinger equation.

Acknowledgments: We would like to thank Johannes Sjöstrand for helpful discussions during the preparation of this paper. The first author also thanks Victor Ivrii for supporting visits to Université Paris Sud, Orsay, and the Department of Mathematics at Orsay for the extended hospitality.

the scattering matrix $\mathcal{S}(E,h)$ at energy E can be given in terms of the wave operators (see Section 4 below). Then, writing

$$\mathcal{S}(E,h) = Id - 2i\pi\mathcal{T}(E,h), \tag{1.2}$$

one can see that $\mathcal{T}(E,h)$ is a compact operator on $L^2(\mathbb{S}^{n-1})$, whose kernel $\mathcal{T}(\omega,\theta,E,h)$ is smooth away from the diagonal in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$. Then, the scattering amplitude is given for $\theta\neq\omega$, by

$$\mathcal{A}(\omega,\theta,E,h) = c(E)h^{(n-1)/2}\mathcal{T}(\omega,\theta,E,h), \tag{1.3}$$

where

$$c(E) = -2\pi(2E)^{-\frac{n-1}{4}}(2\pi)^{\frac{n-1}{2}}e^{-i\frac{(n-3)\pi}{4}}. \tag{1.4}$$

We proceed here as in [32], where D. Robert and H. Tamura have studied the semiclassical behavior of the scattering amplitude for short range potentials at a non-trapping energy E . An energy E is said to be non-trapping when $K(E)$, the trapped set at energy E, is empty. This trapped set is defined as

$$K(E) = \big\{(x,\xi)\in p^{-1}(E);\ \exp(tH_p)(x,\xi)\not\to\infty \text{ as } t\to\pm\infty\big\}, \tag{1.5}$$

where H_p is the Hamiltonian vector field associated to the principal symbol $p(x,\xi)=\frac{1}{2}\xi^2+V(x)$ of the operator P. Notice that the scattering amplitude has been first studied, in the semiclassical regime, by B. Vainberg [34] and Y. Protas [29] in the case of compactly supported potential and for non-trapping energies, where they obtained the same type of result.

Under the non-trapping assumption, and some other non-degeneracy condition (in fact our assumption (A4) below), D. Robert and H. Tamura have shown that the scattering amplitude has an asymptotic expansion with respect to h. This non-degeneracy assumption implies in particular that there is a finite number N_∞ of classical trajectories for the Hamiltonian p, with asymptotic direction ω for $t\to-\infty$ and asymptotic direction θ as $t\to+\infty$. Robert and Tamura's result is the following asymptotic expansion for the scattering amplitude:

$$\mathcal{A}(\omega,\theta,E,h) = \sum_{j=1}^{N_\infty} e^{iS_j^\infty/h}\sum_{m\geq 0} a_{j,m}(\omega,\theta,E)h^m + \mathcal{O}(h^\infty),\ h\to 0, \tag{1.6}$$

where S_j^∞ is the classical action along the corresponding trajectory. Also, they have computed the first term in this expansion, showing that it can be given in terms of quantities attached to the corresponding classical trajectory only.

V. Guillemin [18] has established a similar asymptotic expansion in the setting of smooth compactly-supported metric perturbations of the Laplacian. For short-range potentials, K. Yajima has proved in [35] an asymptotic expansion of the form (1.6) of the scattering amplitude in the L^2 sense. Most recently, A. Hassell and J. Wunsch [19] have shown that the scattering matrix at non-trapping energies on a compact manifold with boundary with a scattering matric is a Legendrian-Lagrangian distribution associated to the total sojourn relation.

There is also a small number of results concerning the scattering amplitude when the non-trapping assumption is not fulfilled. In [26] L. Michel has shown that, if there is no trapped trajectory with incoming direction ω, and θ is ω-regular (see the discussion after (2.7) below), and if there is a resonance free complex neighborhood of E of size $\sim h^N$ for some $N\in\mathbb{N}$, then

$\mathcal{A}(\omega, \theta, E, h)$ is still given by (1.6). The potential is also supposed to be analytic in a sector out of a compact set, and the assumption on the resonance free domain near E amounts to an estimate on the boundary values of the meromorphic extension of the truncated resolvent of the form

$$\|\chi(P-(E\pm i0))^{-1}\chi\| = \mathcal{O}(h^{-N}),\ \chi \in C_0^\infty(\mathbb{R}^n). \tag{1.7}$$

Note that, these assumptions allow the existence of a non-empty trapped set.

In [2] and [3], the first author has shown that at non-trapping energies or in Michel's setting, the scattering amplitude is an h-Fourier integral operator associated to a natural scattering relation. These results imply that the scattering amplitude admits an asymptotic expansion, in the sense of oscillatory integrals, even without the non-degeneracy assumption. In particular, the expansion (1.6) is recovered under the non-degeneracy assumption.

In [23], A. Lahmar-Benbernou and A. Martinez have computed the scattering amplitude at energy $E \sim E_0$, in the case where the trapped set $K(E_0)$ consists in one single point corresponding to a local minimum of the potential (a well in the island situation). In that case, the estimate (1.7) is not true, and their result is obtained through a construction of the resonant states.

In the present work, we compute the scattering amplitude at energy $E \sim E_0$ in the case where the trapped set $K(E_0)$ corresponds to the unique global maximum of the potential. The one-dimensional case has been studied in [30, 14, 15], with specific techniques, and we consider here the general $n > 1$ dimensional case.

Notice that J. Sjöstrand in [33], and P. Briet, J.-M. Combes and P. Duclos in [7, 8] have described the resonances close to E_0 in the case where V is analytic in a sector around $\mathbb{R}^n$. From their result, it follows that Michel's assumption on the existence of a not too small resonance-free neighborhood of E_0 is satisfied. However, we show below (see Proposition 2.5) that for any $\omega \in \mathbb{S}^{n-1}$, there is at least one half-trapped trajectory with incoming direction ω, so that Michel's result never applies here.

Here, we do not assume analyticity for V. We compute the contributions to the scattering amplitude arising from the classical trajectories reaching the unstable equilibrium point, which corresponds to the top of the potential barrier. At the quantum level, tunnel effect occurs, which permits the particle to pass through this point. Our computation here relies heavily on [5], where a precise description of this phenomena has been obtained. In a forthcoming paper, we shall show that in this case also, the scattering amplitude is an h-Fourier integral operator.

This paper is organized in the following way. In Section 2, we describe our assumptions, and state our main results: a resolvent estimate, and the asymptotic expansion of the scattering amplitude in the semiclassical regime. Section 3 is devoted to the proof of the resolvent estimate, from which we deduce in Section 4 estimates similar to those in [32]. In that section, we also recall briefly the representation formula for the scattering amplitude proved by H. Isozaki and H. Kitada, and introduce notations from [32]. The computation of the asymptotic expansion of the scattering amplitude is conducted in sections 5, 6 and 7, following the classical trajectories. Eventually, we have put in four appendices the proofs of some side results or technicalities.

2. Assumptions and main results

We suppose that the potential V satisfies the following assumptions

(A1) V is a C^∞ function on $\mathbb{R}^n$, and, for some $\rho > 1$,

$$\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{-\rho-|\alpha|}).$$

(A2) V has a non-degenerate maximum point at $x = 0$, with $E_0 = V(0) > 0$ and

$$\nabla^2 V(0) = \begin{pmatrix} -\lambda_1^2 & & \\ & \ddots & \\ & & -\lambda_n^2 \end{pmatrix}, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

(A3) The trapped set at energy E_0 is $K(E_0) = \{(0,0)\}$.

Notice that the assumptions **(A1)**–**(A3)** imply that V has an absolute global maximum at $x = 0$. Indeed, if $\mathcal{L} = \{x \neq 0;\ V(x) \geq E_0\}$ was non empty, the geodesic, for the Agmon distance $(E_0 - V(x))_+^{1/2} dx$, between 0 and $\mathcal{L}$ would be the projection of a trapped bicharacteristic (see [1, Theorem 3.7.7]).

As in D. Robert and H. Tamura in [32], one of the key ingredient for the study of the scattering amplitude is a suitable estimate for the resolvent. Using the ideas in [5, Section 4], we have obtained the following result, that we think to be of independent interest.

Theorem 2.1. *Suppose assumptions* **(A1)**, **(A2)** *and* **(A3)** *hold, and let* $\alpha > \frac{1}{2}$ *be a fixed real number. We have*

$$\|(P - (E \pm i0))^{-1}\|_{\alpha,-\alpha} \lesssim h^{-1} |\ln h|, \tag{2.1}$$

uniformly for $|E - E_0| \leq \delta$, *with* $\delta > 0$ *small enough. Here* $\|Q\|_{\alpha,\beta}$ *denotes the norm of the bounded operator* Q *from* $L^2(\langle x \rangle^\alpha\, dx)$ *to* $L^2(\langle x \rangle^\beta\, dx)$.

Moreover, we prove in the Appendix B that our estimate is not far from optimal. Indeed, we have the

Proposition 2.2. *Let* $\psi \in C_0^\infty(\mathbb{R}^n)$ *with* $\psi(0) \neq 0$. *Under the assumptions* **(A1)** *and* **(A2)**, *we have*

$$\|\psi(P - (E_0 \pm i0))^{-1}\psi\| \gtrsim h^{-1}\sqrt{|\ln h|}. \tag{2.2}$$

In particular,

$$\|(P - (E \pm i0))^{-1}\|_{\alpha,-\alpha} \gtrsim h^{-1}\sqrt{|\ln h|}, \tag{2.3}$$

for all $\alpha > \frac{1}{2}$.

We would like to mention that in the case of a closed hyperbolic orbit, the same upper bound has been obtained by N. Burq [9] in the analytic category, and in a recent paper [11] by H. Christianson in the C^∞ setting.

As a matter of fact, in the present setting, S. Nakamura has proved in [28] an $\mathcal{O}(h^{-2})$ bound for the resolvent. Nakamura's estimate would be sufficient for our proof of Theorem

2.6, but it is not sharp enough for the computation of the total scattering cross section along the lines of D. Robert and H. Tamura in [31]. In that paper, the proof relies on a bound $\mathcal{O}(h^{-1})$ for the resolvent, but it is easy to see that an estimate like $\mathcal{O}(h^{-1-\varepsilon})$ for any small enough $\varepsilon > 0$ is sufficient. If we denote

$$\sigma(\omega, E, h) = \int_{\mathbb{S}^{n-1}} |\mathcal{A}(\omega, \theta, E, h)|^2 d\theta, \tag{2.4}$$

the total scattering cross-section, and following D. Robert and H. Tamura's work, our resolvent estimate gives the

Theorem 2.3. *Suppose assumptions* **(A1)**, **(A2)** *and* **(A3)** *hold, and that* $\rho > \frac{n+1}{2}$, $n \geq 2$. *If* $|E - E_0| < \delta$ *for some* $\delta > 0$ *small enough, then*

$$\sigma(\omega, E, h) = 4 \int_{\omega^\perp} \sin^2 \Big\{ 2^{-1} (2E)^{-1/2} h^{-1} \int_{\mathbb{R}} V(y + s\omega) ds \Big\} dy + \mathcal{O}\big(h^{-(n-1)/(\rho-1)}\big). \tag{2.5}$$

Now we state our assumptions concerning the classical trajectories associated with the Hamiltonian p, that is curves $t \mapsto \gamma(t, x, \xi) = \exp(tH_p)(x, \xi)$ for some initial data $(x, \xi) \in T^*\mathbb{R}^n$. Let us recall that, thanks to the decay of V at infinity, for given $\alpha \in \mathbb{S}^{n-1}$ and $z \in \alpha^\perp \sim \mathbb{R}^{n-1}$ (the impact plane), there is a unique bicharacteristic curve

$$\gamma_\pm(t, z, \alpha, E) = (x_\pm(t, z, \alpha, E), \xi_\pm(t, z, \alpha, E)) \tag{2.6}$$

such that

$$\begin{aligned} &\lim_{t \to \pm\infty} |x_\pm(t, z, \alpha, E) - \sqrt{2E}\alpha t - z| = 0, \\ &\lim_{t \to \pm\infty} |\xi_\pm(t, z, \alpha, E) - \sqrt{2E}\alpha| = 0. \end{aligned} \tag{2.7}$$

We shall denote by Λ_ω^- the set of points in $T^*\mathbb{R}^n$ lying on trajectories going to infinity with direction ω as $t \to -\infty$, and Λ_θ^+ the set of those which lie on trajectories going to infinity with direction θ as $t \to +\infty$:

$$\begin{aligned} \Lambda_\omega^- &= \big\{ \gamma_-(t, z, \omega, E_0) \in T^*\mathbb{R}^n;\ z \in \omega^\perp,\ t \in \mathbb{R} \big\}, \\ \Lambda_\theta^+ &= \big\{ \gamma_+(t, z, \theta, E_0) \in T^*\mathbb{R}^n;\ z \in \theta^\perp,\ t \in \mathbb{R} \big\}. \end{aligned} \tag{2.8}$$

From the discussion of Section 4 one can see that Λ_ω^- and Λ_θ^+ are Lagrangian submanifolds of $T^*\mathbb{R}^n$.

Under the assumptions **(A1)**, **(A2)** and **(A3)** there are only two possible behaviors for $x_\pm(t, z, \alpha, E_0)$ as $t \to \mp\infty$: either it escapes to ∞, or it goes to 0.

First we state our assumptions for the first kind of trajectories. For these, we also have, for some $(r_\infty(z, \omega, E_0), \xi_\infty(z, \omega, E_0))$,

$$\begin{aligned} &\lim_{t \to +\infty} \xi_-(t, z, \omega, E_0) = \xi_\infty(z, \omega, E_0), \\ &\lim_{t \to +\infty} x_-(t, z, \omega, E_0) - \xi_\infty(z, \omega, E_0) t = r_\infty(z, \omega, E_0), \end{aligned} \tag{2.9}$$

and we shall say that the trajectory $\gamma_-(t, z, \omega, E_0)$ has initial direction ω and final direction $\theta = \xi_\infty(z, \omega, E_0)/\sqrt{2E_0}$. As in [32] we shall make some non-degeneracy assumption on the trajectories with initial direction ω. This assumption can be given in terms of the angular density

$$\widehat{\sigma}(z) = |\det(\xi_\infty(z,\omega,E_0), \partial_{z_1}\xi_\infty(z,\omega,E_0), \dots, \partial_{z_{n-1}}\xi_\infty(z,\omega,E_0))|. \tag{2.10}$$

Definition 2.4. *The outgoing direction $\theta \in \mathbb{S}^{n-1}$ is called regular for the incoming direction $\omega \in \mathbb{S}^{n-1}$, or ω-regular, if $\theta \neq \omega$ and, for all $z' \in \omega^\perp$ with $\xi_\infty(z',\omega,E_0) = \sqrt{2E_0}\theta$, the map $\omega^\perp \ni z \mapsto \xi_\infty(z,\omega,E_0) \in \mathbb{S}^{n-1}$ is non-degenerate at z', i.e. $\widehat{\sigma}(z') \neq 0$.*

We fix the incoming direction $\omega \in \mathbb{S}^{n-1}$, and we assume that

(A4) $\Lambda_\omega^- \cap \Lambda_\theta^+$ is a finite set of Hamiltonian trajectories $(\gamma_j^\infty)_{1\le j\le N_\infty}$, and the direction $\theta \in \mathbb{S}^{n-1}$ is ω-regular.

We denote $\gamma_j^\infty(t) = \gamma^\infty(t, z_j^\infty) = (x_j^\infty(t), \xi_j^\infty(t))$. Then one can show that Λ_ω^- and Λ_θ^+ intersect transversely along each of these trajectories.

We now turn to trapped bicharacteristics. Let us notice that the linearization F_p at $(0,0)$ of the Hamilton vector field H_p has eigenvalues $-\lambda_n, \dots, -\lambda_1, \lambda_1, \dots, \lambda_n$. Thus $(0,0)$ is a hyperbolic fixed point for H_p, and the Stable Manifold Theorem gives the existence of a stable incoming Lagrangian manifold Λ_- and a stable outgoing Lagrangian manifold Λ_+ characterized by

$$\Lambda_\pm = \{(x,\xi) \in T^*\mathbb{R}^n;\ \exp(tH_p)(x,\xi) \to 0 \text{ as } t \to \mp\infty\}. \tag{2.11}$$

In this paper, we shall describe the contribution to the scattering amplitude of the trapped trajectories, that is those going from infinity to the fixed point $(0,0)$. We have proved in Appendix A the following result, which shows that there are always such trajectories.

Proposition 2.5. *For every $\omega, \theta \in \mathbb{S}^{n-1}$, we have*

$$\Lambda_\omega^- \cap \Lambda_- \neq \emptyset \quad \textit{and} \quad \Lambda_\theta^+ \cap \Lambda_+ \neq \emptyset. \tag{2.12}$$

We suppose that

(A5) Λ_ω^- and Λ_- (resp. Λ_θ^+ and Λ_+) intersect in a finite number N_- (resp N_+) of bicharacteristic curves, with each intersection transverse.

We denote these curves, respectively,

$$\gamma_k^- : t \mapsto \gamma^-(t, z_k^-) = (x_k^-(t), \xi_k^-(t)), \quad 1 \le k \le N_-, \tag{2.13}$$

and

$$\gamma_\ell^+ : t \mapsto \gamma^+(t, z_\ell^+) = (x_\ell^+(t), \xi_\ell^+(t)), \quad 1 \le \ell \le N_+. \tag{2.14}$$

Here, the z_k^- (resp. the z_ℓ^+) belong to $\omega^\perp$ (resp. $\theta^\perp$) and determine the corresponding curve by (2.7).

We recall from [20, Section 3] (see also [5, Section 5]), that each integral curve $\gamma^\pm(t) = (x^\pm(t), \xi^\pm(t)) \in \Lambda_\pm$ satisfies, in the sense of expandible functions (see Definition 6.1 below),

$$\gamma^\pm(t) \sim \sum_{j\ge 1} \gamma_j^\pm(t) e^{\pm\mu_j t}, \text{ as } t \to \mp\infty, \tag{2.15}$$

where $\mu_1 = \lambda_1 < \mu_2 < \dots$ is the strictly increasing sequence of linear combinations over $\mathbb{N}$ of the λ_j's. Here, the functions $\gamma_j^\pm : \mathbb{R} \to \mathbb{R}^{2n}$ are polynomials, that we write

$$\gamma_j^\pm(t) = \sum_{m=0}^{M'_j} \gamma_{j,m}^\pm t^m. \tag{2.16}$$

Considering the base space projection of these trajectories, we denote

$$x^\pm(t) \sim \sum_{j=1}^{+\infty} g_j^\pm(t) e^{\pm\mu_j t}, \text{ as } t \to \mp\infty, \qquad g_j^\pm(t) = \sum_{m=0}^{M'_j} g_{j,m}^\pm t^m. \tag{2.17}$$

Let us denote by $\widehat{\jmath}$ the (only) integer such that $\mu_{\widehat{\jmath}} = 2\lambda_1$. We prove in Proposition 6.11 below that if $j < \widehat{\jmath}$, then $M'_j = 0$, or more precisely, that $\gamma_j^\pm(t) = \gamma_j^\pm$ is a constant vector in $\mathrm{Ker}(F_p \mp \lambda_j)$. We also have $M'_{\widehat{\jmath}} \leq 1$, and $g_{\widehat{\jmath},1}^-$ can be computed in terms of g_1^-.

In this paper, we will denote the objects associated to the k-th incoming or ℓ-th outgoing trajectory by attaching z_k^- or z_ℓ^+ to the notation. Concerning the incoming trajectories, we shall assume that

(A6) For each $k \in \{1, \dots, N_-\}$, $g_1^-(z_k^-) \neq 0$.

Finally, we state our assumptions for the outgoing trajectories $\gamma_\ell^+ \subset \Lambda_+ \cap \Lambda_\theta^+$. First of all, it is easy to see, using Hartman's linearization theorem, that, for all ℓ, there always exists $m \in \mathbb{N}$ such that $g_m^+(z_\ell^+) \neq 0$. We let

$$\boldsymbol{\ell} = \boldsymbol{\ell}(\ell) = \min\{m;\ g_m^+(z_\ell^+) \neq 0\} \tag{2.18}$$

be the smallest of these m's. We know that $\mu_{\boldsymbol{\ell}}$ is one of the λ_j's, and that $M'_{\boldsymbol{\ell}} = 0$.

In [5], we have been able to describe the branching process between an incoming curve $\gamma^- \subset \Lambda_-$ and an outgoing curve $\gamma^+ \subset \Lambda_+$ provided $\langle g_1^- | g_1^+ \rangle \neq 0$ (see the definition for $\widetilde{\Lambda}_+(\rho_-)$ before [5, Theorem 2.6]). Here, for the computation of the scattering amplitude, we can relax this assumption a lot, and analyze the branching in other cases which we now describe. Let us denote, for a given pair of paths $(\gamma^-(z_k^-), \gamma^+(z_\ell^+))$ in $(\Lambda_\omega^- \cap \Lambda_-) \times (\Lambda_\theta^+ \cap \Lambda_+)$,

$$\mathcal{M}_2(k,\ell) = -\frac{1}{8\lambda_1} \sum_{\substack{j \in \mathcal{I}_1(2\lambda_1) \\ \alpha,\beta \in \mathcal{I}_2(\lambda_1)}} \partial_j \partial^\beta V(0) \frac{(g_1^-(z_k^-))^\beta}{\beta!}\, \partial_j \partial^\alpha V(0) \frac{(g_1^+(z_\ell^+))^\alpha}{\alpha!}, \tag{2.19}$$

and

$$\begin{aligned} \mathcal{M}_1(k,\ell) = &- \sum_{\substack{j \in \mathcal{I}_1 \\ \alpha \in \mathcal{I}_2(\lambda_1)}} \frac{\partial_j \partial^\alpha V(0)}{\alpha!} \big((g_1^-(z_k^-))^\alpha (g_{\widehat{\jmath},0}^+(z_\ell^+))_j + (g_{\widehat{\jmath},0}^-(z_k^-))_j (g_1^+(z_\ell^+))^\alpha \big) \\ &+ \sum_{\alpha,\beta \in \mathcal{I}_2(\lambda_1)} \frac{(g_1^-(z_k^-))^\alpha}{\alpha!} \frac{(g_1^+(z_\ell^+))^\beta}{\beta!} C_{\alpha,\beta}, \end{aligned} \tag{2.20}$$

where

$$
\begin{aligned}
C_{\alpha,\beta} = &- \partial^{\alpha+\beta} V(0) + \sum_{j\in\mathcal{I}_1\setminus\mathcal{I}_1(2\lambda_1)} \frac{4\lambda_1^2}{\lambda_j^2(4\lambda_1^2-\lambda_j^2)} \partial_j\partial^\alpha V(0)\partial_j\partial^\beta V(0) \\
&- \sum_{\substack{j\in\mathcal{I}_1 \\ \gamma,\delta\in\mathcal{I}_2(\lambda_1) \\ \gamma+\delta=\alpha+\beta}} \frac{(\gamma+\delta)!}{\gamma!\,\delta!}\frac{1}{2\lambda_j^2}\partial_j\partial^\gamma V(0)\partial_j\partial^\delta V(0). \qquad (2.21)
\end{aligned}
$$

Here, we have set $\mathcal{I}_1 = \{1,\dots,n\}$, that we sometimes identify with $\{1_j, j = 1\dots n\}$, $1_j = (\delta_{ij})_{i=1,\dots,n} \in \mathbb{N}^n$ and

$$
\mathcal{I}_m(\mu) = \{\beta\in\mathbb{N}^n;\ \beta = 1_{k_1}+\dots+1_{k_m} \text{ with } \lambda_{k_1}=\dots=\lambda_{k_m}=\mu\}, \qquad (2.22)
$$

the set of multi-indices β of length $|\beta| = m$ with each index of its non-vanishing components in the set $\{j\in\mathbb{N};\ \lambda_j=\mu\}$. We also denote $\mathcal{I}_m\subset\mathbb{N}^n$ the set of all multi-indices of length m.

We will suppose that

(A7) For each pair of paths $(\gamma^-(z_k^-),\gamma^+(z_\ell^+))$, $k\in\{1,\dots,N_-\}$, $\ell\in\{1,\dots,N_+\}$, one of the three following cases occurs:

- **(a)** The set $\{m<\widehat{\jmath};\ \langle g_m^-(z_k^-)|g_m^+(z_\ell^+)\rangle\neq 0\}$ is not empty. Then we denote
$$\mathbf{k} = \min\{m<\widehat{\jmath};\ \langle g_m^-(z_k^-)|g_m^+(z_\ell^+)\rangle\neq 0\}.$$
- **(b)** For all $m<\widehat{\jmath}$, we have $\langle g_m^-(z_k^-)|g_m^+(z_\ell^+)\rangle = 0$, and $\mathcal{M}_2(k,\ell)\neq 0$.
- **(c)** For all $m<\widehat{\jmath}$, we have $\langle g_m^-(z_k^-)|g_m^+(z_\ell^+)\rangle = 0$, $\mathcal{M}_2(k,\ell)=0$ and $\mathcal{M}_1(k,\ell)\neq 0$.

As one could expect (see [32], [30] or [15]), action integrals appear in our formula for the scattering amplitude. We shall denote

$$
S_j^\infty = \int_{-\infty}^{+\infty} (|\xi_j^\infty(t)|^2 - 2E_0)dt - \langle r_\infty(z_j^\infty,\omega,E_0)|\sqrt{2E_0}\theta\rangle,\ j\in\{1,\dots,N_\infty\}, \qquad (2.23)
$$

$$
S_k^- = \int_{-\infty}^{+\infty} \big(|\xi_k^-(t)|^2 - 2E_0 1_{t<0}\big)dt, \quad k\in\{1,\dots,N_-\}, \qquad (2.24)
$$

$$
S_\ell^+ = \int_{-\infty}^{+\infty} \big(|\xi_\ell^+(t)|^2 - 2E_0 1_{t>0}\big)dt, \quad \ell\in\{1,\dots,N_+\}, \qquad (2.25)
$$

and ν_j^∞, ν_ℓ^+, ν_k^- the Maslov indexes of the curves γ_j^∞, γ_ℓ^+, γ_k^- respectively. Let also

$$
D_k^- = \lim_{t\to+\infty}\Big|\det\frac{\partial x_-(t,z,\omega,E_0)}{\partial(t,z)}|_{z=z_k^-}\Big|\, e^{-(\Sigma_j\lambda_j-2\lambda_1)t}, \qquad (2.26)
$$

$$
D_\ell^+ = \lim_{t\to-\infty}\Big|\det\frac{\partial x_+(t,z,\omega,E_0)}{\partial(t,z)}|_{z=z_\ell^+}\Big|\, e^{(\Sigma_j\lambda_j-2\lambda_{\boldsymbol{\ell}})t}, \qquad (2.27)
$$

be the Maslov determinants for γ_k^-, and γ_ℓ^+respectively. We show below that $0<D_k^-,D_\ell^+<+\infty$. Eventually we set

$$
\Sigma(E,h) = \sum_{j=1}^n\frac{\lambda_j}{2} - i\frac{E-E_0}{h}\cdot \qquad (2.28)
$$

Then, the main result of this paper is the

Theorem 2.6. *Suppose assumptions* **(A1)** *to* **(A7)** *hold, and that $E \in \mathbb{R}$ is such that $E - E_0 = \mathcal{O}(h)$. Then*

$$\mathcal{A}(\omega,\theta,E,h) = \sum_{j=1}^{N_\infty} \mathcal{A}_j^{\rm reg}(\omega,\theta,E,h) + \sum_{k=1}^{N_-}\sum_{\ell=1}^{N_+} \mathcal{A}_{k,\ell}^{\rm sing}(\omega,\theta,E,h) + \mathcal{O}(h^\infty), \tag{2.29}$$

where

$$\mathcal{A}_j^{\rm reg}(\omega,\theta,E,h) \sim e^{iS_j^\infty/h} \sum_{m\geq 0} a_{j,m}^{\rm reg}(\omega,\theta,E)h^m, \tag{2.30}$$

with

$$a_{j,0}^{\rm reg}(\omega,\theta,E) = \frac{e^{-i\nu_j^\infty \pi/2}}{\widehat{\sigma}(z_j^\infty)^{1/2}} e^{-\langle r_\infty(z_j^\infty,\omega,E_0)|\sqrt{2E_0}^{-1}\theta\rangle \frac{E-E_0}{h}}. \tag{2.31}$$

Moreover we have

- *In case **(a)***

$$\mathcal{A}_{k,\ell}^{\rm sing}(\omega,\theta,E,h) \sim e^{i(S_k^-+S_\ell^+)/h} \sum_{m\geq 0} a_{k,\ell,m}^{\rm sing}(\omega,\theta,E,\ln h) h^{(\Sigma(E)+\widehat{\mu}_m)/\mu_{\mathbf{k}}-1/2}, \tag{2.32}$$

where the $a_{k,\ell,m}^{\rm sing}(\omega,\theta,E,\ln h)$ are polynomials with respect to $\ln h$, and

$$\begin{aligned} a_{k,\ell,0}^{\rm sing}(\omega,\theta,E,\ln h) =& \frac{e^{i\pi/4}E^{(n-1)/4}}{2^{(n+1)/4}\sqrt{\pi}} \Big(\prod_{j=1}^n \lambda_j\Big)^{-1/2} \Gamma\Big(\frac{\Sigma(E)}{\mu_{\mathbf{k}}}\Big) \frac{(2\lambda_1\lambda_{\boldsymbol{\ell}})^{3/2}}{\mu_{\mathbf{k}}} \\ &\times e^{-i\nu_\ell^+\pi/2} e^{-i\nu_k^-\pi/2} (D_k^- D_\ell^+)^{-1/2} \\ &\times |g_1^-(z_k^-)|\, |g_{\boldsymbol{\ell}}^+(z_\ell^+)|\, \big(2i\mu_{\mathbf{k}} \big\langle g_{\mathbf{k}}^-(z_k^-)\big| g_{\mathbf{k}}^+(z_\ell^+)\big\rangle\big)^{-\Sigma(E)/\mu_{\mathbf{k}}}. \end{aligned} \tag{2.33}$$

- *In case **(b)***

$$\mathcal{A}_{k,\ell}^{\rm sing}(\omega,\theta,E,h) = e^{i(S_\ell^++S_k^-)/h} a_{k,\ell}^{\rm sing}(\omega,\theta,E) \frac{h^{\Sigma(E)/2\lambda_1-1/2}}{|\ln h|^{\Sigma(E)/\lambda_1}}(1+o(1)), \tag{2.34}$$

where

$$\begin{aligned} a_{k,\ell}^{\rm sing}(\omega,\theta,E) =& \frac{e^{i\pi/4}E^{(n-1)/4}}{2^{(n+1)/4}\sqrt{\pi}} \Big(\prod_{j=1}^n \lambda_j\Big)^{-1/2} \Gamma\Big(\frac{\Sigma(E)}{2\lambda_1}\Big) (2\lambda_1\lambda_{\boldsymbol{\ell}})^{3/2} (2\lambda_1)^{\Sigma(E)/\lambda_1-1} \\ &\times e^{-i\nu_\ell^+\pi/2} e^{-i\nu_k^-\pi/2} (D_k^- D_\ell^+)^{-1/2} \\ &\times |g_1^-(z_k^-)|\, |g_{\boldsymbol{\ell}}^+(z_\ell^+)| \big(-i\mathcal{M}_2(k,\ell)\big)^{-\Sigma(E)/2\lambda_1}. \end{aligned} \tag{2.35}$$

- *In case **(c)***

$$\mathcal{A}_{k,\ell}^{\rm sing}(\omega,\theta,E,h) = e^{i(S_\ell^++S_k^-)/h} a_{k,\ell}^{\rm sing}(\omega,\theta,E) \frac{h^{\Sigma(E)/2\lambda_1-1/2}}{|\ln h|^{\Sigma(E)/2\lambda_1}}(1+o(1)), \tag{2.36}$$

where

$$\begin{aligned} a^{\mathrm{sing}}_{k,\ell}(\omega,\theta,E) = & \frac{e^{i\pi/4}E^{(n-1)/4}}{2^{(n+1)/4}\sqrt{\pi}}\Big(\prod_{j=1}^{n}\lambda_j\Big)^{-1/2}\Gamma\big(\frac{\Sigma(E)}{2\lambda_1}\big)(2\lambda_1\lambda_{\boldsymbol{\ell}})^{3/2}(2\lambda_1)^{\Sigma(E)/2\lambda_1-1} \\ & \times e^{-i\nu_\ell^+\pi/2}e^{-i\nu_k^-\pi/2}(D_k^-D_\ell^+)^{-1/2} \\ & \times |g_1^-(z_k^-)|\,|g_{\boldsymbol{\ell}}^+(z_\ell^+)|\big(-i\mathcal{M}_1(k,\ell)\big)^{-\Sigma(E)/2\lambda_1}. \end{aligned} \tag{2.37}$$

Here, the $\widehat{\mu}_j$ are the linear combinations over $\mathbb{N}$ of the λ_k's and $\mu_k-\mu_{\mathbf{k}}$'s for $k\geq\mathbf{k}$, and the function $z\mapsto z^{-\Sigma(E)/\mu_k}$ is defined on $\mathbb{C}\backslash]-\infty,0]$ and real positive on $]0,+\infty[$.

Of course the assumption that $\langle g_1^-|g_1^+\rangle\neq 0$ (a subcase of **(a)**) is generic. Without the assumption **(A4)**, the regular part $\mathcal{A}^{reg}$ of the scattering amplitude has an integral representation as in [3]. When the assumption **(A7)** is not fulfilled, that is when the terms corresponding to the μ_j with $j\leq\widehat{\jmath}$ do not contribute, we don't know if the scattering amplitude can be given only in terms of the $g_j^{\pm}$'s and of the derivatives of the potential at the critical point.

3. Proof of the main resolvent estimate

Here we prove Theorem 2.1 using Mourre's Theory. We start with the construction of an escape function close to the stationary point $(0,0)$ in the spirit of [10] and [5]. Since Λ_+ and Λ_- are Lagrangian manifolds, one can find a local symplectic map $\kappa:(x,\xi)\mapsto(y,\eta)$ such that

$$p(x,\xi)-E_0=B(y,\eta)y\cdot\eta, \tag{3.1}$$

where $(y,\eta)\mapsto B(y,\eta)$ is a C^∞ mapping from a neighborhood of (0,0) in $T^*\mathbb{R}^n$ to the space $\mathcal{M}_n(\mathbb{R})$ of $n\times n$ matrices with real entries, such that,

$$B(0,0)=\begin{pmatrix}\lambda_1/2 & & \\ & \ddots & \\ & & \lambda_n/2\end{pmatrix}. \tag{3.2}$$

We denote by U a unitary Fourier integral operator (FIO) microlocally defined in a neighborhood of $(0,0)$, whose canonical transformation is κ, and we set

$$\widehat{P}=U(P-E_0)U^*. \tag{3.3}$$

Here the FIO U^* is the adjoint of U, and we have $UU^*=\mathrm{Id}+\mathcal{O}(h^\infty)$ and $U^*U=\mathrm{Id}+\mathcal{O}(h^\infty)$ microlocally near $(0,0)$. Then $\widehat{P}$ is a pseudodifferential operator, with a real (modulo $O(h^\infty)$) symbol $\widehat{p}(y,\eta)=\sum_j\widehat{p}_j(y,\eta)h^j$, such that

$$\widehat{p}_0=B(y,\eta)y\cdot\eta. \tag{3.4}$$

We set $B_1=\mathrm{Op}_h(b_1)$,

$$b_1(y,\eta)=\Big(\ln\Big\langle\frac{y}{\sqrt{hM}}\Big\rangle-\ln\Big\langle\frac{\eta}{\sqrt{hM}}\Big\rangle\Big)\widetilde{\chi}_2(y,\eta), \tag{3.5}$$

where $M>1$ will be fixed later and $\widetilde{\chi}_1\prec\widetilde{\chi}_2\in C_0^\infty(T^*\mathbb{R}^n)$ with $\widetilde{\chi}_1=1$ near $(0,0)$. In what follows, we will assume that $hM<1$. In particular, $b_1\in S^{1/2}(|\ln h|)$. Here and in what follows, we use the usual notation for classes of symbols. For m an order function, a function $a(x,\xi,h)\in C^\infty(T^*\mathbb{R}^n)$ belongs to $S^\delta(m)$ when

$$\forall\alpha\in\mathbb{N}^{2n},\ \exists C_\alpha>0,\ \forall h\in]0,1],\ |\partial^\alpha_{x,\xi}a(x,\xi,h)|\leq C_\alpha h^{-\delta|\alpha|}m(x,\xi). \tag{3.6}$$

We also recall that, with $\operatorname{Op}_h(a)$ denoting the Weyl quantization, if $a\in S^\alpha(1)$ and $b\in S^\beta(1)$, with α, $\beta<1/2$, we have

$$\big[\operatorname{Op}_h(a),\operatorname{Op}_h(b)\big]=\operatorname{Op}_h\big(ih\{b,a\}\big)+h^{3(1-\alpha-\beta)}\operatorname{Op}_h(r), \tag{3.7}$$

with $r\in S^{\min(\alpha,\beta)}(1)$: In particular the term of order 2 vanishes.

Hence, we have here

$$[B_1,\widehat{P}]=\operatorname{Op}_h\big(ih\{\widehat{p}_0,b_1\}\big)+|\ln h|\,h^{3/2}\operatorname{Op}_h(r_M), \tag{3.8}$$

with $r_M\in S^{1/2}(1)$. The semi-norms of r_M may depend on M. We have

$$\{\widehat{p}_0,b_1\}=c_1+c_2, \tag{3.9}$$

with

$$c_1=\Big(\ln\Big\langle\frac{y}{\sqrt{hM}}\Big\rangle-\ln\Big\langle\frac{\eta}{\sqrt{hM}}\Big\rangle\Big)\{\widehat{p}_0,\widetilde{\chi}_2\} \tag{3.10}$$

$$\begin{aligned} c_2=&\Big\{\widehat{p}_0,\ln\Big\langle\frac{y}{\sqrt{hM}}\Big\rangle-\ln\Big\langle\frac{\eta}{\sqrt{hM}}\Big\rangle\Big\}\widetilde{\chi}_2\\ =&\Big(\big(By+(\partial_\eta B)y\cdot\eta\big)\cdot\frac{y}{hM+y^2}+\big(B\eta+(\partial_y B)y\cdot\eta\big)\cdot\frac{\eta}{hM+\eta^2}\Big)\widetilde{\chi}_2. \end{aligned} \tag{3.11}$$

The symbols $c_1\in S^{1/2}(|\ln h|)$, $c_2\in S^{1/2}(1)$ satisfy $\operatorname{supp}(c_1)\subset\operatorname{supp}(\nabla\widetilde{\chi}_2)$. Let $\widetilde{\varphi}\in C^\infty_0(T^*\mathbb{R}^n)$ be a function such that $\widetilde{\varphi}=0$ near $(0,0)$ and $\widetilde{\varphi}=1$ near the support of $\nabla\widetilde{\chi}_2$. We have

$$\begin{aligned} \operatorname{Op}_h(c_1)=&\operatorname{Op}_h(\widetilde{\varphi})\operatorname{Op}_h(c_1)\operatorname{Op}_h(\widetilde{\varphi})+\mathcal{O}(h^\infty)\\ \geq&-C_1h|\ln h|\operatorname{Op}_h(\widetilde{\varphi})\operatorname{Op}_h(\widetilde{\varphi})+\mathcal{O}(h^\infty)\\ \geq&-C_1h|\ln h|\operatorname{Op}_h(\widetilde{\varphi}^2)+\mathcal{O}(h^2|\ln h|), \end{aligned} \tag{3.12}$$

for some $C_1>0$. On the other hand, using [5, (4.96)–(4.97)], we get

$$\operatorname{Op}_h(c_2)\geq\varepsilon M^{-1}\operatorname{Op}_h(\widetilde{\chi}_1)+\mathcal{O}(M^{-2}), \tag{3.13}$$

for some $\varepsilon>0$. With the notation $A_1=U^*B_1U$, the formulas (3.8), (3.9), (3.12) and (3.13) imply

$$\begin{aligned} -i[A_1,P]=&-iU^*[B_1,\widehat{P}]U+\mathcal{O}(h^\infty)\\ \geq&\varepsilon hM^{-1}U^*\operatorname{Op}_h(\widetilde{\chi}_1)U-C_1h|\ln h|U^*\operatorname{Op}_h(\widetilde{\varphi}^2)U\\ &+\mathcal{O}(hM^{-2})+\mathcal{O}_M(h^{3/2}|\ln h|). \end{aligned} \tag{3.14}$$

Here, $\chi_j=\widetilde{\chi}_j\circ\kappa$, $j=1,2$ and $\varphi=\widetilde{\varphi}\circ\kappa$ are $C^\infty_0(T^*\mathbb{R}^n,[0,1])$ functions which satisfy $\chi_1=1$ near $(0,0)$ and $\varphi=0$ near $(0,0)$. Using Egorov's Theorem, (3.14) becomes

$$-i[A_1,P]\geq\varepsilon hM^{-1}\operatorname{Op}_h(\chi_1)-C_1h|\ln h|\operatorname{Op}_h(\varphi^2)+\mathcal{O}(hM^{-2})+\mathcal{O}_M(h^{3/2}|\ln h|). \tag{3.15}$$

Now, we build an escape function outside of $\operatorname{supp}(\chi_1)$ as in [24]. Let $\mathbf{1}_{(0,0)}\prec\chi_0\prec\chi_1\prec\chi_2\prec\chi_3\prec\chi_4\prec\chi_5$ be $C^\infty_0(T^*\mathbb{R}^n,[0,1])$ functions with $\varphi\prec\chi_4$. We define $a_3=$

$g(\xi)(1-\chi_3(x,\xi))x\cdot\xi$ where $g\in C_0^\infty(\mathbb{R}^n)$ satisfies $\mathbf{1}_{p^{-1}([E_0-\delta,E_0+\delta])}\prec g$. Using [6, Lemma 3.1], we can find a bounded, C^∞ function $a_2(x,\xi)$ such that

$$H_pa_2\geq\begin{cases}0 & \text{for all } (x,\xi)\in p^{-1}([E_0-\delta,E_0+\delta]),\\ 1 & \text{for all } (x,\xi)\in \operatorname{supp}(\chi_4-\chi_0)\cap p^{-1}([E_0-\delta,E_0+\delta]),\end{cases}\tag{3.16}$$

and we set $A_2=\operatorname{Op}_h(a_2\chi_5)$, $A_3=\operatorname{Op}_h(a_3)$. We denote

$$A=A_1+C_2|\ln h|A_2+|\ln h|A_3,\tag{3.17}$$

where $C_2>1$ will be fixed later. Now let $\widetilde{\psi}\in C_0^\infty([E_0-\delta,E_0+\delta],[0,1])$ with $\widetilde{\psi}=1$ near E_0. We recall that $\widetilde{\psi}(P)$ is a classical pseudodifferential operator of class $\Psi^0(\langle\xi\rangle^{-\infty})$ with principal symbol $\widetilde{\psi}(p)$. Then, from (3.15), we obtain

$$\begin{aligned}-i\widetilde{\psi}(P)[A,P]\widetilde{\psi}(P)\geq&\varepsilon hM^{-1}\widetilde{\psi}(P)\operatorname{Op}_h(\chi_1)\widetilde{\psi}(P)-C_1h|\ln h|\widetilde{\psi}(P)\operatorname{Op}_h(\varphi^2)\widetilde{\psi}(P)\\&+C_2h|\ln h|\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(\chi_4-\chi_0)\big)+C_2h|\ln h|\operatorname{Op}_h\big(\widetilde{\psi}^2(p)a_2H_p\chi_5\big)\\&+h|\ln h|\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(\xi^2-x\cdot\nabla V)(1-\chi_3)\big)\\&+h|\ln h|\operatorname{Op}_h\big(\widetilde{\psi}^2(p)x\cdot\xi H_p(g\chi_3)\big)+\mathcal{O}(hM^{-2})+\mathcal{O}_M(h^{3/2}|\ln h|).\end{aligned}\tag{3.18}$$

From **(A1)**, we have $x\cdot\nabla V(x)\to0$ as $x\to\infty$. In particular, if χ_3 is equal to 1 in a sufficiently large zone, we have

$$\widetilde{\psi}^2(p)(\xi^2-x\cdot\nabla V)(1-\chi_3)\geq E_0\widetilde{\psi}^2(p)(1-\chi_3).\tag{3.19}$$

If $C_2>0$ is large enough, the Gårding inequality implies

$$\begin{aligned}C_2\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(\chi_4-\chi_0)\big)-C_1\operatorname{Op}_h\big(\widetilde{\psi}^2(p)\varphi^2\big)+\operatorname{Op}_h\big(\widetilde{\psi}^2(p)x\cdot\xi H_p(g\chi_3)\big)&\\ \geq\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(\chi_4-\chi_0)\big)+\mathcal{O}(h).&\end{aligned}\tag{3.20}$$

As in [24], we take $\chi_5(x,\xi)=\widetilde{\chi}_5(\mu x)g(\xi)$ with μ small and $\widetilde{\chi}_5\in C_0^\infty(\mathbb{R}^n,[0,1])$, $\widetilde{\chi}_5=1$ near 0. Since a_2 is bounded, we get

$$\big|C_2\widetilde{\psi}^2(p)a_2H_p\chi_5\big|\leq\mu C_2\|a_2\|_{L^\infty}\|H_p\widetilde{\chi}_5\|_{L^\infty}\lesssim\mu.\tag{3.21}$$

Therefore, if μ is small enough, (3.19) implies

$$\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(\xi^2-x\cdot\nabla V)(1-\chi_3)\big)+C_2\operatorname{Op}_h\big(\widetilde{\psi}^2(p)a_2H_p\chi_5\big)\geq\frac{E_0}{2}\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(1-\chi_3)\big).\tag{3.22}$$

Then (3.18), (3.20), (3.22) and the Gårding inequality give, for some $\varepsilon>0$,

$$\begin{aligned}-i\widetilde{\psi}(P)[A,P]\widetilde{\psi}(P)\geq&\varepsilon hM^{-1}\operatorname{Op}_h\big(\widetilde{\psi}^2(p)\chi_1\big)+h|\ln h|\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(\chi_4-\chi_0)\big)\\&+\frac{E_0}{2}h|\ln h|\operatorname{Op}_h\big(\widetilde{\psi}^2(p)(1-\chi_3)\big)+\mathcal{O}(hM^{-2})+\mathcal{O}_M(h^{3/2}|\ln h|)\\ \geq&\varepsilon hM^{-1}\operatorname{Op}_h\big(\widetilde{\psi}^2(p)\big)+\mathcal{O}(hM^{-2})+\mathcal{O}_M(h^{3/2}|\ln h|).\end{aligned}\tag{3.23}$$

Choosing M large enough and $\mathbf{1}_{E_0}\prec\psi\prec\widetilde{\psi}$, we have proved the

Lemma 3.1. *Let M be large enough and $\psi\in C_0^\infty([E_0-\delta,E_0+\delta])$, $\delta>0$ small enough, with $\psi=1$ near E_0. Then, we have*

$$-i\psi(P)[A,P]\psi(P)\geq\varepsilon h\psi^2(P),\tag{3.24}$$

for some $\varepsilon > 0$. *Moreover*

$$[A, P] = \mathcal{O}(h|\ln h|). \tag{3.25}$$

Now we estimate $[[P, A], A]$. From the properties of the support of the χ_j, we have

$$\begin{aligned}[[P, A], A] =&[[P, A_1], A_1] + C_2|\ln h|[[P, A_1], A_2]\\ &+ C_2|\ln h|[[P, A_2], A_1] + C_2^2|\ln h|^2[[P, A_2], A_2] + C_2|\ln h|^2[[P, A_2], A_3]\\ &+ C_2|\ln h|^2[[P, A_3], A_2] + |\ln h|^2[[P, A_3], A_3] + \mathcal{O}(h^\infty).\end{aligned} \tag{3.26}$$

We also know that $P \in \Psi^0(\langle\xi\rangle^2)$, $A_2 \in \Psi^0(\langle\xi\rangle^{-\infty})$ and $A_3 \in \Psi^0(\langle x\rangle\langle\xi\rangle^{-\infty})$. Then, we can show that all the terms in (3.26) with $j, k = 2, 3$ satisfy

$$[[P, A_j], A_k] \in \Psi^0(h^2). \tag{3.27}$$

On the other hand,

$$[[P, A_1], A_2] = U^*[[\widehat{P}, B_1], UA_2U^*]U + \mathcal{O}(h^\infty), \tag{3.28}$$

with $UA_2U^* \in \Psi^0(1)$. From (3.8) – (3.11), we have $[\widehat{P}, B_1] \in \Psi^{1/2}(h|\ln h|)$ and then

$$[[P, A_1], A_2] = \mathcal{O}(h^{3/2}|\ln h|). \tag{3.29}$$

The term $[[P, A_2], A_1]$ gives the same type of contribution. It remains to study

$$[[P, A_1], A_1] = U^*[[\widehat{P}, B_1], B_1]U + \mathcal{O}(h^\infty). \tag{3.30}$$

Let $\widetilde{\chi}_3 \in C_0^\infty(T^*\mathbb{R}^n), [0, 1])$ with $\widetilde{\chi}_2 \prec \widetilde{\chi}_3$ and

$$f = \Big(\ln\Big\langle\frac{y}{\sqrt{hM}}\Big\rangle - \ln\Big\langle\frac{\eta}{\sqrt{hM}}\Big\rangle\Big)\widetilde{\chi}_3(y, \eta) \in S^{1/2}(|\ln h|). \tag{3.31}$$

Then, with a remainder $r_M \in S^{1/2}(1)$ which differs from line to line,

$$\begin{aligned}i[\widehat{P}, B_1] =&h \operatorname{Op}_h\big(f\{\widetilde{\chi}_2, \widehat{p}_0\} + c_2\big) - h^{3/2}|\ln h| \operatorname{Op}_h(r_M)\\ =&h \operatorname{Op}_h(f) \operatorname{Op}_h(\{\widetilde{\chi}_2, \widehat{p}_0\}) + h \operatorname{Op}_h(c_2) + h^{3/2}|\ln h| \operatorname{Op}_h(r_M).\end{aligned} \tag{3.32}$$

In particular, since $[\widehat{P}, B_1] \in \Psi^{1/2}(h|\ln h|)$, $c_2 \in S^{1/2}(1)$ and $f \in S^{1/2}(|\ln h|)$,

$$\begin{aligned}[[\widehat{P}, B_1], B_1] =&[[\widehat{P}, B_1], \operatorname{Op}_h(f\widetilde{\chi}_2)]\\ =&-ih[\operatorname{Op}_h(f) \operatorname{Op}_h(\{\widetilde{\chi}_2, \widehat{p}_0\}), \operatorname{Op}_h(f\widetilde{\chi}_2)] - ih[\operatorname{Op}_h(c_2), \operatorname{Op}_h(f\widetilde{\chi}_2)]\\ &+ \mathcal{O}(h^{3/2}|\ln h|^2)\\ =&-ih[\operatorname{Op}_h(f) \operatorname{Op}_h(\{\widetilde{\chi}_2, \widehat{p}_0\}), \operatorname{Op}_h(f) \operatorname{Op}_h(\widetilde{\chi}_2)] + \mathcal{O}(h|\ln h|)\\ =&-ih \operatorname{Op}_h(f)[\operatorname{Op}_h(\{\widetilde{\chi}_2, \widehat{p}_0\}), \operatorname{Op}_h(f)] \operatorname{Op}_h(\widetilde{\chi}_2)\\ &- ih \operatorname{Op}_h(f) \operatorname{Op}_h(f)[\operatorname{Op}_h(\{\widetilde{\chi}_2, \widehat{p}_0\}), \operatorname{Op}_h(\widetilde{\chi}_2)]\\ &- ih \operatorname{Op}_h(f)[\operatorname{Op}_h(f), \operatorname{Op}_h(\widetilde{\chi}_2)] \operatorname{Op}_h(\{\widetilde{\chi}_2, \widehat{p}_0\}) + \mathcal{O}(h|\ln h|)\\ =&\mathcal{O}(h|\ln h|).\end{aligned} \tag{3.33}$$

From (3.26), (3.27), (3.29) and (3.33), we get

$$[[P, A], A] = \mathcal{O}(h|\ln h|). \tag{3.34}$$

As a matter of fact, using [5], one can show that $[[P, A], A] = \mathcal{O}(h)$. Now we can use the following proposition which is an adaptation of the limiting absorption principle of Mourre [27] (see also [12, Theorem 4.9], [21, Proposition 2.1] and [4, Theorem 7.4.1]).

Proposition 3.2. *Let $(P, D(P))$ and $(\mathcal{A}, D(\mathcal{A}))$ be self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Assume the following assumptions:*

i) P is of class $C^2(\mathcal{A})$. Recall that P is of class $C^r(\mathcal{A})$ if there exists $z \in \mathbb{C} \setminus \sigma(P)$ such that

$$\mathbb{R} \ni t \to e^{it\mathcal{A}}(P - z)^{-1}e^{-it\mathcal{A}}, \tag{3.35}$$

is C^r for the strong topology of $\mathcal{L}(\mathcal{H})$.

ii) The form $[P, \mathcal{A}]$ defined on $D(\mathcal{A}) \cap D(P)$ extends to a bounded operator on $\mathcal{H}$ and

$$\|[P, \mathcal{A}]\| \lesssim \beta. \tag{3.36}$$

iii) The form $[[P, \mathcal{A}], \mathcal{A}]$ defined on $D(\mathcal{A})$ extends to a bounded operator on $\mathcal{H}$ and

$$\|[[P, \mathcal{A}], \mathcal{A}]\| \lesssim \gamma. \tag{3.37}$$

iv) There exist a compact interval $I \subset \mathbb{R}$ and $g \in C_0^\infty(\mathbb{R})$ with $\mathbf{1}_I \prec g$ such that

$$ig(P)[P, \mathcal{A}]g(P) \gtrsim \gamma g^2(P). \tag{3.38}$$

v) $\beta^2 \lesssim \gamma \lesssim 1$.

Then, for all $\alpha > 1/2$, $\lim_{\varepsilon \to 0}\langle \mathcal{A}\rangle^{-\alpha}(P - E \pm i\varepsilon)^{-1}\langle \mathcal{A}\rangle^{-\alpha}$ exists and

$$\big\|\langle \mathcal{A}\rangle^{-\alpha}(P - E \pm i0)^{-1}\langle \mathcal{A}\rangle^{-\alpha}\big\| \lesssim \gamma^{-1}, \tag{3.39}$$

uniformly for $E \in I$.

Remark 3.3. *From Theorem 6.2.10 of [4], we have the following useful characterization of the regularity $C^2(\mathcal{A})$. Assume that (ii) and (iv) hold. Then, P is of class $C^2(\mathcal{A})$ if and only if, for some $z \in \mathbb{C} \setminus \sigma(P)$, the set $\{u \in D(\mathcal{A});\ (P - z)^{-1}u \in D(\mathcal{A}) \text{ and } (P - \overline{z})^{-1}u \in D(\mathcal{A})\}$ is a core for $\mathcal{A}$.*

Proof. The proof follows the work of Hislop and Nakamura [21]. For $\varepsilon > 0$, we define $M^2 = ig(P)[P, \mathcal{A}]g(P)$ and $G_\varepsilon(z) = (P - i\varepsilon M^2 - z)^{-1}$ which is analytic for $\operatorname{Re} z \in I$ and $\operatorname{Im} z > 0$. Following [12, Lemma 4.14] with (3.35), we get

$$\|g(P)G_\varepsilon(z)\varphi\| \lesssim (\varepsilon\gamma)^{-1/2}|(\varphi, G_\varepsilon(z)\varphi)|^{1/2}, \tag{3.40}$$

$$\|(1 - g(P))G_\varepsilon(z)\| \lesssim 1 + \varepsilon\beta\|G_\varepsilon(z)\|, \tag{3.41}$$

and then

$$\|G_\varepsilon(z)\| \lesssim (\varepsilon\gamma)^{-1}, \tag{3.42}$$

for $\varepsilon < \varepsilon_0$ with ε_0 small enough, but independent of β, γ.

As in [21], let $D_\varepsilon = (1 + |\mathcal{A}|)^{-\alpha}(1 + \varepsilon|\mathcal{A}|)^{\alpha-1}$ for $\alpha \in]1/2, 1]$ and $F_\varepsilon(z) = D_\varepsilon G_\varepsilon(z)D_\varepsilon$. Of course, from (3.42),

$$\|F_\varepsilon(z)\| \lesssim (\varepsilon\gamma)^{-1}, \tag{3.43}$$

and (3.40) and (3.41) with $\varphi = D_\varepsilon \psi$ give

$$\|G_\varepsilon(z) D_\varepsilon\| \lesssim 1 + (\varepsilon\gamma)^{-1/2} \|F_\varepsilon\|^{1/2}. \tag{3.44}$$

The derivative of $F_\varepsilon(z)$ is given by (see [12, Lemma 4.15])

$$\partial_\varepsilon F_\varepsilon(z) = i D_\varepsilon G_\varepsilon M^2 G_\varepsilon D_\varepsilon = Q_0 + Q_1 + Q_2 + Q_3, \tag{3.45}$$

with

$$\begin{aligned} Q_0 =&(\alpha-1)|\mathcal{A}|(1+|\mathcal{A}|)^{-\alpha}(1+\varepsilon|\mathcal{A}|)^{\alpha-2} G_\varepsilon(z) D_\varepsilon \\ &+ (\alpha-1) D_\varepsilon G_\varepsilon(z) |\mathcal{A}|(1+|\mathcal{A}|)^{-\alpha}(1+\varepsilon|\mathcal{A}|)^{\alpha-2} \end{aligned} \tag{3.46}$$

$$Q_1 = D_\varepsilon G_\varepsilon (1-g(P))[P,\mathcal{A}](1-g(P)) G_\varepsilon D_\varepsilon \tag{3.47}$$

$$Q_2 = D_\varepsilon G_\varepsilon (1-g(P))[P,\mathcal{A}] g(P) G_\varepsilon D_\varepsilon + D_\varepsilon G_\varepsilon g(P)[P,\mathcal{A}](1-g(P)) G_\varepsilon D_\varepsilon \tag{3.48}$$

$$Q_3 = - D_\varepsilon G_\varepsilon [P,\mathcal{A}] G_\varepsilon D_\varepsilon. \tag{3.49}$$

From (3.44), we obtain

$$\|Q_0\| \lesssim \varepsilon^{\alpha-1}\big(1 + (\varepsilon\gamma)^{-1/2}\|F_\varepsilon\|^{1/2}\big), \tag{3.50}$$

and from (3.36), *v)* of Proposition 3.2, (3.41), and (3.42), we have

$$\|Q_1\| \lesssim \gamma^{-1}. \tag{3.51}$$

Using in addition (3.44), we obtain

$$\|Q_2\| \lesssim 1 + (\varepsilon\gamma)^{-1/2}\|F_\varepsilon\|^{1/2}. \tag{3.52}$$

Now we write $Q_3 = Q_4 + Q_5$ with

$$Q_4 = -D_\varepsilon G_\varepsilon [P - i\varepsilon M^2 - z, \mathcal{A}] G_\varepsilon D_\varepsilon \tag{3.53}$$

$$Q_5 = -i\varepsilon D_\varepsilon G_\varepsilon [M^2, \mathcal{A}] G_\varepsilon D_\varepsilon. \tag{3.54}$$

For Q_4, we have the estimate

$$\|Q_4\| \lesssim \varepsilon^{\alpha-1}\big(1 + (\varepsilon\gamma)^{-1/2}\|F_\varepsilon\|^{1/2}\big) \tag{3.55}$$

On the other hand, (3.36), (3.37) and *v)* imply

$$\|[M^2, \mathcal{A}]\| \lesssim \gamma. \tag{3.56}$$

Then (3.44) gives

$$\|Q_5\| \lesssim 1 + \|F_\varepsilon\|. \tag{3.57}$$

Using the estimates on the Q_j, we get

$$\|\partial_\varepsilon F_\varepsilon\| \lesssim \varepsilon^{\alpha-1}\big(\gamma^{-1} + (\varepsilon\gamma)^{-1/2}\|F_\varepsilon\|^{1/2} + \|F_\varepsilon\|\big). \tag{3.58}$$

Using (3.43) and integrating (3.37) N times with respect to ε, we get

$$\|F_\varepsilon\| \lesssim \gamma^{-1}\big(1 + \varepsilon^{2\alpha(1-2^{-N})-1}\big), \tag{3.59}$$

so that, for N large enough,

$$\limsup_{\delta\to 0} \sup_{E\in I} \|\langle\mathcal{A}\rangle^{-\alpha}(P - E \pm i\delta)^{-1}\langle\mathcal{A}\rangle^{-\alpha}\| \lesssim \gamma^{-1}. \tag{3.60}$$

Using, as in [21], the fact that $z \mapsto F_0(z)$ is Hölder continuous, we prove the existence of the limit $\lim_{\operatorname{Im} z\to 0} F_0(z)$ for $\operatorname{Re} z \in I$ and the proposition follows from (3.60). ☐

From Lemma 3.1 and (3.34), we can apply Proposition 3.2 with $\mathcal{A} = A/|\ln h|$, $\beta = h$ and $\gamma = h/|\ln h|$. Therefore we have the estimate

$$\big\|\langle \mathcal{A}\rangle^{-\alpha}(P - E \pm i0)^{-1}\langle \mathcal{A}\rangle^{-\alpha}\big\| \lesssim h^{-1}|\ln h|, \tag{3.61}$$

for $E \in [E_0 - \delta, E_0 + \delta]$. As usual, we have

$$\|\langle x\rangle^{-\alpha}\langle \mathcal{A}\rangle^{\alpha}\| = \mathcal{O}(1), \tag{3.62}$$

for $\alpha \geq 0$. Indeed, (3.62) is clear for $\alpha \in 2\mathbb{N}$, and the general case follows by complex interpolation. Then, (3.61) and (3.26) imply Theorem 2.1.

4. Representation of the scattering amplitude

As in [32], our starting point for the computation of the scattering amplitude is the representation given by Isozaki and Kitada in [22]. We recall briefly their formula, that they obtained writing parametrices for the wave operators $W_\pm$ as Fourier integral operators, taking advantage of the well-known intertwining property $W_\pm P = P_0 W_\pm$, $P = P_0 + V$, with $P_0 = -\frac{h^2}{2}\Delta$. The wave operators are defined by

$$W_\pm = \operatorname*{s\text{--}lim}_{t\to\pm\infty} e^{itP/h} e^{-itP_0/h}, \tag{4.1}$$

where the limits exist in $L^2(\mathbb{R}^n)$ thanks to the short-range assumption **(A1)**. The scattering operator is by definition $\mathcal{S} = (W_+)^* W_-$, and the scattering matrix $\mathcal{S}(E,h)$ is then given by the decompostion of $\mathcal{S}$ with respect to the spectral measure of P_0. Now we recall briefly the discussion in [32, Section 1,2] (see also [3]), and we start with some notations.

If Ω is an open subset of $T^*\mathbb{R}^n$, we denote by $A_m(\Omega)$ the class of symbols a such that $(x,\xi) \mapsto a(x,\xi,h)$ belongs to $C^\infty(\Omega)$ and

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha\beta}\langle x\rangle^{m-|\alpha|}\langle \xi\rangle^{-L}, \text{ for all } L > 0, (x,\xi) \in \Omega, (\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d. \tag{4.2}$$

We also denote by

$$\Gamma_\pm(R,d,\sigma) = \Big\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n;\ |x| > R, \frac{1}{d} < |\xi| < d, \pm\cos(x,\xi) > \pm\sigma\Big\} \tag{4.3}$$

with $R > 1$, $d > 1$, $\sigma \in (-1,1)$, and $\cos(x,\xi) = \frac{x\cdot\xi}{|x|\,|\xi|}$, the outgoing and incoming subsets of $T^*\mathbb{R}^n$, respectively. Eventually, for $\alpha > \frac{1}{2}$, we denote the bounded operator $\mathcal{F}_0(E,h)$: $L^2_\alpha(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$ given by

$$\left(\mathcal{F}_0(E,h)f\right)(\omega) = (2\pi h)^{-\frac{n}{2}}(2E)^{\frac{n-2}{4}} \int_{\mathbb{R}^n} e^{-\frac{i}{h}\sqrt{2E}\omega\cdot x} f(x)dx, E > 0. \tag{4.4}$$

Isozaki and Kitada have constructed phase functions $\Phi_\pm$ and symbols $a_\pm$ and $b_\pm$ such that, for some $R_0 >> 0$, $1 < d_4 < d_3 < d_2 < d_1 < d_0$, and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 < 1$:

i) $\Phi_\pm \in C^\infty(T^*\mathbb{R}^n)$ solve the eikonal equation

$$\frac{1}{2}\left|\nabla_x \Phi_\pm(x,\xi)\right|^2 + V(x) = \frac{1}{2}|\xi|^2 \tag{4.5}$$

in $(x,\xi) \in \Gamma_\pm(R_0, d_0, \pm\sigma_0)$, respectively.

ii) $(x,\xi) \mapsto \Phi_\pm(x,\xi) - x\cdot\xi \in A_0\left(\Gamma_\pm(R_0,d_0,\pm\sigma_0)\right)$.

iii) For all $(x,\xi)\in T^*\mathbb{R}^n$

$$\left|\frac{\partial^2\Phi_\pm}{\partial_{x_j}\partial_{\xi_k}}(x,\xi)-\delta_{jk}\right|<\varepsilon(R_0),\tag{4.6}$$

where δ_{jk} is the Kronecker delta and $\varepsilon(R_0)\to 0$ as $R_0\to+\infty$.

iv) $a_\pm\sim\sum_{j=0}^\infty h^j a_{\pm j}$, where $a_{\pm j}\in A_{-j}(\Gamma_\pm(3R_0,d_1,\mp\sigma_1))$, $\operatorname{supp} a_{\pm j}\subset\Gamma_\pm(3R_0,d_1,\mp\sigma_1)$, $a_{\pm j}$ solve

$$\langle\nabla_x\Phi_\pm|\nabla_x a_{\pm 0}\rangle+\frac{1}{2}\left(\Delta_x\Phi_\pm\right)a_{\pm 0}=0\tag{4.7}$$

$$\langle\nabla_x\Phi_\pm|\nabla_x a_{\pm j}\rangle+\frac{1}{2}\left(\Delta_x\Phi_\pm\right)a_{\pm j}=\frac{i}{2}\Delta_x a_{\pm j-1}, j\geq 1,\tag{4.8}$$

with the conditions at infinity

$$a_{\pm 0}\to 1, a_{\pm j}\to 0, j\geq 1, \text{ as } |x|\to\infty.\tag{4.9}$$

in $\Gamma_\pm(2R_0,d_2,\mp\sigma_2)$, and solve (4.7) and (4.8) in $\Gamma_\pm(4R_0,d_1,\mp\sigma_2)$.

v) $b_\pm\sim\sum_{j=0}^\infty h^j b_{\pm j}$, where $b_{\pm j}\in A_{-j}(\Gamma_\pm(5R_0,d_3,\pm\sigma_4))$, $\operatorname{supp} b_{\pm j}\subset\Gamma_\pm(5R_0,d_3,\pm\sigma_4)$, $b_{\pm j}$ solve (4.7) and (4.8) with the conditions at infinity (4.9) in $\Gamma_\pm(6R_0,d_4,\pm\sigma_3)$, and solve (4.7) and (4.8) in $\Gamma_\pm(6R_0,d_3,\pm\sigma_3)$.

For a symbol c and a phase function φ, we denote by $I_h(c,\varphi)$ the oscillatory integral

$$I_h(c,\varphi)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}e^{\frac{i}{h}(\varphi(x,\xi)-y\cdot\xi)}c(x,\xi)d\xi\tag{4.10}$$

and we set

$$\begin{aligned}K_{\pm a}(h)&=P(h)I_h(a_\pm,\Phi_\pm)-I_h(a_\pm,\Phi_\pm)P_0(h),\\ K_{\pm b}(h)&=P(h)I_h(b_\pm,\Phi_\pm)-I_h(b_\pm,\Phi_\pm)P_0(h).\end{aligned}\tag{4.11}$$

The operator $\mathcal{T}(E,h)$ for $E\in]\frac{1}{2d_4^2},\frac{d_4^2}{2}[$ is then given by (see [22, Theorem 3.3])

$$\mathcal{T}(E,h)=\mathcal{T}_{+1}(E,h)+\mathcal{T}_{-1}(E,h)-\mathcal{T}_2(E,h),\tag{4.12}$$

where

$$\mathcal{T}_{\pm 1}(E,h)=\mathcal{F}_0(E,h)I_h(a_\pm,\Phi_\pm)^*K_{\pm b}(h)\mathcal{F}_0^*(E,h)\tag{4.13}$$

and

$$\mathcal{T}_2(E,h)=\mathcal{F}_0(E,h)K_{+a}^*(h)\mathcal{R}(E+i0,h)\left(K_{+b}(h)+K_{-b}(h)\right)\mathcal{F}_0^*(E,h),\tag{4.14}$$

where we denote from now on $\mathcal{R}(E\pm i0,h)=(P-(E\pm i0))^{-1}$.

Writing explicitly their kernel, it is easy to see, by a non-stationary phase argument, that the operators $\mathcal{T}_{\pm 1}$ are $\mathcal{O}(h^\infty)$ when $\theta\neq\omega$. Therefore we have

$$\mathcal{A}(\omega,\theta,E,h)=-c(E)h^{(n-1)/2}\mathcal{T}_2(\omega,\theta,E,h)+\mathcal{O}(h^\infty),\tag{4.15}$$

where $c(E)$ is given in (1.4).

As in [32], we shall use our resolvent estimate (Theorem 2.1) in a particular form. It was noticed by L. Michel in [26, Proposition 3.1] that, in the present trapping case, the following proposition follows easily from the corresponding one in the non-trapping setting. Indeed, if φ is a compactly supported smooth function, it is clear that $\widetilde{P}=-h^2\Delta+(1-\varphi(x/R))V(x)$

satisfies the non-trapping assumption for R large enough, thanks to the decay of V at ∞. Writing [32, Lemma 2.3] for $\widetilde{P}$, one gets the

Proposition 4.1. *Let $\omega_\pm \in A_0$ have support in $\Gamma_\pm(R, d, \sigma_\pm)$ for $R > R_0$. For $E \in [E_0 - \delta, E_0 + \delta]$, we have*

(i) *For any $\alpha > 1/2$ and $M > 1$, then, for any $\varepsilon > 0$,*

$$\|\mathcal{R}(E \pm i0, h)\omega_\pm(x, hD_x)\|_{-\alpha+M,-\alpha} = \mathcal{O}(h^{-3-\varepsilon}). \tag{4.16}$$

(ii) *If $\sigma_+ > \sigma_-$, then for any $\alpha \gg 1$,*

$$\|\omega_\mp(x, hD_x)\mathcal{R}(E \pm i0, h)\omega_\pm(x, hD_x)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty). \tag{4.17}$$

(iii) *If $\omega(x, \xi) \in A_0$ has support in $|x| < (9/10)R$, then for any $\alpha \gg 1$*

$$\|\omega(x, hD_x)\mathcal{R}(E \pm i0, h)\omega_\pm(x, hD_x)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty). \tag{4.18}$$

Then we can follow line by line the discussion after Lemma 2.1 of D. Robert and H. Tamura, and we obtain (see Equations 2.2-2.4 there):

$$\mathcal{A}(\omega, \theta, E, h) = \widetilde{c}(E)h^{-(n+1)/2}\langle\mathcal{R}(E + i0, h)g_-e^{i\psi_-/h}, g_+e^{i\psi_+/h}\rangle + \mathcal{O}(h^\infty), \tag{4.19}$$

where $\widetilde{c}(E) = (2\pi)^{(1-n)/2}(2E)^{(n-3)/4}e^{-i\frac{(n-3)\pi}{4}}$,

$$g_\pm = e^{-i\psi_\pm/h}[\chi_\pm, P]a_\pm(x, h)e^{i\psi_\pm/h}, \tag{4.20}$$

and

$$\psi_+(x) = \Phi_+(x, \sqrt{2E}\theta), \qquad \psi_-(x) = \Phi_-(x, \sqrt{2E}\omega). \tag{4.21}$$

Moreover the functions $\chi_\pm$ are $C_0^\infty(\mathbb{R}^n)$ functions such that $\chi_\pm = 1$ near some ball $B(0, R_\pm)$, with support in $B(0, R_\pm + 1)$.

Eventually, we shall need the following version of Egorov's Theorem, which is also used in Robert and Tamura's paper.

Proposition 4.2 ([32, Proposition 3.1])**.** *Let $\omega(x, \xi) \in A_0$ be of compact support. Assume that, for some fixed $t \in \mathbb{R}$, ω_t is a function in A_0 which vanishes in a small neighborhood of*

$$\{(x, \xi);\ (x, \xi) = \exp(tH_p)(y, \eta),\ (y, \eta) \in \operatorname{supp} \omega\}.$$

Then

$$\| \operatorname{Op}_h(\omega_t)e^{-itP/h} \operatorname{Op}_h(\omega)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty),$$

for any $\alpha \gg 1$. Moreover, the order relation is uniform in t when t ranges over a compact interval of $\mathbb{R}$.

In the three next sections, we prove Theorem 2.6 using (4.19). We set

$$u_- = u_-^h = \mathcal{R}(E + i0, h)g_-e^{i\psi_-/h}, \tag{4.22}$$

and our proof consists in the computation of u_- in different region of the phase space, following the classical trajectories γ_j^∞, or γ_k^- and γ_ℓ^+. It is important to notice that we have $(P-E)u_- = 0$ out of the support of g_-.

5. Computations before the critical point

5.1. Computation of u_- in the incoming region.

We start with the computation of u_- in an incoming region which contains the microsupport of g_-. Notice that, thanks to Theorem 2.1, $\langle x\rangle^{-\alpha}u_-(x)$ is a semiclassical family of distributions for $\alpha > 1/2$.

Lemma 5.1. *Let $P = -\frac{h^2}{2}\Delta + V$, where V satisfies assumption* **(A1)** *with $\rho > 0$. Let also I be a compact interval of $]0,+\infty[$, and $d > 0$ such that $I \subset]\frac{1}{2d^2}, \frac{d^2}{2}[$. For any $0 < \sigma_+ < 1$, there exists $R(\sigma_+) > 0$ such that, for all $R > R(\sigma_+)$ and any compact subset $K \subset T^*\mathbb{R}^n$ of $p^{-1}(I)$, there exists $T > 0$ such that, if $\rho \in K$ and $t > T$,*

$$\exp(tH_p)(\rho) \in \Gamma_+(R/2, d, \sigma_+) \cup (B(0, R/2) \times \mathbb{R}^n). \tag{5.1}$$

Proof. We recall from the constructions of C. Gérard and J. Sjöstrand in [17] that for any $\delta > 0$, there exist $R_\delta > 0$ and a function $G(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that,

$$(H_pG)(x,\xi) \geq 0 \quad \text{for all } (x,\xi) \in p^{-1}(]\frac{1}{2d^2}, \frac{d^2}{2}[), \tag{5.2}$$

$$(H_pG)(x,\xi) > 2E(1-\delta) \quad \text{for } |x| > R_\delta \text{ and } p(x,\xi) = E \in]\frac{1}{2d^2}, \frac{d^2}{2}[, \tag{5.3}$$

$$G(x,\xi) = x\cdot\xi \quad \text{for } |x| > R_\delta. \tag{5.4}$$

We choose $\delta > 0$ such that $1 - 3\delta > \sigma_+$. We can assume that

$$|\xi| \leq \sqrt{2E}(1+\delta), \tag{5.5}$$

for $|x| \geq R_\delta$, $(x,\xi) \in p^{-1}(]\frac{1}{2d^2}, \frac{d^2}{2}[)$. We first assume that $R > 4R_\delta$, and that K is a compact subset of $p^{-1}(I)$. For $\rho \in K$ and $\gamma(t) = (x(t), \xi(t)) = \exp(tH_p)(\rho)$, the corresponding Hamiltonian curve, we distinguish between 2 cases:

1) *For all $t > 0$, we have $|x(t)| > R_\delta$.*

Then $G(\gamma(t)) > 2E(1-\delta)t + G(\rho)$ and, for $t > T_1$ with T_1 large enough,

$$G(\gamma(t)) > 2 \sup_{\substack{x\in B(0,R_\delta)\\ p(x,\xi)\in I}} G(x,\xi). \tag{5.6}$$

By continuity, there exists a neighborhood $\mathcal{U}$ of ρ such that, for all $\widetilde{\rho} \in \mathcal{U}$, we have

$$G(\widetilde{\gamma}(T_1)) > \sup_{\substack{x\in B(0,R_\delta)\\ p(x,\xi)\in I}} G(x,\xi). \tag{5.7}$$

Since G is non-decreasing along $\widetilde{\gamma}(t)$, we have $|\widetilde{x}(t)| > R_\delta$ for all $t > T_1$, and then

$$G(\widetilde{\gamma}(t)) > 2E(1-\delta)(t - T_1) + G(\widetilde{\gamma}(T_1)) > 2E(1-\delta)t - C. \tag{5.8}$$

From (5.5) and (5.8), we get $|\widetilde{x}(t)| > \frac{1}{C}t - C$ for all $\widetilde{\rho} \in \mathcal{U}$, and then $|\widetilde{\xi}(t)| = \sqrt{2E} + o_{t\to\infty}(1)$. On the other hand using (5.5) we have $|\widetilde{x}(t)| \leq \sqrt{2E}(1+\delta)t + C$, for some $C > 0$ independent of $\widetilde{\rho} \in \mathcal{U}$. In particular, the previous estimates give, for $t > T_{\mathcal{U}}$ with $T_{\mathcal{U}}$ large enough but independent of $\widetilde{\rho} \in \mathcal{U}$

$$|\widetilde{x}(t)| > R/2, \tag{5.9}$$

$$\cos\big(\widetilde{x},\widetilde{\xi}\big)(t) > \frac{2E(1-\delta)t-C}{(\sqrt{2E}(1+\delta)t+C)(\sqrt{2E}+o_{t\to\infty}(1))} = \frac{1-\delta}{1+\delta}+o_{t\to\infty}(1) > 1-3\delta, \tag{5.10}$$

Thus, for $t > T_{\mathcal{U}}$ and $\widetilde{\rho}\in\mathcal{U}$, we have

$$\widetilde{\gamma}(t)\in\Gamma_+(R/2,d,\sigma_+), \tag{5.11}$$

where we recall that $\sigma_+ < 1-3\delta$.

2) *There exists* $T_2>0$ *such that* $|x(T_2)| = R_\delta$.

Then there exists a neighborhood $\mathcal{V}$ of ρ such that for all $\widetilde{\rho}\in\mathcal{V}$ we have $|\widetilde{x}(T_2)| < 2R_\delta$, where $(\widetilde{x}(t),\widetilde{\xi}(t)) = \exp tH_p(\widetilde{\rho})$. Now let $t>T_2$.

a) If $|\widetilde{x}(t)|\le R/2$, then $\widetilde{\gamma}(t)\in B(0,R/2)\times\mathbb{R}^n$.

b) Assume now $|\widetilde{x}(t)| > R/2$. Denote by T_3 ($>T_2$) the last time (before t) such that $|\widetilde{x}(T_3)| = 2R_\delta$. Then

$$\begin{aligned} G(\widetilde{\gamma}(t)) >& 2E(1-\delta)(t-T_3)+G(\widetilde{\gamma}(T_3)) \\ >& 2E(1-\delta)(t-T_3)-C, \end{aligned} \tag{5.12}$$

where C depends only on R_δ. On the other hand, the have $|\widetilde{x}(t)| < \sqrt{2E}(1+\delta)(t-T_3)+C$, where the constant C depends only on R_δ. Then,

$$t-T_3 > \frac{|\widetilde{x}(t)|}{\sqrt{2E}(1+\delta)} - \frac{C}{\sqrt{2E}(1+\delta)}, \tag{5.13}$$

$$|\widetilde{\xi}(t)| = \sqrt{2E}+o_{R\to\infty}(1), \tag{5.14}$$

$$\begin{aligned} \cos\big(\widetilde{x},\widetilde{\xi}\big)(t) >& \frac{2E(1-\delta)|\widetilde{x}(t)|}{|\widetilde{x}(t)|(\sqrt{2E}(1+\delta))(\sqrt{2E}+o_{R\to\infty}(1))} + \mathcal{O}(R^{-1}) \\ >& \frac{1-\delta}{1+\delta}+o_{R\to\infty}(1) > 1-2\delta+o_{R\to\infty}(1). \end{aligned} \tag{5.15}$$

So, if R is large enough, $\widetilde{\gamma}(t)\in\Gamma_+(R/2,d,\sigma_+)$.

Then a) and b) imply that, for all $\widetilde{\rho}\in\mathcal{V}$ and $t>T:=T_2$, we have

$$\widetilde{\gamma}(t)\in\Gamma_+(R/2,d,\sigma_+)\cup(B(0,R/2)\times\mathbb{R}^n). \tag{5.16}$$

The lemma follows from (5.11), (5.16) and a compactness argument. □

Recall that the microsupport of $g_-(x)e^{i\psi_-(x)/h}\in C_0^\infty(\mathbb{R}^n)$ is contained in $\Gamma_-(R_-,d_1,\sigma_1)$. Let $\omega_-(x,\xi)\in A_0$ with $\omega_-=1$ near $\Gamma_-(R_-/2,d_1,\sigma_1)$ and $\operatorname{supp}(\omega_-)\subset\Gamma_-(R_-/3,d_0,\sigma_0)$. Using the identity

$$u_- = \frac{i}{h}\int_0^T e^{-it(P-E)/h}(g_-e^{i\psi_-/h})dt + \mathcal{R}(E+i0,h)e^{-iT(P-E)/h}(g_-e^{i\psi_-/h}), \tag{5.17}$$

and Proposition 4.1, Proposition 4.2 and Lemma 5.1, we get

$$\operatorname{Op}_h(\omega_-)u_- = \operatorname{Op}_h(\omega_-)\frac{i}{h}\int_0^T e^{-it(P-E)/h}(g_-e^{i\psi_-/h})dt + \mathcal{O}(h^\infty), \tag{5.18}$$

for some $T > 0$ large enough. In particular,

$$\mathrm{MS}(\mathrm{Op}_h(\omega_-)u_-) \subset \Lambda_\omega^- \cap (B(0, R_- + 1) \times \mathbb{R}^n). \tag{5.19}$$

5.2. Computation of u_- along γ_k^-.

Here we compute u_- microlocally along a trajectory γ_k^-. We recall that γ_k^- is a bicharacteristic curve $(x_k^-(t), \xi_k^-(t))$ such that $(x_k^-(t), \xi_k^-(t)) \to (0, 0)$ as $t \to +\infty$, and such that, as $t \to -\infty$,

$$\begin{aligned} &|x_k^-(t) - \sqrt{2E_0}\omega t - z_k^-| \to 0, \\ &|\xi_k^-(t) - \sqrt{2E_0}\omega| \to 0. \end{aligned} \tag{5.20}$$

The symbol a_- solves (4.7) and (4.8) near $\gamma_k^- \cap \mathrm{MS}(g_- e^{i\psi_-/h})$. In particular, if R_- is large enough, microlocally near $\gamma_k^- \cap \Gamma_-(R_-/2, d_1, \sigma_1) \cap (B(0, R_-) \times \mathbb{R}^n)$, u_- is given by (5.18) and

$$\begin{aligned} u_- =& \frac{i}{h} \int_0^T e^{-it(P-E)/h} ([\chi_-, P] a_- e^{i\psi_-/h}) dt + \mathcal{O}(h^\infty) \\ =& \frac{i}{h} \int_0^T e^{-it(P-E)/h} (\chi_-(P - E) a_- e^{i\psi_-/h}) dt \\ &- \frac{i}{h} \int_0^T e^{-it(P-E)/h} ((P - E)\chi_- a_- e^{i\psi_-/h}) dt + \mathcal{O}(h^\infty) \\ =& - \frac{i}{h} \int_0^T (P - E) e^{-it(P-E)/h} (\chi_- a_- e^{i\psi_-/h}) dt + \mathcal{O}(h^\infty) \\ =& - (P - E)\mathcal{R}(E + i0, h) a_- e^{i\psi_-/h} + \mathcal{O}(h^\infty) \\ =& - a_- e^{i\psi_-/h} + \mathcal{O}(h^\infty). \end{aligned} \tag{5.21}$$

Now, using (5.21), and the fact that u_- is a semiclassical distribution satisfying

$$(P - E)u_- = 0, \tag{5.22}$$

near $B(0, R_-)$, we can compute u_- microlocally near $\gamma_k^- \cap B(0, R_-)$ using Maslov's theory (see [25] for more details). Moreover, it is proved in Proposition C.1 (see also [5, Lemma 5.8]) that the Lagrangian manifold Λ_ω^- has a nice projection with respect to x in a neighborhood of γ_k^- close to $(0, 0)$. Then, in such a neighborhood, u_-can be written as

$$u_-(x) = -a_-(x, h) e^{-i\nu_k^- \pi/2} e^{i\psi_-(x)/h}, \tag{5.23}$$

where ν_k^- denotes the Maslov index of γ_k^-. The phase ψ_- satisfies the usual eikonal equation

$$p(x, \nabla\psi_-) = E_0. \tag{5.24}$$

Here, to the contrary of (4.21), we have written $E = E_0 + E_1 h$ with $E_1 = \mathcal{O}(1)$, and we choose to work with E_1 in the amplitudes instead of the phases. As usual, we have

$$\partial_t(\psi_-(x_k^-(t))) = \nabla\psi_-(x_k^-(t)) \cdot \partial_t x_k^-(t) = \nabla\psi_-(x_k^-(t)) \cdot \xi_k^-(t) = |\xi_k^-(t)|^2, \tag{5.25}$$

so that

$$\psi_-(x_k^-(t)) = \psi_-(x_k^-(s)) + \int_s^t |\xi_k^-(u)|^2 du \tag{5.26}$$

We also have $\psi_-(x_k^-(s)) = (\sqrt{2E_0}\omega s + z_k^-)\cdot\sqrt{2E_0}\omega + o(1)$ as $s\to-\infty$, and then

$$\psi_-(x_k^-(t)) = 2E_0 s + \int_s^t |\xi_k^-(u)|^2 du + o(1), \quad s\to-\infty. \tag{5.27}$$

We have obtained in particular that

$$\psi_-(x_k^-(t)) = \int_{-\infty}^t |\xi_k^-(u)|^2 - 2E_0 1_{u<0}\, du = \int_{-\infty}^t \frac{1}{2}|\xi_k^-(u)|^2 - V(x_k^-(u)) + E_0 \operatorname{sgn}(u)\, du. \tag{5.28}$$

We turn to the computation of the symbol. The function $a_-(x,h)\sim\sum_{k=0}^\infty a_{-,k}(x)h^k$ satisfies the usual transport equations:

$$\begin{cases} \nabla\psi_-\cdot\nabla a_{-,0} + \frac{1}{2}(\Delta\psi_- - 2iE_1)a_{-,0} = 0, \\ \nabla\psi_-\cdot\nabla a_{-,k} + \frac{1}{2}(\Delta\psi_- - 2iE_1)a_{-,k} = i\frac{1}{2}\Delta a_{-,k-1}, \quad k\geq 1, \end{cases} \tag{5.29}$$

In particular, we get for the principal symbol

$$\partial_t(a_{-,0}(x_k^-(t))) = \nabla a_{-,0}(x_k^-(t))\cdot\xi_k^-(t) = \nabla a_{-,0}(x_k^-(t))\cdot\nabla\psi_-(x_k^-(t)), \tag{5.30}$$

so that,

$$\partial_t(a_{-,0}(x_k^-(t))) = -\frac{1}{2}\big(\Delta\psi_-(x_k^-(t)) - 2iE_1\big)a_{-,0}(x_k^-(t)) \tag{5.31}$$

and then

$$a_{-,0}(x_k^-(t)) = a_{-,0}(x_k^-(s))\exp\left(-\frac{1}{2}\int_s^t \Delta\psi_-(x_-(u))\,du + i(t-s)E_1\right). \tag{5.32}$$

On the other hand, from [32, Lemma 4.3], based on Maslov theory, we have

$$a_{-,0}(x_k^-(t)) = (2E_0)^{1/4}D_k^-(t)^{-1/2}e^{itE_1}, \tag{5.33}$$

where

$$D_k^-(t) = \Big|\det\frac{\partial x_-(t,z,\omega,E_0)}{\partial(t,z)}|_{z=z_k^-}\Big|. \tag{5.34}$$

6. Computation of u_- at the critical point

In this section we use the results of [5] to obtain a representation of u_- in a whole neighborhood of the critical point. Indeed we saw already that $(P-E)u_-=0$ outside the support of g_-, in particular in a neighborhood of the critical point. First, we need to recall some terminology from [20] and [5].

We recall from Section 2 that $(\mu_j)_{j\geq 0}$ is the strictly increasing sequence of linear combinations over $\mathbb{N}$ of the λ_j's, with $\mu_0 = 0$. Let $u(t,x)$ be a function defined on $[0,+\infty[\times U$, $U\subset\mathbb{R}^m$.

Definition 6.1. *We say that $u : [0,+\infty[\times U\to\mathbb{R}$, a smooth function, is expandible, if, for any $N\in\mathbb{N}$, $\varepsilon>0$, $(\alpha,\beta)\in\mathbb{N}^{1+m}$,*

$$\partial_t^\alpha\partial_x^\beta\Big(u(t,x) - \sum_{j=1}^N u_j(t,x)e^{-\mu_j t}\Big) = \mathcal{O}\big(e^{-(\mu_{N+1}-\varepsilon)t}\big), \tag{6.1}$$

for a sequence $(u_j)_j$ *of smooth functions, which are polynomials in* t*. We shall write*

$$u(t,x) \sim \sum_{j\geq 1} u_j(t,x) e^{-\mu_j t},$$

when (6.1) holds.

We say that $f(t,x) = \widetilde{\mathcal{O}}(e^{-\mu t})$ if for all $(\alpha,\beta) \in \mathbb{N}^{1+m}$ and $\varepsilon > 0$ we have

$$\partial_t^\alpha \partial_x^\beta f(t,x) = \mathcal{O}(e^{-(\mu-\varepsilon)t}). \tag{6.2}$$

Definition 6.2. *We say that* $u(t,x,h)$*, a smooth function, is of class* $\mathcal{S}^{A,B}$ *if, for any* $\varepsilon > 0$*,* $(\alpha,\beta) \in \mathbb{N}^{1+m}$*,*

$$\partial_t^\alpha \partial_x^\beta u(t,x,h) = \mathcal{O}\big(h^A e^{-(B-\varepsilon)t}\big). \tag{6.3}$$

Let $\mathcal{S}^{\infty,B} = \bigcap_{A\in\mathbb{R}} \mathcal{S}^{A,B}$*. We shall say that* $u(t,x,h)$ *is a classical expandible function of order* (A,B)*, if, for any* $K \in \mathbb{N}$*,*

$$u(t,x,h) - \sum_{k=A}^{K} u_k(t,x) h^k \in \mathcal{S}^{K+1,B}, \tag{6.4}$$

for a sequence $(u_k)_k$ *of expandible functions. We shall write*

$$u(t,x,h) \sim \sum_{k\geq A} u_k(t,x) h^k,$$

in that case.

Now, since the intersection between Λ_ω^- and Λ_- is transverse along the trajectories $\gamma_k^-(z_k^-)$, and since $g_1^-(z_k^-) \neq 0$, Theorem 2.1 and Theorem 5.4 of [5] imply that one can write, microlocally near $(0,0)$,

$$u_- = \frac{1}{\sqrt{2\pi h}} \int \sum_{k=1}^{N_-} \alpha^k(t,x,h) e^{i\varphi^k(t,x)/h} dt, \tag{6.5}$$

where the $\alpha^k(t,x,h)$'s are classical expandible functions in $\mathcal{S}^{0,2\operatorname{Re}\Sigma(E)}$:

$$\begin{aligned} \alpha^k(t,x,h) &\sim \sum_{m\geq 0} \alpha_m^k(t,x) h^m, \\ \alpha_m^k(t,x) &\sim \sum_{j\geq 0} \alpha_{m,j}^k(t,x) e^{-2(\Sigma(E)+\mu_j)t}, \end{aligned} \tag{6.6}$$

and where the $\alpha_{m,j}^k(t,x)$'s are polynomial with respect to t. We recall from (2.28) that, for $E = E_0 + hE_1$,

$$\Sigma(E) = \sum_{j=1}^{n} \frac{\lambda_j}{2} - iE_1. \tag{6.7}$$

Following line by line [5, Section 6], we obtain (see [5, (6.26)])

$$\alpha_{0,0}^k(0) = -e^{i\pi/4}(2\lambda_1)^{3/2} e^{-i\nu_k^- \pi/2} |g(\gamma_k^-)| (D_k^-)^{-1/2} (2E_0)^{1/4}. \tag{6.8}$$

Notice that from (5.32) and Proposition C.1, we have $0 < D_k^- < +\infty$.

From [5, Section 5], we recall that the phases $\varphi^k(t,x)$ satisfy the eikonal equation

$$\partial_t \varphi^k + p(x, \nabla_x \varphi^k) = E_0, \tag{6.9}$$

and that they have the asymptotic expansions

$$\varphi^k(t,x) \sim \sum_{j=0}^{+\infty} \sum_{m=0}^{M_j^k} \varphi_{j,m}^k(x) t^m e^{-\mu_j t}, \tag{6.10}$$

with $M_j^k < +\infty$. In the following, we set

$$\varphi_j^k(t,x) = \sum_{m=0}^{M_j^k} \varphi_{j,m}^k(x) t^m, \tag{6.11}$$

and, still from [5, Section 5], we have that the first φ_j^k's are of the form

$$\varphi_0^k(t,x) = \varphi_+(x) + c_k \tag{6.12}$$

$$\varphi_1^k(t,x) = -2\lambda_1 g_1^-(z_k^-) \cdot x + \mathcal{O}(x^2), \tag{6.13}$$

where $c_k \in \mathbb{R}$ is the constant depending on k given by

$$c_k = \text{“}\psi_-(0)\text{”} = \lim_{t \to +\infty} \psi_-(x_k^-(t)) = S_k^-, \tag{6.14}$$

thanks to (5.28) (see also [5, Lemma 5.10]). Moreover φ_+ is the generating function of the outgoing stable Lagrangian submanifold Λ_+ with $\varphi_+(0) = 0$, and

$$\varphi_+(x) = \sum_j \frac{\lambda_j}{2} x_j^2 + \mathcal{O}(x^3). \tag{6.15}$$

The fact that $\varphi_1^k(t,x)$ does not depend on t and the expression (6.13) follows also from Corollary 6.6 and (6.109).

6.1. **Study of the transport equations for the phases.**

Now, we examine the equations satisfied by the functions $\varphi_j^k(t,x)$, defined in (6.10) and (6.11), for the integers $j \leq \widehat{\jmath}$ (recall that $\widehat{\jmath}$ is defined by $\mu_{\widehat{\jmath}} = 2\lambda_1$). For clearer notation, we omit the superscript k until further notice.

Recall that $\varphi(t,x)$ satisfies the eikonal equation (6.9), which implies (see (6.10))

$$\sum_j \sum_{m=0}^{M_j} e^{-\mu_j t} \varphi_{j,m}(x)(-\mu_j t^m + m t^{m-1}) + \frac{1}{2}\Big(\sum_j \sum_{m=0}^{M_j} \nabla \varphi_{j,m}(x) t^m e^{-\mu_j t}\Big)^2 + V(x) \sim E_0, \tag{6.16}$$

and then

$$\sum_j \sum_{m=0}^{M_j} e^{-\mu_j t} \varphi_{j,m}(x)(-\mu_j t^m + m t^{m-1}) + \frac{1}{2} \sum_{j,\widetilde{\jmath}} \sum_{m=0}^{M_j} \sum_{\widetilde{m}=0}^{M_{\widetilde{\jmath}}} \nabla \varphi_{j,m} \nabla \varphi_{\widetilde{\jmath},\widetilde{m}}(x) e^{-(\mu_j + \mu_{\widetilde{\jmath}})t} t^{m+\widetilde{m}} + V(x) \sim E_0. \tag{6.17}$$

When $\mu_j < 2\lambda_1$, the cross product in the previous formula provides a term of the form $e^{-\mu_j t}$ if and only if $\mu_j = 0$ or $\mu_{\widetilde{\jmath}} = 0$. In particular, the term of order $e^{-\mu_j t}$ in (6.17) gives

$$\sum_{m=0}^{M_j} \varphi_{j,m}(x)(-\mu_j t^m + m t^{m-1}) + \nabla\varphi_+(x) \cdot \sum_{m=0}^{M_j} \nabla\varphi_{j,m}(x) t^m = 0. \tag{6.18}$$

When $\mu_j = 2\lambda_1$, one gets also a term of order $e^{-2\lambda_1 t}$ for $\mu_j = \mu_{\widetilde{\jmath}} = \lambda_1$ and then

$$\begin{aligned}\sum_{m=0}^{M_j} \varphi_{j,m}(x)(-\mu_j t^m + m t^{m-1}) + \nabla\varphi_+(x) \cdot \sum_{m=0}^{M_j} \nabla\varphi_{j,m}(x) t^m \\ + \frac{1}{2} \sum_{m=0}^{M_1} \sum_{\widetilde{m}=0}^{M_1} t^{m+\widetilde{m}} \nabla\varphi_{1,m}(x) \nabla\varphi_{1,\widetilde{m}}(x) = 0.\end{aligned} \tag{6.19}$$

To study these equations, we denote by

$$L = \nabla\varphi_+(x) \cdot \nabla \tag{6.20}$$

the vector field that appears in (6.18) and (6.19). We set also $L_0 = \sum_j \lambda_j x_j \partial_j$ its linear part at $x = 0$, and we begin with the study of the solution of

$$(L - \mu) f = g, \tag{6.21}$$

with $\mu \in \mathbb{R}$ and $f,\, g \in C^\infty(\mathbb{R}^n)$. First of all, we show that it is sufficient to solve (6.21) for formal series.

Proposition 6.3. *Let $g \in C^\infty(\mathbb{R}^n)$ and g_0 be the Taylor series of g at 0. For each formal series f_0 such that $(L - \mu) f_0 = g_0$, there exists a unique function $f \in C^\infty(\mathbb{R}^n)$ defined near 0 such that f has Taylor series f_0 at 0 and*

$$(L - \mu) f = g, \tag{6.22}$$

near 0.

Proof. Let $\widetilde{f}_0$ be a C^∞ function having f_0 has Taylor series at 0. With the notation $f = \widetilde{f}_0 + r$, the problem (6.22) is equivalent to finding $r = \mathcal{O}(x^\infty)$ with

$$(L - \mu) r = g - (L - \mu) \widetilde{f}_0 = \widetilde{r}, \tag{6.23}$$

where $\widetilde{r} \in C^\infty$ has $g_0 - (L - \mu) f_0 = 0$ as Taylor series at 0. Let $y(t, x)$ be the solution of

$$\begin{cases} \partial_t y(t, x) = \nabla\varphi_+(y(t, x)), \\ y(0, x) = x. \end{cases} \tag{6.24}$$

Thus, (6.23) is equivalent to

$$r(x) = \int_t^0 e^{-\mu s} \widetilde{r}(y(s, x)) ds + e^{-\mu t} r(y(t, x)). \tag{6.25}$$

Since $r(x)$, $\widetilde{r}(x) = \mathcal{O}(x^\infty)$ and $y(s, x) = \mathcal{O}(e^{\lambda_1 t}|x|)$ for $t < 0$, the functions $e^{-\mu t} r(y(t, x))$, $e^{-\mu t} \widetilde{r}(y(t, x))$ are $\mathcal{O}(e^{Nt})$ as $t \to -\infty$ for all $N > 0$. Then

$$r(x) = \int_{-\infty}^0 e^{-\mu s} \widetilde{r}(y(s, x)) ds, \tag{6.26}$$

and $r(x) = \mathcal{O}(x^\infty)$. The uniqueness follows and it is enough to prove that r given by (6.26) is C^∞. We have

$$\partial_t(\nabla_x y) = (\nabla_x^2 \varphi_+(y))(\nabla_x y), \tag{6.27}$$

and since $\nabla_x^2 \varphi_+$ is bounded, there exists $C > 0$ such that

$$|\nabla_x y(t,x)| \lesssim e^{-Ct}, \tag{6.28}$$

has $t \to -\infty$. Then, $e^{-\mu s}(\nabla \widetilde{r})(y(s,x))(\partial_j y(t,x)) = \mathcal{O}(e^{Nt})$ as $t \to -\infty$ for all $N > 0$ and $\partial_j r(x) = \int_{-\infty}^0 e^{-\mu s}(\nabla \widetilde{r})(y(s,x))(\partial_j y(t,x)) ds$. The derivatives of order greater than 1 can be treated in the same way. □

We let

$$L_\mu = L - \mu : \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!], \tag{6.29}$$

where we use the standard notation $\mathbb{C}[\![x]\!]$ for formal series, and $\mathbb{C}_p[\![x]\!]$ for formal series of degree $\geq p$. We notice that

$$L_\mu x^\alpha = (L_0 - \mu)x^\alpha + \mathbb{C}_{|\alpha|+1}[\![x]\!] = (\lambda \cdot \alpha - \mu)x^\alpha + \mathbb{C}_{|\alpha|+1}[\![x]\!]. \tag{6.30}$$

Recall that $\mathcal{I}_\ell(\mu)$ has been defined in (2.22). The number of elements in $\mathcal{I}_\ell(\mu)$ will be denoted

$$n_\ell(\mu) = \# \mathcal{I}_\ell(\mu). \tag{6.31}$$

One has for example $n_2(\mu) = \frac{n_1(\mu)(n_1(\mu)+1)}{2}$.

Proposition 6.4. *Suppose $\mu \in]0, 2\lambda_1[$. With the above notations, one has $\operatorname{Ker} L_\mu \oplus \operatorname{Im} L_\mu = \mathbb{C}[\![x]\!]$. More precisely:*

i) The kernel of L_μ has dimension $n_1(\mu)$, and one can find a basis $(E_{j_1}, \ldots, E_{j_{n_1(\mu)}})$ of $\operatorname{Ker} L_\mu$ such that $E_j(x) = x_j + \mathbb{C}_2[\![x]\!]$, $j \in \mathcal{I}_1(\mu)$.

ii) A formal series $F = F_0 + \sum_{j=1}^n F_j x_j + \mathbb{C}_2[\![x]\!]$ belongs to $\operatorname{Im} L_\mu$ if and only if $F_j = 0$ for all $j \in \mathcal{I}_1(\mu)$.

Remark 6.5. *Thanks to Propostion 6.3, the same result is true for germs of C^∞ functions at 0. Notice that when $\mu \neq \mu_j$ for all j, L_μ is invertible.*

Proof. For a given $F = \sum_\alpha F_\alpha x^\alpha \in \mathbb{C}[\![x]\!]$, we look for solutions $E = \sum_\alpha E_\alpha x^\alpha \in \mathbb{C}[\![x]\!]$ to the equation

$$L_\mu \Big(\sum_\alpha E_\alpha x^\alpha \Big) = \sum_\alpha F_\alpha x^\alpha. \tag{6.32}$$

The calculus of the term of order x^0 in (6.32) leads to the equation

$$E_0 = -\frac{F_0}{\mu}. \tag{6.33}$$

With this value for E_0, (6.32) becomes, using again (6.30),

$$\sum_{|\alpha|=1} (\lambda \cdot \alpha - \mu) E_\alpha x^\alpha = \sum_{|\alpha|=1} F_\alpha x^\alpha + \mathbb{C}_2[\![x]\!]. \tag{6.34}$$

We have two cases:

If $\alpha \notin \mathcal{I}_1(\mu)$, one should have

$$E_\alpha = \frac{F_\alpha}{\lambda \cdot \alpha - \mu}. \tag{6.35}$$

If $\alpha \in \mathcal{I}_1(\mu)$, the formula (6.34) gives $F_\alpha = 0$. In that case, the corresponding E_α can be chosen arbitrarily.

Now suppose that the E_α are fixed for all $|\alpha| \leq p-1$ (with $p \geq 2$), and such that

$$L_\mu\Big(\sum_{|\alpha| \leq p-1} E_\alpha x^\alpha \Big) = \sum_\alpha F_\alpha x^\alpha + \mathbb{C}_p[\![x]\!]. \tag{6.36}$$

We can write (6.32) as

$$L_\mu\Big(\sum_{|\alpha| = p} E_\alpha x^\alpha \Big) = \sum_\alpha F_\alpha x^\alpha - L_\mu\Big(\sum_{|\alpha| \leq p-1} E_\alpha x^\alpha \Big) + \mathbb{C}_{p+1}[\![x]\!], \tag{6.37}$$

or, using again (6.30),

$$\sum_{|\alpha| = p} (\lambda \cdot \alpha - \mu) E_\alpha x^\alpha = \sum_{|\alpha| \leq p} F_\alpha x^\alpha - L_\mu\Big(\sum_{|\alpha| \leq p-1} E_\alpha x^\alpha \Big) + \mathbb{C}_{p+1}[\![x]\!]. \tag{6.38}$$

Since $|\alpha| \geq 2$, one has $\lambda \cdot \alpha \geq 2\lambda_1 > \mu$, so that (6.38) determines by induction all the E_α's for $|\alpha| = p$ in a unique way. □

Corollary 6.6. *If $j < \widehat{\jmath}$, the function $\varphi_j(t,x)$ does not depend on t, i.e. we have $M_j = 0$.*

Proof. Suppose that $M_j \geq 1$, then (6.18) gives the system

$$\begin{cases} (L - \mu_j)\varphi_{j,M_j} = 0, \\ (L - \mu_j)\varphi_{j,M_j - 1} = -M_j \varphi_{j,M_j}, \end{cases} \tag{6.39}$$

with $\varphi_{j,M_j} \neq 0$. But this would imply that $\varphi_{j,M_j} \in \operatorname{Ker} L_\mu \cap \operatorname{Im} L_\mu$, a contradiction. □

As a consequence, for $j < \widehat{\jmath}$, the equation (6.18) on φ_j reduces to

$$(L - \mu_j)\varphi_{j,0} = 0, \tag{6.40}$$

and, from Proposition 6.4, we get that

$$\varphi_j(t,x) = \varphi_{j,0}(x) = \sum_{k \in \mathcal{I}_1(\mu_j)} d_{j,k} x_k + \mathcal{O}(x^2). \tag{6.41}$$

We now consider the case $j = \widehat{\jmath}$, and we study (6.19). We have already seen that φ_1 does not depend on t, so that this equation can be written

$$\sum_{m=0}^{M_j} \varphi_{j,m}(x)(-\mu_j t^m + m t^{m-1}) + \nabla\varphi_+(x) \cdot \sum_{m=0}^{M_j} \nabla\varphi_{j,m}(x) t^m + \frac{1}{2}\big|\nabla\varphi_1(x)\big|^2 = 0. \tag{6.42}$$

As for the study of (6.18), we begin with that of (6.21), with $\mu = 2\lambda_1$. We denote by $\Psi : \mathbb{R}^{n_1(2\lambda_1)} \longrightarrow \mathbb{R}^{n_2(\lambda_1)}$ the linear map

$$\Psi(E_{\beta_1}, \dots, E_{\beta_{n_1(2\lambda_1)}}) = \Big(\sum_{\beta \in \mathcal{I}_1(2\lambda_1)} E_\beta \frac{1}{\alpha!} \big(\partial^\alpha (L - \mu) x^\beta \big) |_{x=0} \Big)_{\alpha \in \mathcal{I}_2(\lambda_1)}, \tag{6.43}$$

and we set

$$n(\Psi) = \dim \operatorname{Ker} \Psi. \tag{6.44}$$

Recalling that $L = \nabla \varphi_+(x) \cdot \nabla$, we see that

$$\Psi(E_{\beta_1}, \dots, E_{\beta_{n_1(2\lambda_1)}}) = \Big(\sum_{\beta \in \mathcal{I}_1(2\lambda_1)} E_\beta \frac{\partial^\alpha \partial^\beta \varphi_+(0)}{\alpha!} \Big)_{\alpha \in \mathcal{I}_2(\lambda_1)}. \tag{6.45}$$

More generally, for any $|\alpha| = 2$, we denote

$$\Psi_\alpha((E_\beta)_{\beta \in \mathcal{I}_1(2\lambda_1)}) = \sum_{\beta \in \mathcal{I}_1(2\lambda_1)} E_\beta \frac{\partial^\alpha \partial^\beta \varphi_+(0)}{\alpha!}. \tag{6.46}$$

Then, at the level of formal series, we have the

Proposition 6.7. *Suppose $\mu = 2\lambda_1$. Then*

i) $\operatorname{Ker} L_\mu$ *has dimension* $n_2(\lambda_1) + n(\Psi)$.
ii) *A formal series* $F = \sum_\alpha F_\alpha x^\alpha$ *belongs to* $\operatorname{Im} L_\mu$ *if and only if*

$$\forall \alpha \in \mathcal{I}_1(2\lambda_1), \quad F_\alpha = 0, \tag{6.47}$$

$$\Big(\sum_{\substack{|\beta|=1 \\ \beta \notin \mathcal{I}_1(2\lambda_1)}} \frac{\partial^\beta \partial^\alpha \varphi_+(0)}{\alpha!} \frac{F_\beta}{2\lambda_1 - \lambda \cdot \beta} + F_\alpha \Big)_{\alpha \in \mathcal{I}_2(\lambda_1)} \in \operatorname{Im} \Psi. \tag{6.48}$$

iii) *If* $F \in \operatorname{Im} L_\mu$, *any formal series* $E = \sum_\alpha E_\alpha x^\alpha$ *with* $L_\mu E = F$ *satisfies*

$$E_0 = \frac{1}{-2\lambda_1} F_0, \tag{6.49}$$

$$E_\alpha = \frac{1}{\lambda \cdot \alpha - 2\lambda_1} F_\alpha, \qquad \text{for } \alpha \in \mathcal{I}_1 \setminus \mathcal{I}_1(2\lambda_1), \tag{6.50}$$

$$\Psi\big((E_\beta)_{\beta \in \mathcal{I}_1(2\lambda_1)}\big) = \Big(\sum_{\substack{|\beta|=1 \\ \beta \notin \mathcal{I}_1(2\lambda_1)}} \frac{\partial^\beta \partial^\alpha \varphi_+(0)}{\alpha!} \frac{F_\beta}{2\lambda_1 - \lambda \cdot \beta} + F_\alpha \Big)_{\alpha \in \mathcal{I}_2(\lambda_1)}. \tag{6.51}$$

Moreover for $\alpha \in \mathcal{I}_2 \setminus \mathcal{I}_2(\lambda_1)$, *one has*

$$E_\alpha = \frac{1}{\lambda \cdot \alpha - 2\lambda_1} \Big(F_\alpha - \Psi_\alpha((E_\beta)_{\beta \in \mathcal{I}_1(2\lambda_1)}) + \sum_{\substack{|\beta|=1 \\ \beta \notin \mathcal{I}_1(2\lambda_1)}} \frac{F_\beta}{2\lambda_1 - \lambda \cdot \beta} \frac{\partial^{\alpha+\beta} \varphi_+(0)}{\alpha!} \Big). \tag{6.52}$$

Lastly, E *is completely determined by* F *and a choice of the* E_α *for* $|\alpha| \leq 2$ *such that* (6.49)– (6.52) *are satisfied.*
iv) $\operatorname{Ker} L_\mu \cap \operatorname{Im}(L_\mu)^2 = \{0\}$.

Proof. For a given $F = \sum_\alpha F_\alpha x^\alpha$ we look for a $E = \sum_\alpha E_\alpha x^\alpha$ such that $L_{2\lambda_1} E = F$. First of all, we must have

$$E_0 = -\frac{F_0}{2\lambda_1}. \tag{6.53}$$

When this is true, we get

$$\sum_{|\alpha|=1} E_\alpha (L_0 - 2\lambda_1) x^\alpha = \sum_{|\alpha|=1} F_\alpha (L - 2\lambda_1) x^\alpha + \mathbb{C}_2[\![x]\!], \tag{6.54}$$

and we obtain as necessary condition that $F_\alpha = 0$ for any $\alpha \in \mathcal{I}_1(2\lambda_1)$. So far, the E_α for $\alpha \in \mathcal{I}_1(2\lambda_1)$ can be chosen arbitrarily, and we must have

$$E_\alpha = \frac{F_\alpha}{\lambda \cdot \alpha - 2\lambda_1}, \quad \alpha \in \mathcal{I}_1 \setminus \mathcal{I}_1(2\lambda_1). \tag{6.55}$$

We suppose that (6.53) and (6.55) hold. Then we have

$$\sum_{|\alpha|=2} E_\alpha (L_0 - 2\lambda_1) x^\alpha = \sum_{|\alpha|=2} F_\alpha x^\alpha + \Big(\sum_{\substack{|\alpha|=1 \\ \alpha \notin \mathcal{I}_1(2\lambda_1)}} F_\alpha x^\alpha - \sum_{|\alpha|=1} E_\alpha (L - 2\lambda_1) x^\alpha \Big) + \mathbb{C}_3[\![x]\!]. \tag{6.56}$$

Notice that the second term in the R.H.S of (6.56) belongs to $\mathbb{C}_2[\![x]\!]$ thanks to (6.55). Again, we have two cases:

- When $\alpha \in \mathcal{I}_2(\lambda_1)$, the corresponding E_α can be chosen arbitrarily, but one must have

$$F_\alpha = \sum_{|\beta|=1} E_\beta \big(\frac{1}{\alpha!} \partial^\alpha (L - 2\lambda_1) x^\beta \big) \vert_{x=0} \tag{6.57}$$

$$= \Psi_\alpha \big((E_\beta)_{\beta \in \mathcal{I}_1(2\lambda_1)} \big) + \sum_{\substack{|\beta|=1 \\ \beta \notin \mathcal{I}_1(2\lambda_1)}} E_\beta \frac{\partial^{\alpha+\beta} \varphi_+(0)}{\alpha!}, \tag{6.58}$$

  and this, with (6.55), gives (6.51).
- When $|\alpha| = 2$, $\alpha \notin \mathcal{I}_2(\lambda_1)$, one obtains

$$\begin{aligned} E_\alpha =& \frac{1}{\lambda \cdot \alpha - 2\lambda_1} \Big(F_\alpha - \sum_{|\beta|=1} E_\beta \big(\frac{1}{\alpha!} \partial^\alpha (L - 2\lambda_1) x^\beta \big) \vert_{x=0} \Big) \\ =& \frac{1}{\lambda \cdot \alpha - 2\lambda_1} \Big(F_\alpha - \Psi_\alpha \big((E_\beta)_{\beta \in \mathcal{I}_1(2\lambda_1)} \big) - \sum_{\substack{|\beta|=1 \\ \beta \notin \mathcal{I}_1(2\lambda_1)}} E_\beta \frac{\partial^{\alpha+\beta} \varphi_+(0)}{\alpha!} \Big), \end{aligned} \tag{6.59}$$

  and this, with (6.55), gives (6.52).

Now suppose that (6.53), (6.55), (6.57) and (6.59) hold, and that we have chosen a value for the free variables E_α for $\alpha \in \mathcal{I}_1(2\lambda_1) \cup \mathcal{I}_2(\lambda_1)$. Thanks to the fact that $\lambda \cdot \alpha \neq 2\lambda_1$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = 3$, we see as in the proof of Propostion 6.4, that the equation (6.54) has a unique solution, and the points (i), (ii) and (iii) follow easily.

To prove the last point of the proposition, suppose that

$$E = \sum_{\alpha \in \mathbb{N}^n} E_\alpha x^\alpha \in \operatorname{Ker} L_\mu \cap \operatorname{Im}(L_\mu)^2. \tag{6.60}$$

First, we have $E \in \operatorname{Ker} L_\mu \cap \operatorname{Im} L_\mu$. Thus, $E_0 = 0$ by (6.49), $E_\alpha = 0$ for $\alpha \in \mathcal{I}_1(2\lambda_1)$ by (6.47), and $E_\alpha = 0$ for $\alpha \in \mathcal{I}_1 \setminus \mathcal{I}_1(2\lambda_1)$ by (6.50). Last, since $L_\mu E = 0$, we also have $E_\alpha = 0$ for $\alpha \in \mathcal{I}_2 \setminus \mathcal{I}_2(\lambda_1)$, and finally,

$$E = \sum_{\alpha\in\mathcal{I}_2(\lambda_1)} E_\alpha x^\alpha + \mathbb{C}_3[\![x]\!]. \tag{6.61}$$

Moreover, one can write $E = L_\mu G$ for some $G \in \operatorname{Im} L_\mu$. Since $E_0 = 0$, we must have $G_0 = 0$. Since $G \in \operatorname{Im} L_\mu$, by (6.47), we have $G_\alpha = 0$ for $\alpha \in \mathcal{I}_1(2\lambda_1)$. Finally, since $E_\alpha = 0$ for $|\alpha| = 1, \alpha \notin \mathcal{I}_1(2\lambda_1)$, the same is true for the corresponding G_α, and

$$G = \sum_{|\alpha|\geq 2} G_\alpha x^\alpha. \tag{6.62}$$

Then, since $L_\mu x^\alpha = 0 + \mathbb{C}_3[x]$ for $\alpha \in \mathcal{I}_2(\lambda_1)$, we obtain $E_\alpha = 0$ for $\alpha \in \mathcal{I}_2(\lambda_1)$. As above, we then get that, for $|\alpha| \geq 3$, $E_\alpha = 0$, and this ends the proof. □

Corollary 6.8. *We have $M_{\widehat{\jmath}} \leq 2$. If, in addition, $\lambda_k \neq 2\lambda_1$ for all $k \in \{1, \dots, n\}$, then $M_{\widehat{\jmath}} \leq 1$.*

Proof. Suppose that $M_{\widehat{\jmath}} \geq 3$. Then (6.42) gives

$$(L - \mu_{\widehat{\jmath}})\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} = 0 \tag{6.63}$$

$$(L - \mu_{\widehat{\jmath}})\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}-1} = -M_{\widehat{\jmath}}\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} \tag{6.64}$$

$$(L - \mu_{\widehat{\jmath}})\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}-2} = -(M_{\widehat{\jmath}} - 1)\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}-1}, \tag{6.65}$$

with $\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} \neq 0$. Notice that we have used the fact that $M_{\widehat{\jmath}} - 2 > 0$ in (6.65). But this gives $\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} \in \operatorname{Ker}(L - \mu_{\widehat{\jmath}})$ and $(L - \mu_{\widehat{\jmath}})^2\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}-2} = M_{\widehat{\jmath}}(M_{\widehat{\jmath}} - 1)\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}}$, so that $\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} \in \operatorname{Im}(L - \mu_{\widehat{\jmath}})^2$. This contradicts point (iv) of Proposition 6.7.

Now we suppose that $\lambda_k \neq 2\lambda_1$ for all $k \in \{1, \dots n\}$, that is $\mathcal{I}_1(2\lambda_1) = \emptyset$, and that $M_{\widehat{\jmath}} = 2$. Then (6.42) gives

$$(L - \mu_{\widehat{\jmath}})\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} = 0 \tag{6.66}$$

$$(L - \mu_{\widehat{\jmath}})\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}-1} = -M_{\widehat{\jmath}}\,\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} \tag{6.67}$$

with $\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} \neq 0$. Therefore we have $\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} \in \operatorname{Ker} L_{\mu_{\widehat{\jmath}}} \cap \operatorname{Im} L_{\mu_{\widehat{\jmath}}}$, and we get the same conclusion as in (6.61): $\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}}(x) = \mathcal{O}(x^2)$. Then, we write

$$\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} = (L - \mu_{\widehat{\jmath}})g, \tag{6.68}$$

and we see, as in (6.62), that $g = \mathcal{O}(x^2)$, here because $\mathcal{I}_1(2\lambda_1) = \emptyset$. Finally, we conclude also that $\varphi_{\widehat{\jmath},M_{\widehat{\jmath}}} = 0$, a contradiction. □

6.2. Taylor expansions of φ_+ and φ_1^k.

Now we compute the Taylor expansions of the leading terms with respect to t, of the phase functions $\varphi(t, x) = \varphi^k(t, x)$.

Lemma 6.9. *The smooth function* $\varphi_+(x) = \sum_{j=1}^{n} \frac{\lambda_j}{2} x_j^2 + \mathcal{O}(x^3)$ *satisfies*

$$\partial^\alpha \varphi_+(0) = -\frac{1}{\lambda \cdot \alpha} \partial^\alpha V(0), \tag{6.69}$$

for $|\alpha| = 3$, *and*

$$\partial^\alpha \varphi_+(0) = -\frac{1}{2(\lambda \cdot \alpha)} \sum_{j=1}^{n} \sum_{\substack{\beta,\gamma \in \mathcal{I}_2 \\ \alpha = \beta+\gamma}} \frac{\alpha!}{\beta!\,\gamma!} \frac{\partial_j \partial^\beta V(0)}{\lambda_j + \lambda \cdot \beta} \frac{\partial_j \partial^\gamma V(0)}{\lambda_j + \lambda \cdot \gamma} - \frac{1}{\lambda \cdot \alpha} \partial^\alpha V(0), \tag{6.70}$$

for $|\alpha| = 4$, *where* $\alpha, \beta, \gamma \in \mathbb{N}^n$.

Proof. The smooth function $x \mapsto \varphi_+(x)$ is defined in a neighborhood of 0, and it is characterized (up to a constant: we have chosen $\varphi_+(0) = 0$) by

$$\begin{cases} p(x, \nabla\varphi_+(x)) = \frac{1}{2} |\nabla \varphi_+(x)|^2 + V(x) = E_0 \\ \nabla \varphi_+(x) = (\lambda_j x_j)_{j=1,\dots,n} + \mathcal{O}(x^2) \end{cases} \tag{6.71}$$

The Taylor expansion of φ_+ at $x = 0$ is

$$\varphi_+(x) = \sum_{j=1}^{n} \frac{\lambda_j}{2} x_j^2 + \sum_{|\alpha|=3,4} \frac{\partial^\alpha \varphi_+(0)}{\alpha!} x^\alpha + \mathcal{O}(x^5), \tag{6.72}$$

and we have

$$\partial_j \varphi_+(x) = \lambda_j x_j + \sum_{|\alpha|=3,4} \alpha_j \frac{\partial^\alpha \varphi_+(0)}{\alpha!} x^{\alpha - 1_j} + \mathcal{O}(x^4). \tag{6.73}$$

Therefore

$$\begin{aligned} |\nabla \varphi_+(x)|^2 = & \sum_{j=1}^{n} \lambda_j^2 x_j^2 + 2 \sum_{|\alpha|=3} \Big(\sum_{j=1}^{n} \lambda_j \alpha_j \Big) \frac{\partial^\alpha \varphi_+(0)}{\alpha!} x^\alpha + 2 \sum_{|\alpha|=4} \Big(\sum_{j=1}^{n} \lambda_j \alpha_j \Big) \frac{\partial^\alpha \varphi_+(0)}{\alpha!} x^\alpha \\ & + \sum_{j=1}^{n} \Big(\sum_{|\alpha|=3} \alpha_j \frac{\partial^\alpha \varphi_+(0)}{\alpha!} x^{\alpha - 1_j} \Big)^2 + \mathcal{O}(x^5). \end{aligned} \tag{6.74}$$

Let us compute further the last term in (6.74):

$$\begin{aligned} \sum_{j=1}^{n} \Big(\sum_{|\alpha|=3} \alpha_j \frac{\partial^\alpha \varphi_+(0)}{\alpha!} x^{\alpha - 1_j} \Big)^2 &= \sum_{j=1}^{n} \sum_{|\beta|,|\gamma|=3} \beta_j \gamma_j \frac{\partial^\beta \varphi_+(0)}{\beta!} \frac{\partial^\gamma \varphi_+(0)}{\gamma!} x^{\beta+\gamma-21_j} \\ &= \sum_{j=1}^{n} \sum_{|\alpha|=4} x^\alpha \Big(\sum_{\substack{\alpha=\beta+\gamma \\ |\beta|,|\gamma|=2}} \frac{\partial_j \partial^\beta \varphi_+(0)}{\beta!} \frac{\partial_j \partial^\gamma \varphi_+(0)}{\gamma!} \Big). \end{aligned} \tag{6.75}$$

Writing the Taylor expansion of V at $x = 0$ as

$$V(x) = E_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{2} x_j^2 + \sum_{|\alpha|=3,4} \frac{\partial^\alpha V(0)}{\alpha!} x^\alpha + \mathcal{O}(x^5), \tag{6.76}$$

and using the eikonal equation (6.71), we obtain first, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = 3$,

$$\partial^\alpha \varphi_+(0) = -\frac{1}{\lambda\cdot\alpha}\partial^\alpha V(0). \tag{6.77}$$

Then, (6.74) and (6.75) give

$$\partial^\alpha \varphi_+(0) = -\frac{1}{\lambda\cdot\alpha}\partial^\alpha V(0) - \frac{1}{2(\lambda\cdot\alpha)}\sum_{j=1}^n \sum_{\substack{\beta,\gamma\in\mathcal{I}_2\\ \alpha=\beta+\gamma}} \frac{\alpha!}{\beta!\gamma!}\frac{\partial_j\partial^\beta V(0)}{\lambda_j+\lambda\cdot\beta}\frac{\partial_j\partial^\gamma V(0)}{\lambda_j+\lambda\cdot\gamma}, \tag{6.78}$$

for $|\alpha| = 4$. □

Now we turn to the function φ_1. This function is a solution, in a neighborhood of $x = 0$, of the transport equation

$$L\varphi_1(x) = \lambda_1\varphi_1(x), \tag{6.79}$$

where L is given in (6.20).

Lemma 6.10. *The C^∞ function $\varphi_1(x) = -2\lambda_1 g_1^-(z^-)\cdot x + \mathcal{O}(x^2)$ satisfies*

$$\partial^\alpha\varphi_1(0) = \frac{2\lambda_1\alpha!}{(\lambda_1-\lambda\cdot\alpha)(\lambda_1+\lambda\cdot\alpha)}\sum_{j=1}^n\frac{\partial_j\partial^\alpha V(0)}{\alpha!}\big(g_1^-(z^-)\big)_j, \tag{6.80}$$

for $|\alpha| = 2$, and

$$\begin{aligned}\partial^\alpha\varphi_1(0) = &-\frac{2\lambda_1}{\lambda_1-\lambda\cdot\alpha}\sum_{\substack{1_k\in\mathcal{I}_1(\lambda_1), j\in\mathcal{I}_1\\ \beta,\gamma\in\mathcal{I}_2\\ \alpha+1_j=\beta+\gamma}}\frac{\alpha!\gamma_j}{\beta!\gamma!}\frac{\partial_j\partial^\beta V(0)}{\lambda_j+\lambda\cdot\beta}\frac{\partial_k\partial^\gamma V(0)}{(\lambda_1-\lambda\cdot\gamma)(\lambda_1+\lambda\cdot\gamma)}\big(g_1^-(z^-)\big)_k\\ &+\frac{\lambda_1}{(\lambda_1-\lambda\cdot\alpha)(\lambda_1+\lambda\cdot\alpha)}\sum_{\substack{k\in\mathcal{I}_1, j\in\mathcal{I}_1(\lambda_1)\\ \beta,\gamma\in\mathcal{I}_2\\ 1_j+\alpha=\beta+\gamma}}\frac{(\alpha+1_j)!}{\beta!\gamma!}\frac{\partial_k\partial^\beta V(0)}{\lambda_k+\lambda\cdot\beta}\frac{\partial_k\partial^\gamma V(0)}{\lambda_k+\lambda\cdot\gamma}\big(g_1^-(z^-)\big)_j\\ &+\frac{2\lambda_1}{(\lambda_1-\lambda\cdot\alpha)(\lambda_1+\lambda\cdot\alpha)}\sum_{1_j\in\mathcal{I}_1(\lambda_1)}\partial_j\partial^\alpha V(0)\big(g_1^-(z^-)\big)_j.\end{aligned} \tag{6.81}$$

for $|\alpha| = 3$.

Proof. We write

$$\varphi_1(x) = \sum_{j=1}^n a_jx_j + \sum_{|\alpha|=2,3}a_\alpha x^\alpha + \mathcal{O}(x^4), \tag{6.82}$$

and Lemma 6.9 together with (6.73) give all the coefficients in the expansion

$$\nabla\varphi_+(x) = \Big(\lambda_jx_j + \sum_{|\alpha|=2,3}A_{j,\alpha}x^\alpha + \mathcal{O}(x^4)\Big)_{j=1,\dots,n}. \tag{6.83}$$

In fact, we have

$$A_{j,\alpha} = \frac{\partial^{\alpha+1_j}\varphi_+(0)}{\alpha!} \quad \text{and} \quad a_\alpha = \frac{\partial^\alpha\varphi_1(0)}{\alpha!}. \tag{6.84}$$

We get

$$\begin{aligned}
L\varphi_1(x) =& \sum_{j=1}^{n} \partial_j \varphi_+(x) \partial_j \varphi_1(x) \\
=& \sum_{j=1}^{n} \Big(a_j \lambda_j x_j + \sum_{|\alpha|=2} (\alpha_j \lambda_j a_\alpha + a_j A_{j,\alpha}) x^\alpha \\
&+ \sum_{|\alpha|=3} \alpha_j \lambda_j a_\alpha x^\alpha + \sum_{|\beta|=|\gamma|=2} A_{j,\beta} \gamma_j a_\gamma x^{\beta+\gamma-1_j} + \sum_{|\alpha|=3} a_j A_{j,\alpha} x^\alpha \Big) + \mathcal{O}(x^4) \\
=& \sum_{j=1}^{n} a_j \lambda_j x_j + \sum_{|\alpha|=2} \big(\lambda \cdot \alpha \, a_\alpha + \sum_{j=1}^{n} A_{j,\alpha} a_j \big) x^\alpha \\
&+ \sum_{|\alpha|=3} \Big(\lambda \cdot \alpha \, a_\alpha + \sum_{j=1}^{n} \big(\sum_{\substack{\alpha=\beta+\gamma-1_j \\ |\beta|,|\gamma|=2}} A_{j,\beta} \gamma_j a_\gamma + a_j A_{j,\alpha} \big) \Big) x^\alpha + \mathcal{O}(x^4). \qquad (6.85)
\end{aligned}$$

Thus, (6.79) gives, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = 2$,

$$a_\alpha = \frac{1}{\lambda_1 - \lambda \cdot \alpha} \sum_{j=1}^{n} A_{j,\alpha} a_j, \qquad (6.86)$$

and, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| = 3$,

$$a_\alpha = \frac{1}{\lambda_1 - \lambda \cdot \alpha} \sum_{j=1}^{n} \Big(\sum_{\substack{\beta,\gamma \in \mathcal{I}_2 \\ \alpha+1_j=\beta+\gamma}} \gamma_j A_{j,\beta} a_\gamma + a_j A_{j,\alpha} \Big). \qquad (6.87)$$

Then, the lemma follows from (6.84). □

6.3. **Asymptotics near the critical point for the trajectories.**

The knowledge obtained so far is not sufficient for the computation of the φ_j's. We shall obtain here some more information by studying the behavior of the incoming trajectory $\gamma^-(t)$ as $t \to +\infty$. We recall from [20, Section 3] (see also [5, Section 5]), that the curve $\gamma^-(t) = (x^-(t), \xi^-(t)) \in \Lambda_- \cap \Lambda_\omega^-$ satisfies, in the sense of expandible functions,

$$\gamma^-(t) = \sum_{j \geq 1} \sum_{m=0}^{M_j'} \gamma_{j,m}^- t^m e^{-\mu_j t}, \qquad (6.88)$$

Notice that we continue to omit the subscript k for $\gamma_k^- = (x_k^-, \xi_k^-)$, z_k^-, ... Writing also

$$x^-(t) \sim \sum_{j=1}^{+\infty} g_j^-(t, z^-) e^{-\mu_j t}, \quad g_j^-(t, z^-) = \sum_{m=0}^{M_j'} g_{j,m}^-(z^-) t^m, \qquad (6.89)$$

for some integers M_j', we know that $g_1^-(t, z^-) = g_{1,0}^-(z^-) \neq 0$. Since $\xi^-(t) = \partial_t x^-(t)$, we have

$$\xi^-(t) \sim \sum_{j=1}^{+\infty} \sum_{m=0}^{M_j'} g_{j,m}^-(z^-)(-\mu_j t^m + m t^{m-1}) e^{-\mu_j t}. \qquad (6.90)$$

Proposition 6.11. *If $j < \widehat{\jmath}$, then $M'_j = 0$. We also have $M'_{\widehat{\jmath}} \leq 1$, and $M'_{\widehat{\jmath}} = 0$ when $\mathcal{I}_1(2\lambda_1) = \emptyset$. Moreover*

$$(g^-_{\widehat{\jmath},1}(z^-))^\beta = \begin{cases} \dfrac{1}{4\lambda_1} \displaystyle\sum_{|\alpha|=2} \frac{\partial^{\alpha+\beta} V(0)}{\alpha!} (g^-_1(z^-))^\alpha & \text{for } \beta \in \mathcal{I}_1(2\lambda_1), \\ 0 & \text{for } \beta \notin \mathcal{I}_1(2\lambda_1), \end{cases} \tag{6.91}$$

and, for $|\beta| = 1$, $\beta \notin \mathcal{I}_1(2\lambda_1)$,

$$(g^-_{\widehat{\jmath},0}(z^-))^\beta = \frac{1}{(2\lambda_1 + \lambda\cdot\beta)(2\lambda_1 - \lambda\cdot\beta)} \sum_{|\alpha|=2} \frac{\partial^{\alpha+\beta} V(0)}{\alpha!} (g^-_1(z^-))^\alpha. \tag{6.92}$$

Proof. First of all, since $\partial_t \gamma^-(t) = H_p(\gamma^-(t))$, we can write

$$\partial_t \gamma^-(t) = F_p(\gamma^-(t)) + \mathcal{O}(t^{2M'_1} e^{-2\lambda_1 t}), \tag{6.93}$$

where

$$F_p = d_{(0,0)} H_p = \begin{pmatrix} 0 & I \\ \Lambda^2 & 0 \end{pmatrix}, \ \Lambda^2 = \operatorname{diag}(\lambda_1^2, \dots, \lambda_n^2). \tag{6.94}$$

We obtain

$$\sum_{1 \leq j < \widehat{\jmath}} \sum_{m=0}^{M'_j} (F_p + \mu_j) \gamma^-_{j,m} t^m e^{-\mu_j t} = \sum_{1 \leq j < \widehat{\jmath}} \sum_{m=0}^{M'_j} \gamma^-_{j,m} m t^{m-1} e^{-\mu_j t}. \tag{6.95}$$

Now suppose $j < \widehat{\jmath}$ and $M'_j \geq 1$. We get, for this j, for some $\gamma^-_{j,M'_j} \neq 0$,

$$\begin{cases} (F_p + \mu_j) \gamma^-_{j,M'_j} = 0, \\ (F_p + \mu_j) \gamma^-_{j,M'_j - 1} = M'_j \gamma^-_{j,M'_j}, \end{cases} \tag{6.96}$$

so that $\operatorname{Ker}(F_p + \mu_j) \cap \operatorname{Im}(F_p + \mu_j) \neq \{0\}$. Since F_p is a diagonizable matrix, this can easily be seen to be a contradiction.

Now we study $M'_{\widehat{\jmath}}$. So far we have obtained that

$$\gamma^-(t) = \sum_{1 \leq j < \widehat{\jmath}} \gamma^-_j e^{-\mu_j t} + \sum_{m=0}^{M'_{\widehat{\jmath}}} \gamma^-_{\widehat{\jmath},m} t^m e^{-2\lambda_1 t} + \mathcal{O}(t^C e^{-\mu_{\widehat{\jmath}+1} t}), \tag{6.97}$$

and we can write

$$H_p(x,\xi) = \begin{pmatrix} \xi \\ \Lambda^2 x - \displaystyle\sum_{|\alpha|=2} \frac{\partial^\alpha \nabla V(0)}{\alpha!} x^\alpha + \mathcal{O}(x^3) \end{pmatrix}. \tag{6.98}$$

Thus we have

$$H_p(\gamma^-(t)) = F_p\Big(\sum_{j<\widehat{\jmath}} \gamma^-_j e^{-\mu_j t} + \sum_{m=0}^{M'_{\widehat{\jmath}}} \gamma^-_{\widehat{\jmath},m} t^m e^{-2\lambda_1 t} \Big) + e^{-2\lambda_1 t} A(\gamma^-_1) + \mathcal{O}(e^{-(2\lambda_1+\varepsilon)t}), \tag{6.99}$$

where, noticing that $\mu_j + \mu_{j'} = 2\lambda_1$ if and only if $j = j' = 1$,

$$A(\gamma_1^-) = \begin{pmatrix} 0 \\ -\displaystyle\sum_{|\alpha|=2} \frac{\partial^\alpha \nabla V(0)}{\alpha!} (g_1^-)^\alpha \end{pmatrix}. \tag{6.100}$$

For the terms of order $e^{-2\lambda_1 t}$, we have, since $\partial_t \gamma^-(t) = H_p(\gamma^-(t))$,

$$(F_p + 2\lambda_1) \sum_{m=0}^{M'_{\widehat{j}}} \gamma^-_{\widehat{j},m} t^m = \sum_{m=0}^{M'_{\widehat{j}}} \gamma^-_{\widehat{j},m} m t^{m-1} - A(\gamma_1^-). \tag{6.101}$$

Thus, if we suppose that $M'_{\widehat{j}} \geq 2$, we obtain

$$\begin{cases} (F_p + 2\lambda_1)\gamma^-_{\widehat{j},M'_{\widehat{j}}} = 0, \\ (F_p + 2\lambda_1)\gamma^-_{\widehat{j},M'_{\widehat{j}-1}} = M'_{\widehat{j}}\, \gamma^-_{\widehat{j},M'_{\widehat{j}}}. \end{cases} \tag{6.102}$$

Then again we have $\gamma^-_{\widehat{j},M'_{\widehat{j}}} \in \operatorname{Ker}(F_p + 2\lambda_1) \cap \operatorname{Im}(F_p + 2\lambda_1)$, a contradiction.

Finally, if $\lambda_j \neq 2\lambda_1$ for all j, then $\operatorname{Ker}(F_p + 2\lambda_1) = \{0\}$. Therefore, if we suppose that $M'_{\widehat{j}} = 1$, we see that $\gamma_{\widehat{j},1} \neq 0$ satisfies the first equation in (6.102) and we obtain a contradiction.

Now we compute $\gamma^-_{\widehat{j}}(t) = \gamma^-_{\widehat{j},1} t + \gamma^-_{\widehat{j},0}$. We have

$$\begin{cases} (F_p + 2\lambda_1)\gamma^-_{\widehat{j},1} = 0, \\ (F_p + 2\lambda_1)\gamma^-_{\widehat{j},0} = \gamma^-_{\widehat{j},1} - A(\gamma_1^-), \end{cases} \tag{6.103}$$

and we see that $\gamma^-_{\widehat{j},1} = \Pi \gamma^-_{\widehat{j},1} = \Pi A(\gamma_1^-)$, where Π is the projection on the eigenspace of F_p associated to $-2\lambda_1$. We denote by $e_j = (\delta_{i,j} \otimes 0)_{i=1,\dots,n}$ and $\varepsilon_j = (0 \otimes \delta_{i,j})_{i=1,\dots,n}$ for $j = 1, \dots, n$, so that $(e_1, \dots e_n, \varepsilon_1, \dots, \varepsilon_n)$ is the canonical basis of $\mathbb{R}^{2n} = T_{(0,0)} T^* \mathbb{R}^n$. Then it is easy to check that, for all j, $v_j^\pm = e_j \pm \lambda_j \varepsilon_j$ is an eigenvector of F_p for the eigenvalue $\pm\lambda_j$. In the basis $\{e_1, \varepsilon_1, \dots, e_n, \varepsilon_n\}$ the projector Π is block diagonal and, if $K_j = \operatorname{span}(e_j, \varepsilon_j)$, we have

$$\Pi_{|K_j} = \begin{cases} \begin{pmatrix} 1/2 & -1/4\lambda_1 \\ -\lambda_1 & 1/2 \end{pmatrix} & \text{for } j \in \mathcal{I}_1(2\lambda_1), \\ 0 & \text{for } j \notin \mathcal{I}_1(2\lambda_1). \end{cases} \tag{6.104}$$

Therefore, we obtain

$$(g^-_{\widehat{j},1})^\beta = \begin{cases} -\dfrac{1}{4\lambda_1} \displaystyle\sum_{|\alpha|=2} \frac{\partial^\beta \partial^\alpha V(0)}{\alpha!} (g_1^-(z^-))^\alpha & \text{for } \beta \in \mathcal{I}_1(2\lambda_1), \\ 0 & \text{for } \beta \notin \mathcal{I}_1(2\lambda_1). \end{cases} \tag{6.105}$$

Now suppose that $k \notin \mathcal{I}_1(2\lambda_1)$. Then the second equality in (6.103) restricted to K_k gives

$$\begin{pmatrix} 2\lambda_1 & 1 \\ \lambda_k^2 & 2\lambda_1 \end{pmatrix} \Pi_k \gamma_{\widehat{j},0} = -\Pi_k A(\gamma_1^-), \tag{6.106}$$

where Π_k denotes the projection onto K_k. Solving this system, one gets

$$(g^-_{\widehat{\jmath},0})_k = \frac{1}{4\lambda_1^2 - \lambda_k^2} \Pi_x \Pi_k A(\gamma_1^-), \tag{6.107}$$

and, together with (6.100), this ends the proof of Proposition 6.11. □

6.4. **Computation of the φ_j^k's.**

Here we compute the φ_j^k's for $j \leq \widehat{\jmath}$. We continue to omit the superscript k. From [5], we know that $\xi^-(t) = \nabla_x \varphi\big(t, x^-(t)\big)$, so that, using Corollary 6.6, and Corollary 6.8,

$$\begin{aligned} \xi^-(t) =& \nabla\varphi_+(x^-(t)) + \nabla\varphi_1(x^-(t))e^{-\lambda_1 t} + \sum_{2\leq j<\widehat{\jmath}} \nabla\varphi_j(0)e^{-\mu_j t} \\ &+ \nabla\varphi_{\widehat{\jmath},2}(0)t^2 e^{-2\lambda_1 t} + \nabla\varphi_{\widehat{\jmath},1}(0)te^{-2\lambda_1 t} + \nabla\varphi_{\widehat{\jmath},0}(0)e^{-2\lambda_1 t} + \widetilde{\mathcal{O}}(e^{-\mu_{\widehat{\jmath}+1}t}). \end{aligned} \tag{6.108}$$

Since $\varphi_+ = -\varphi_-$ and $(x^-, \xi^-) \in \Lambda_-$, we have $\nabla\varphi_+(x^-(t)) = -\xi^-(t)$, and we obtain first, by (6.90),

$$\nabla\varphi_j(0) = -2\mu_j g_j^-(z^-), \tag{6.109}$$

for $1 \leq j < \widehat{\jmath}$.

Now we study $\varphi_{\widehat{\jmath}}(t,x) = \varphi_{\widehat{\jmath},0}(x) + t\varphi_{\widehat{\jmath},1}(x) + t^2\varphi_{\widehat{\jmath},2}(x)$ when $\mathcal{I}_1(2\lambda_1) \neq \emptyset$. It follows from (6.108) that we have

$$\begin{cases} -4\lambda_1 g^-_{\widehat{\jmath},1}(z^-) = \nabla\varphi_{\widehat{\jmath},1}(0), \\ -4\lambda_1 g^-_{\widehat{\jmath},0}(z^-) + 2g^-_{\widehat{\jmath},1}(z^-) = \nabla\varphi_{\widehat{\jmath},0}(0) + \nabla^2\varphi_1(0)g_1^-(z^-). \end{cases} \tag{6.110}$$

On the other hand, we have seen that, by (6.42), the functions $\varphi_{\widehat{\jmath},2}$, $\varphi_{\widehat{\jmath},1}$ and $\varphi_{\widehat{\jmath},0}$ satisfy

$$\begin{cases} (L-2\lambda_1)\varphi_{\widehat{\jmath},2} = 0, \\ (L-2\lambda_1)\varphi_{\widehat{\jmath},1} = -2\varphi_{\widehat{\jmath},2}, \\ (L-2\lambda_1)\varphi_{\widehat{\jmath},0} = -\varphi_{\widehat{\jmath},1} - \dfrac{1}{2}|\nabla\varphi_1(0)|^2. \end{cases} \tag{6.111}$$

In particular $\varphi_{\widehat{\jmath},2} \in \operatorname{Ker}(L-2\lambda_1) \cap \operatorname{Im}(L-2\lambda_1)$ so that (see (6.61)),

$$\varphi_{\widehat{\jmath},2}(x) = \sum_{\alpha\in\mathcal{I}_2(\lambda_1)} c_{2,\alpha}x^\alpha + \mathcal{O}(x^3). \tag{6.112}$$

Going back to (6.108), we now obtain

$$\begin{aligned} \xi^-(t) =& \nabla\varphi_+(x^-(t)) + \nabla\varphi_1(x^-(t))e^{-\lambda_1 t} + \sum_{2\leq j<\widehat{\jmath}} \nabla\varphi_j(0)e^{-\mu_j t} \\ & \nabla\varphi_{\widehat{\jmath},1}(0)te^{-2\lambda_1 t} + \nabla\varphi_{\widehat{\jmath},0}(0)e^{-2\lambda_1 t} + \widetilde{\mathcal{O}}(e^{-\mu_{\widehat{\jmath}+1}t}), \end{aligned} \tag{6.113}$$

and this equality is consistent with Proposition 6.11.

Then, (6.49) and (6.50) give

$$\varphi_{\widehat{\jmath},1}(x) = \sum_{\alpha\in\mathcal{I}_1(2\lambda_1)} c_{1,\alpha}x^\alpha + \sum_{|\alpha|=2} c_{1,\alpha}x^\alpha + \mathcal{O}(x^3), \tag{6.114}$$

and, by (6.51), we have

$$\Psi((c_{1,\beta})_{\beta\in\mathcal{I}_1(2\lambda_1)}) = (-2c_{2,\alpha})_{\alpha\in\mathcal{I}_2(\lambda_1)}. \tag{6.115}$$

By (6.52), we also have for $|\alpha| = 2$, $\alpha \notin \mathcal{I}_2(\lambda_1)$,

$$c_{1,\alpha} = \frac{1}{2\lambda_1 - \lambda\cdot\alpha} \sum_{\beta\in\mathcal{I}_1(2\lambda_1)} \frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!} c_{1,\beta}. \tag{6.116}$$

The function $\varphi_{\widehat{\jmath},0}(x) = \sum_{|\alpha|\leq 2} c_{0,\alpha}x^\alpha + \mathcal{O}(x^3)$ satisfies (see (6.42))

$$(L - 2\lambda_1)\varphi_{\widehat{\jmath},0} = -\varphi_{\widehat{\jmath},1} - \frac{1}{2}\big|\nabla\varphi_1(x)\big|^2. \tag{6.117}$$

First of all, the compatibility condition (6.47) gives

$$\forall\alpha\in\mathcal{I}_1(2\lambda_1), c_{1,\alpha} = -\nabla\varphi_1(0)\cdot\partial^\alpha\nabla\varphi_1(0), \tag{6.118}$$

so that in particular, by (6.115), the function $\varphi_{\widehat{\jmath},2}$ is known up to $\mathcal{O}(x^3)$ terms:

$$\forall\alpha\in\mathcal{I}_2(\lambda_1),\ c_{2,\alpha} = \frac{1}{2}\sum_{\beta\in\mathcal{I}_1(2\lambda_1)} \frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!}\nabla\varphi_1(0)\cdot\partial^\beta\nabla\varphi_1(0), \tag{6.119}$$

and

$$\forall\alpha\notin\mathcal{I}_2(\lambda_1), |\alpha| = 2,\ c_{1,\alpha} = -\frac{1}{2\lambda_1 - \lambda\cdot\alpha}\sum_{\beta\in\mathcal{I}_1(2\lambda_1)} \frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!}\nabla\varphi_1(0)\cdot\partial^\beta\nabla\varphi_1(0). \tag{6.120}$$

Now (6.49) and (6.50) give

$$c_{0,0} = \varphi_{\widehat{\jmath},0}(0) = \frac{1}{4\lambda_1}|\nabla\varphi_1(0)|^2, \tag{6.121}$$

and

$$\forall\alpha\notin\mathcal{I}_1(2\lambda_1), |\alpha| = 1, c_{0,\alpha} = \frac{1}{2\lambda_1 - \lambda\cdot\alpha}\nabla\varphi_1(0)\cdot\partial^\alpha\nabla\varphi_1(0). \tag{6.122}$$

From the other compatibility condition (6.48), we know that

$$\begin{aligned}\Big(c_{1,\alpha} + \frac{1}{\alpha!}\nabla\varphi_1(0)\cdot\partial^\alpha\nabla\varphi_1(0) + \frac{1}{2}\sum_{\substack{\beta,\gamma\in\mathcal{I}_1(\lambda_1)\\ \beta+\gamma=\alpha}} \partial^\beta\nabla\varphi_1(0)\cdot\partial^\gamma\nabla\varphi_1(0)&\\ + \sum_{\substack{|\beta|=1\\ \beta\notin\mathcal{I}_1(2\lambda_1)}} \frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!}\frac{\nabla\varphi_1(0)\cdot\partial^\beta\nabla\varphi_1(0)}{2\lambda_1-\lambda\cdot\beta}\Big)_{\alpha\in\mathcal{I}_2(\lambda_1)} &\in \operatorname{Im}\Psi,\end{aligned} \tag{6.123}$$

and, from (6.51), we obtain the following relation between the $(c_{0,\beta})_{\beta\in\mathcal{I}_1(2\lambda_1)}$ and the $(c_{1,\alpha})_{\alpha\in\mathcal{I}_2(\lambda_1)}$

$$\begin{aligned}\forall\alpha\in\mathcal{I}_2(\lambda_1),\ c_{1,\alpha} = -\frac{1}{\alpha!}\partial^\alpha\nabla\varphi_1(0)\cdot\nabla\varphi_1(0) - \frac{1}{2}\sum_{\substack{\beta,\gamma\in\mathcal{I}_1(\lambda_1)\\ \beta+\gamma=\alpha}} \partial^\beta\nabla\varphi_1(0)\cdot\partial^\gamma\nabla\varphi_1(0)&\\ - \sum_{\beta\in\mathcal{I}_1(2\lambda_1)}\frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!}c_{0,\beta} - \sum_{\substack{|\beta|=1\\ \beta\notin\mathcal{I}_1(2\lambda_1)}} \frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!}\frac{\nabla\varphi_1(0)\cdot\partial^\beta\nabla\varphi_1(0)}{2\lambda_1-\lambda\cdot\beta}&.\end{aligned} \tag{6.124}$$

Using the second equation in (6.110), we obtain, for $|\beta| = 1$,

$$c_{0,\beta} = -4\lambda_1 (g^-_{\widehat{\jmath},0}(z^-))^\beta + 2(g^-_{\widehat{\jmath},1}(z^-))^\beta - \partial^\beta \nabla\varphi_1(0)\cdot g^-_1(z^-). \tag{6.125}$$

So far, we have computed the functions $\varphi_{\widehat{\jmath},1}(x)$ and $\varphi_{\widehat{\jmath},2}(x)$ up to $\mathcal{O}(x^3)$, in terms of derivatives of φ_+ and φ_1, and of the $g^-_{\widehat{\jmath},m}(z^-)$. We shall now use the expressions we have obtained in Section 6.2 and in Section 6.3 to give these functions in terms of g^-_1 and of derivatives of V only.

First, by (6.112), (6.119), Lemma 6.9 and Lemma 6.10, we obtain

$$\varphi_{\widehat{\jmath},2}(x) = -\frac{1}{8\lambda_1}\sum_{\substack{\gamma\in\mathcal{I}_1(2\lambda_1)\\ \alpha,\beta\in\mathcal{I}_2(\lambda_1)}} \partial^{\beta+\gamma}V(0)\frac{(g^-_1(z^-))^\beta}{\beta!}\,\partial^{\alpha+\gamma}V(0)\frac{x^\alpha}{\alpha!} + \mathcal{O}(x^3). \tag{6.126}$$

Then we have

$$\varphi_{\widehat{\jmath},1}(x) = -4\lambda_1 g^-_{\widehat{\jmath},1}(z^-)\cdot x + \sum_{\alpha\in\mathcal{I}_2(\lambda_1)} c_{1,\alpha}x^\alpha + \sum_{\substack{|\alpha|=2\\ \alpha\notin\mathcal{I}_2(\lambda_1)}} c_{1,\alpha}x^\alpha + \mathcal{O}(x^3), \tag{6.127}$$

where the $c_{1,\alpha}$ are given by (6.124) and (6.125) for $\alpha\in\mathcal{I}_2(\lambda_1)$, and by (6.120) for $\alpha\notin\mathcal{I}_2(\lambda_1)$.

- For $|\alpha| = 2$, $\alpha\notin\mathcal{I}_2(\lambda_1)$, we obtain from (6.116), Lemma 6.9, and Lemma 6.10,

$$\begin{aligned} c_{1,\alpha} =& \frac{4\lambda_1^2}{(2\lambda_1+\lambda\cdot\alpha)(2\lambda_1-\lambda\cdot\alpha)} \\ &\times \sum_{\beta\in\mathcal{I}_1(2\lambda_1)} \frac{\partial^{\alpha+\beta}V(0)}{\alpha!}\sum_{j=1}^n \frac{1}{(\lambda_1+\lambda_j)(3\lambda_1+\lambda_j)}\partial_j\partial^\beta\nabla V(0)\cdot g^-_1(z^-)(g^-_1(z^-))_j. \end{aligned} \tag{6.128}$$

Since $(g^-_1(z^-))_j = 0$ except for $1_j\in\mathcal{I}_1(\lambda_1)$, we get, changing notation a bit,

$$c_{1,\alpha} = \frac{1}{(2\lambda_1+\lambda\cdot\alpha)(2\lambda_1-\lambda\cdot\alpha)}\sum_{\substack{\gamma\in\mathcal{I}_1(2\lambda_1)\\ \beta\in\mathcal{I}_2(\lambda_1)}} \frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}\frac{\partial^{\beta+\gamma}V(0)}{\beta!}(g^-_1(z^-))^\beta. \tag{6.129}$$

- Now we compute $c_{1,\alpha}$ for $\alpha\in\mathcal{I}_2(\lambda_1)$.

For the last term in the R.H.S. of (6.124), we obtain

$$\begin{aligned} &-\sum_{\substack{|\beta|=1\\ \beta\notin\mathcal{I}_1(2\lambda_1)}} \frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!}\frac{\nabla\varphi_1(0)\cdot\partial^\beta\nabla\varphi_1(0)}{2\lambda_1-\lambda\cdot\beta} = \\ &\sum_{\substack{\gamma\in\mathcal{I}_1\setminus\mathcal{I}_1(2\lambda_1)\\ \beta\in\mathcal{I}_2(\lambda_1)}} \frac{8\lambda_1^2}{(2\lambda_1-\lambda\cdot\gamma)(\lambda\cdot\gamma)(2\lambda_1+\lambda\cdot\gamma)^2}\frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}\frac{\partial^{\beta+\gamma}V(0)}{\beta!}(g^-_1(z^-))^\beta. \end{aligned} \tag{6.130}$$

Using (6.91) and (6.125), we have also

$$
\begin{aligned}
&-\sum_{\beta\in\mathcal{I}_1(2\lambda_1)}\frac{\partial^{\alpha+\beta}\varphi_+(0)}{\alpha!}c_{0,\beta}=\\
&-\sum_{\gamma\in\mathcal{I}_1(2\lambda_1)}\frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}(g^-_{\widehat{j},0}(z^-))^{\gamma}+\frac{1}{4\lambda_1^2}\sum_{\substack{\gamma\in\mathcal{I}_1(2\lambda_1)\\ \beta\in\mathcal{I}_2(\lambda_1)}}\frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}\frac{\partial^{\beta+\gamma}V(0)}{\beta!}(g^-_1(z^-))^{\beta}.
\end{aligned}\tag{6.131}
$$

Now we compute $-\frac{1}{\alpha!}\partial^{\alpha}\nabla\varphi_1(0)\cdot\nabla\varphi_1(0)$ for $\alpha\in\mathcal{I}_2(\lambda_1)$. We obtain

$$
\begin{aligned}
-\frac{1}{\alpha!}\partial^{\alpha}\nabla\varphi_1(0)\cdot\nabla\varphi_1(0)=&-\sum_{\beta\in\mathcal{I}_2(\lambda_1)}\frac{\partial^{\alpha+\beta}V(0)}{\alpha!\beta!}(g^-_1(z^-))^{\beta}\\
&-\frac{1}{4}\sum_{j,p,k=1}^{n}\sum_{\substack{\beta,\gamma\in\mathcal{I}_2\\ \beta+\gamma=\alpha+1_p+1_j}}\frac{((\alpha+1_p)_j+1)(\alpha_p+1)}{(\lambda_k+\lambda\cdot\beta)(\lambda_k+\lambda\cdot\gamma)}\frac{\partial^{\beta+1_k}V(0)}{\beta!}\frac{\partial^{\gamma+1_k}V(0)}{\gamma!}(g^-_1(z^-))_j(g^-_1(z^-))_p\\
&+2\lambda_1\sum_{j,p,k=1}^{n}\sum_{\substack{\beta,\gamma\in\mathcal{I}_2\\ \beta+\gamma=\alpha+1_p+1_j}}\frac{(\alpha_p+1)\gamma_j}{(\lambda_1-\lambda\cdot\gamma)(\lambda_1+\lambda\cdot\gamma)(\lambda_j+\lambda\cdot\beta)}\times\\
&\qquad\times\frac{\partial^{\beta+1_j}V(0)}{\beta!}\frac{\partial^{\gamma+1_k}V(0)}{\gamma!}(g^-_1(z^-))_k(g^-_1(z^-))_p\\
=&\,I+II+III.
\end{aligned}\tag{6.132}
$$

Writing $\delta=1_j+1_p$, we get

$$
II=-\frac{1}{2}\sum_{k=1}^{n}\sum_{\substack{\beta,\gamma,\delta\in\mathcal{I}_2\\ \beta+\gamma=\alpha+\delta}}\frac{(\alpha+\delta)!}{(\lambda_k+\lambda\cdot\beta)(\lambda_k+\lambda\cdot\gamma)}\frac{\partial^{\beta+1_k}V(0)}{\beta!}\frac{\partial^{\gamma+1_k}V(0)}{\gamma!}\frac{(g^-_1(z^-))^{\delta}}{\alpha!\,\delta!}.\tag{6.133}
$$

Since $\delta\in\mathcal{I}_2(\lambda_1)$ (otherwise $(g^-_1(z^-))^{\delta}=0$), we have $\beta,\gamma\in\mathcal{I}_2(\lambda_1)$ and, changing notations a bit,

$$
II=-\frac{1}{2}\sum_{\beta\in\mathcal{I}_2(\lambda_1)}\frac{(\alpha+\beta)!}{\alpha!}\sum_{\substack{\gamma,\delta\in\mathcal{I}_2(\lambda_1)\\ \gamma+\delta=\alpha+\beta}}\sum_{j=1}^{n}\frac{1}{(2\lambda_1+\lambda_j)^2}\frac{\partial_j\partial^{\gamma}V(0)}{\gamma!}\frac{\partial_j\partial^{\delta}V(0)}{\delta!}\frac{(g^-_1(z^-))^{\beta}}{\beta!}.\tag{6.134}
$$

In the last term III, we can suppose that $\gamma=1_j+1_q$ for some $q\in\{1,\dots,n\}$. Then $\gamma_j=\gamma!$ and, writing $\beta=1_a+1_b$ we have

$$
\begin{aligned}
III=\lambda_1&\sum_{j,k,p=1}^{n}(\alpha_p+1)(g^-_1(z^-))_k(g^-_1(z^-))_p\\
&\times\sum_{\substack{a,b,q\in\mathcal{I}_1\\ 1_a+1_b+1_q=\alpha+1_p}}\frac{(\alpha_p+1)}{(\lambda_1-\lambda_j-\lambda_q)(\lambda_1+\lambda_j+\lambda_q)(\lambda_j+\lambda_a+\lambda_b)}\partial_{j,a,b}V(0)\partial_{j,q,k}V(0).
\end{aligned}\tag{6.135}
$$

Since $\alpha \in \mathcal{I}_2(\lambda_1)$ and $1_p \in \mathcal{I}_1(\lambda_1)$ (otherwise $(g_1^-(z^-))_p = 0$), we have $1_a, 1_b, 1_q \in \mathcal{I}_1(\lambda_1)$ so that we can write

(6.136)
$$III = -\sum_{j,k,p=1}^{n} (\alpha_p + 1) \frac{\lambda_1}{\lambda_j(2\lambda_1 + \lambda_j)^2} (g_1^-(z^-))_k (g_1^-(z^-))_p \sum_{\substack{a,b,q \in \mathcal{I}_1 \\ 1_a + 1_b + 1_q = \alpha + 1_p}} \partial_{j,a,b} V(0) \partial_{j,q,k} V(0).$$

Now it is easy to check, noticing that $(\alpha + 1_p)_k \in \{0, 1, 2, 3\}$ and examining each case, that

$$\sum_{\substack{a,b,q \in \mathcal{I}_1 \\ 1_a + 1_b + 1_q = \alpha + 1_p}} \partial_{j,a,b} V(0) \partial_{j,q,k} V(0) = \frac{(\alpha + 1_p)_k}{4} \sum_{\substack{a,b,c,d \in \mathcal{I}_1 \\ 1_a + 1_b + 1_c + 1_d = \alpha + 1_p + 1_k}} \partial_{j,a,b} V(0) \partial_{j,c,d} V(0). \tag{6.137}$$

Therefore, we have

$$\begin{aligned} III = -\frac{1}{4} \sum_{j,k,p=1}^{n} \frac{(\alpha + 1_p + 1_k)!}{\alpha!} \frac{\lambda_1}{\lambda_j(2\lambda_1 + \lambda_j)^2} (g_1^-(z^-))_k (g_1^-(z^-))_p \\ \times \sum_{\substack{a,b,c,d \in \mathcal{I}_1 \\ 1_a + 1_b + 1_c + 1_d = \alpha + 1_p + 1_k}} \partial_{j,a,b} V(0) \partial_{j,c,d} V(0). \end{aligned} \tag{6.138}$$

Eventually, setting $\beta = 1_p + 1_k$, $\gamma = 1_a + 1_b$ and $\delta = 1_c + 1_d$, we get

(6.139)
$$III = -\sum_{\beta \in \mathcal{I}_2(\lambda_1)} \frac{(\alpha + \beta)!}{\alpha!} \sum_{\substack{\gamma, \delta \in \mathcal{I}_2(\lambda_1) \\ \gamma + \delta = \alpha + \beta}} \sum_{j=1}^{n} \frac{2\lambda_1}{\lambda_j(2\lambda_1 + \lambda_j)^2} \frac{\partial_j \partial^\gamma V(0)}{\gamma!} \frac{\partial_j \partial^\delta V(0)}{\delta!} \frac{(g_1^-(z^-))^\beta}{\beta!}.$$

We are left with the computation of

$$\begin{aligned} -\frac{1}{2} \sum_{\substack{\beta, \gamma \in \mathcal{I}_1(\lambda_1) \\ \beta + \gamma = \alpha}} \partial^\beta \nabla \varphi_1(0) \cdot \partial^\gamma \nabla \varphi_1(0) &= -\frac{1}{2} \sum_{j=1}^{n} \sum_{\substack{\beta, \gamma \in \mathcal{I}_1(\lambda_1) \\ \beta + \gamma = \alpha}} \partial_j \partial^\beta \varphi_1(0) \cdot \partial_j \partial^\gamma \varphi_1(0) \\ &= -\frac{1}{2} \sum_{j=1}^{n} \frac{4\lambda_1^2}{\lambda_j^2(2\lambda_1 + \lambda_j)^2} \sum_{\substack{\beta, \gamma \in \mathcal{I}_1(\lambda_1) \\ \beta + \gamma = \alpha}} \sum_{k,\ell=1}^{n} \partial_j \partial_k \partial^\beta V(0) (g_1^-(z^-))_k \partial_j \partial_\ell \partial^\gamma V(0) (g_1^-(z^-))_\ell. \end{aligned} \tag{6.140}$$

At this point, we notice that

$$
\begin{aligned}
-\frac{1}{2} \sum_{\alpha\in\mathcal{I}_2(\lambda_1)} \sum_{\substack{\beta,\gamma\in\mathcal{I}_1(\lambda_1)\\ \beta+\gamma=\alpha}} & \partial^\beta \nabla\varphi_1(0)\cdot\partial^\gamma\nabla\varphi_1(0) x^\alpha \\
&= -\frac{1}{2}\sum_{j=1}^n \frac{4\lambda_1^2}{\lambda_j^2(2\lambda_1+\lambda_j)^2} \sum_{\substack{\beta,\gamma\in\mathcal{I}_1(\lambda_1)\\ \alpha\in\mathcal{I}_2(\lambda_1)\\ \beta+\gamma=\alpha}} \sum_{k,\ell=1}^n \partial_j\partial_k\partial^\beta V(0)(g_1^-(z^-))_k \partial_j\partial_\ell\partial^\gamma V(0)(g_1^-(z^-))_\ell \, x^\alpha \\
&= -\frac{1}{2}\sum_{j=1}^n \frac{4\lambda_1^2}{\lambda_j^2(2\lambda_1+\lambda_j)^2}\Big\{ \sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} (\alpha+\beta)! \sum_{\substack{\gamma,\delta\in\mathcal{I}_2(\lambda_1)\\ \gamma+\delta=\alpha+\beta}} \frac{\partial_j\partial^\gamma V(0)}{\gamma!}\frac{\partial_j\partial^\delta V(0)}{\delta!}\frac{x^\alpha}{\alpha!}\frac{(g_1^-(z^-))^\beta}{\beta!} \\
&\qquad - 2\sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} \frac{\partial_j\partial^\alpha V(0)}{\alpha!}\frac{\partial_j\partial^\beta V(0)}{\beta!} x^\alpha (g_1^-(z^-))^\beta \Big\}
\end{aligned}
\tag{6.141}
$$

From (6.124), (6.130), (6.131) (6.139), and (6.141), we finally obtain that

$$
\begin{aligned}
\sum_{\alpha\in\mathcal{I}_2(\lambda_1)} c_{1,\alpha}x^\alpha = & \sum_{\substack{\gamma\in\mathcal{I}_1\setminus\mathcal{I}_1(2\lambda_1)\\ \alpha,\beta\in\mathcal{I}_2(\lambda_1)}} \frac{8\lambda_1^2}{(2\lambda_1-\lambda\cdot\gamma)(\lambda\cdot\gamma)(2\lambda_1+\lambda\cdot\gamma)^2}\frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}\frac{\partial^{\beta+\gamma}V(0)}{\beta!}(g_1^-(z^-))^\beta x^\alpha \\
& - \sum_{\substack{\gamma\in\mathcal{I}_1(2\lambda_1)\\ \alpha\in\mathcal{I}_2(\lambda_1)}} \frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}(g_{\widehat{\jmath},0}^-(z^-))^\gamma x^\alpha + \frac{1}{4\lambda_1^2}\sum_{\substack{\gamma\in\mathcal{I}_1(2\lambda_1)\\ \alpha,\beta\in\mathcal{I}_2(\lambda_1)}} \frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}\frac{\partial^{\beta+\gamma}V(0)}{\beta!}(g_1^-(z^-))^\beta x^\alpha \\
& - \sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} \frac{\partial^{\alpha+\beta}V(0)}{\alpha!\beta!}(g_1^-(z^-))^\beta x^\alpha \\
& - \frac{1}{2}\sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} (\alpha+\beta)! \sum_{\substack{\gamma,\delta\in\mathcal{I}_2\\ \gamma+\delta=\alpha+\beta}} \sum_{j=1}^n \frac{1}{(2\lambda_1+\lambda_j)^2}\frac{\partial_j\partial^\gamma V(0)}{\gamma!}\frac{\partial_j\partial^\delta V(0)}{\delta!}\frac{(g_1^-(z^-))^\beta}{\beta!}\frac{x^\alpha}{\alpha!} \\
& - \sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} (\alpha+\beta)! \sum_{\substack{\gamma,\delta\in\mathcal{I}_2(\lambda_1)\\ \gamma+\delta=\alpha+\beta}} \sum_{j=1}^n \frac{2\lambda_1}{\lambda_j(2\lambda_1+\lambda_j)^2}\frac{\partial_j\partial^\gamma V(0)}{\gamma!}\frac{\partial_j\partial^\delta V(0)}{\delta!}\frac{(g_1^-(z^-))^\beta}{\beta!}\frac{x^\alpha}{\alpha!} \\
& - 2\sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} (\alpha+\beta)! \sum_{\substack{\gamma,\delta\in\mathcal{I}_2(\lambda_1)\\ \gamma+\delta=\alpha+\beta}} \sum_{j=1}^n \frac{\lambda_1^2}{\lambda_j^2(2\lambda_1+\lambda_j)^2}\frac{\partial_j\partial^\gamma V(0)}{\gamma!}\frac{\partial_j\partial^\delta V(0)}{\delta!}\frac{(g_1^-(z^-))^\beta}{\beta!}\frac{x^\alpha}{\alpha!} \\
& + 4\sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} \sum_{j=1}^n \frac{\lambda_1^2}{\lambda_j^2(2\lambda_1+\lambda_j)^2}\frac{\partial_j\partial^\alpha V(0)}{\alpha!}\frac{\partial_j\partial^\beta V(0)}{\beta!} x^\alpha (g_1^-(z^-))^\beta,
\end{aligned}
\tag{6.142}
$$

or, more simply,

$$\sum_{\alpha\in\mathcal{I}_2(\lambda_1)} c_{1,\alpha}x^\alpha = -\sum_{\substack{\gamma\in\mathcal{I}_1(2\lambda_1)\\ \alpha\in\mathcal{I}_2(\lambda_1)}} \frac{\partial^{\alpha+\gamma}V(0)}{\alpha!}(g^-_{\widehat{\jmath},0}(z^-))^\gamma x^\alpha + \sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} \frac{(g_1^-(z^-))^\beta}{\beta!}\frac{x^\alpha}{\alpha!}$$
$$\times\Big\{\sum_{\gamma\in\mathcal{I}_1\setminus\mathcal{I}_1(2\lambda_1)} \frac{8\lambda_1^2}{(2\lambda_1-\lambda\cdot\gamma)(\lambda\cdot\gamma)(2\lambda_1+\lambda\cdot\gamma)^2}\partial^{\alpha+\gamma}V(0)\partial^{\beta+\gamma}V(0)$$
$$+\frac{1}{4\lambda_1^2}\sum_{\gamma\in\mathcal{I}_1(2\lambda_1)}\partial^{\alpha+\gamma}V(0)\partial^{\beta+\gamma}V(0)-\partial^{\alpha+\beta}V(0)$$
$$-\frac{(\alpha+\beta)!}{2}\sum_{\substack{\gamma,\delta\in\mathcal{I}_2\\ \gamma+\delta=\alpha+\beta}}\sum_{j=1}^n\frac{1}{\lambda_j^2}\frac{\partial_j\partial^\gamma V(0)}{\gamma!}\frac{\partial_j\partial^\delta V(0)}{\delta!}$$
$$+4\sum_{j=1}^n\frac{\lambda_1^2}{\lambda_j^2(2\lambda_1+\lambda_j)^2}\partial_j\partial^\alpha V(0)\partial_j\partial^\beta V(0)\Big\}. \tag{6.143}$$

7. Computations after the critical point

7.1. Stationary phase expansion in the outgoing region.

Now we compute the scattering amplitude starting from (4.19). First of all, we change the cut-off function χ_+ so that the support of the right hand side of the scalar product in (4.19) is close to $(0,0)$.

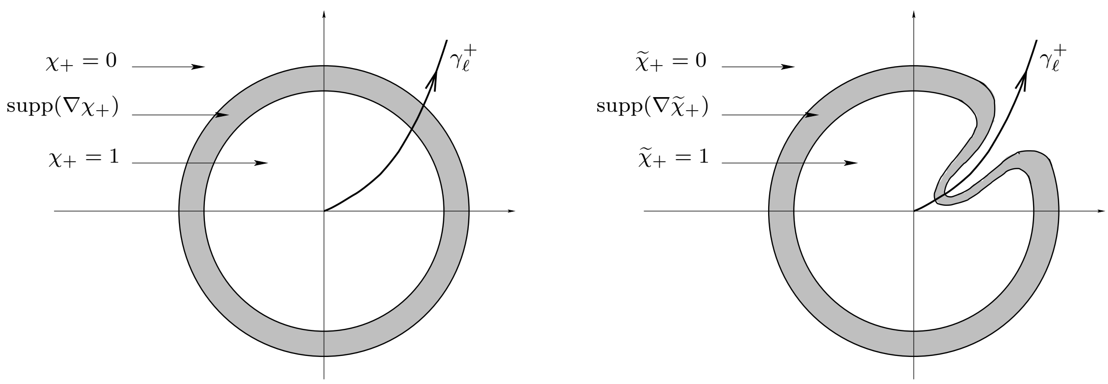

FIGURE 1. The support of χ_+ and $\widetilde{\chi}_+$ in $T^*\mathbb{R}^n$.

Using Maslov's theory, we construct a function v_+ which coincides with $a_+(x,h)e^{i\psi_+(x)/h}$ out of a small neighborhood of $\bigcup_\ell \gamma_\ell^+ \cap (B(0,R_++1)\times\mathbb{R}^n)$ and such that v_+ is a solution of $(P-E)v_+=0$ microlocally near $\bigcup_\ell\gamma_\ell^+$. Let $\widetilde{\chi}_+(x,\xi)\in C^\infty(T^*\mathbb{R}^n)$ be such that $\widetilde{\chi}_+(x,\xi)=\chi_+(x)$ out of a small enough neighborhood of $\bigcup_\ell\gamma_\ell^+\cap(B(0,R_++1)\times\mathbb{R}^n)$. In particular,

$(P-E)v_+$ is microlocally 0 near the support of $\chi_+ - \widetilde{\chi}_+$. So, we have

$$\begin{aligned}\langle u_-, [\chi_+, P]v_+\rangle =& \langle u_-, [\mathrm{Op}(\widetilde{\chi}_+), P]v_+\rangle + \langle u_-, (\chi_+ - \mathrm{Op}(\widetilde{\chi}_+))(P-E)v_+\rangle \\ &- \langle (P-E)u_-, (\chi_+ - \mathrm{Op}(\widetilde{\chi}_+))v_+\rangle \\ =& \langle u_-, [\mathrm{Op}(\widetilde{\chi}_+), P]v_+\rangle + \mathcal{O}(h^\infty) - \langle g_- e^{i\psi_-/h}, (\chi_+ - \mathrm{Op}(\widetilde{\chi}_+))v_+\rangle \\ =& \langle u_-, [\mathrm{Op}(\widetilde{\chi}_+), P]v_+\rangle + \mathcal{O}(h^\infty), \end{aligned} \tag{7.1}$$

since the microsupports of $g_- e^{i\psi_-/h}$ and $\chi_+ - \widetilde{\chi}_+$ are disjoint. Thus, the scattering amplitude is given by

$$\mathcal{A}(\omega, \theta, E, h) = \widetilde{c}(E) h^{-(n+1)/2} \langle u_-, [\mathrm{Op}(\widetilde{\chi}_+), P]v_+\rangle + \mathcal{O}(h^\infty). \tag{7.2}$$

Now we will prove that, modulo $\mathcal{O}(h^\infty)$, the only contribution to the scattering amplitude in (7.2) comes from the values of the functions u_- and v_+ microlocally on the trajectories γ_ℓ^+ and γ_j^∞. From (5.18), the fact that $u_- = \mathcal{O}(h^{-C})$ and $(P-E)u_- = 0$ microlocally out of the microsupport of $g_- e^{-i\psi_-/h}$, and the usual propagation of singularities theorem, we get

$$\mathrm{MS}(u_-) \subset \Lambda_\omega^- \cup \Lambda_+. \tag{7.3}$$

Moreover, we have

$$\mathrm{MS}(v_+) \subset \Lambda_\theta^+. \tag{7.4}$$

Now, let f_j^∞ (resp. f_ℓ^+) be $C_0^\infty(T^*\mathbb{R}^n)$ functions with support in a small enough neighborhood of γ_j^∞ (resp. $\gamma_\ell^+ \cap \mathrm{MS}(v_+)$) such that $f_j^\infty = 1$ (resp. $f_k^+ = 1$) in a neighborhood of γ_j^∞ (resp. $\gamma_\ell^+ \cap \mathrm{MS}(v_+)$). In particular, we assume that all these functions have disjoint support. Since u_- and v_+ have disjoint microsupport out of the support of the f_j^∞ and the f_ℓ^+, we have

$$\begin{aligned}\mathcal{A}(\omega, \theta, E, h) =& \widetilde{c}(E) h^{-(n+1)/2} \sum_j \langle \mathrm{Op}(f_j^\infty) u_-, \mathrm{Op}(f_j^\infty)[\mathrm{Op}(\widetilde{\chi}_+), P]v_+\rangle \\ &+ \widetilde{c}(E) h^{-(n+1)/2} \sum_\ell \langle \mathrm{Op}(f_\ell^+) u_-, \mathrm{Op}(f_\ell^+)[\mathrm{Op}(\widetilde{\chi}_+), P]v_+\rangle + \mathcal{O}(h^\infty) \\ =& \mathcal{A}^{reg} + \mathcal{A}^{sing}. \end{aligned} \tag{7.5}$$

Concerning the terms which contain f_j^∞, $\mathcal{A}^{reg}$, we are in the same setting as in [32, Section 4] with the difference that the calculus is made for any $E = E_0 + hE_1$ with $E_1 = \mathcal{O}(1)$ and not for $E = E_0$.

In Equation (5.33), we have shown that the main term of the symbol appearing in the WKB expansion on u_- differs, from the case $E = E_0$, by a factor $e^{it_-(\rho)E_1}$ for $\rho \in \gamma_j^\infty$. The time $t_-(\rho)$ is the unique time t such that $\gamma_-(t, z_j^\infty, \omega, E_0) = \rho$ (see (2.6) and (2.8)). The same way, the main term of the symbol in the WKB expansion on v_+ differs by a factor $e^{it_+(\rho)E_1}$ on the curve γ_j^∞. Here $t_+(\rho) = t$ is such that $\gamma_+(t, \widetilde{z}_j^\infty, \theta, E_0) = \rho$, where $\widetilde{z}_j^\infty$ is the projection of $r_\infty(z_j^\infty, \omega, E_0)$ on $\theta^\perp$. The bicharacteristic curves $\gamma_-(t, z_j^\infty, \omega, E_0)$, $\gamma_+(t, \widetilde{z}_j^\infty, \theta, E_0)$ and γ_j^∞ are the same sets, and the quantity $t_- - t_+$ does not depend on $\rho \in \gamma_j^\infty$. Moreover, from (2.9), we have

$$t_- - t_+ = -\big\langle r_\infty(z_j^\infty, \omega, E_0) \big| \sqrt{2E_0}^{-1} \theta \big\rangle. \tag{7.6}$$

Then, following [32, Section 4], the computation of the term $\mathcal{A}^{reg}$ gives

$$\mathcal{A}^{reg} = \sum_{j=1}^{N_\infty} \Big(\sum_{m \geq 0} a_{j,m}^{\rm reg}(\omega, \theta, E) h^m \Big) e^{i S_j^\infty / h} + \mathcal{O}(h^\infty), \tag{7.7}$$

with

$$a_{j,0}^{\rm reg}(\omega, \theta, E) = a_{j,0}^{\rm reg}(\omega, \theta, E_0) e^{i(t_- - t_+)E_1}. \tag{7.8}$$

Here, $a_{j,0}^{\rm reg}(\omega, \theta, E_0)$ is the term obtained by Robert and Tamura and equal to

$$a_{j,0}^{\rm reg}(\omega, \theta, E_0) = \frac{e^{-i \nu_j^\infty \pi/2}}{\widehat{\sigma}(z_j^\infty)^{1/2}}. \tag{7.9}$$

Now we compute $\mathcal{A}^{sing}$. Proceeding as in Section 5.2 for u_-, one can show that v_+ can be written as

$$v_+(x) = a_+(x,h) e^{i \nu_\ell^+ \pi/2} e^{i \psi_+(x)/h}, \tag{7.10}$$

microlocally near any $\rho \in \gamma_\ell^+$ close enough to $(0,0)$. Here ν_ℓ^+ is the Maslov index of γ_ℓ^+. The phase ψ_+ and the classical symbol a_+ satisfy the usual eikonal and transport equations. In particular, as in (5.28) and (5.33), we have

$$\psi_+(x_\ell^+(t)) = - \int_t^{+\infty} |\xi_\ell^+(u)|^2 - 2E_0 1_{u>0} \, du = - \int_t^{+\infty} \frac{1}{2} |\xi_\ell^+(u)|^2 - V(x_\ell^+(u)) - E_0 \operatorname{sgn}(u) \, du, \tag{7.11}$$

and $a_+(x,h) \sim \sum_m a_{+,m}(x) h^m$ with

$$a_{+,0}(x_\ell^+(t)) = (2E_0)^{1/4} (D_\ell^+(t))^{-1/2} e^{itE_1}, \tag{7.12}$$

where

$$D_\ell^+(t) = \Big| \det \frac{\partial x_+(t, z, \theta, E_0)}{\partial (t,z)} |_{z = z_\ell^+} \Big|. \tag{7.13}$$

We can choose $\widetilde{\chi}_+$ so that the support of the symbol of $\operatorname{Op}(f_\ell^+)[\operatorname{Op}(\widetilde{\chi}_+), P]$ is contained in a vicinity of such a point $\rho \in \gamma_\ell^+$ (see Figure 1). Then, microlocally near ρ, we have

$$\operatorname{Op}(f_\ell^+)[\operatorname{Op}(\widetilde{\chi}_+), P] v_+ = \widetilde{a}_+(x,h) e^{i \nu_\ell^+ \pi/2} e^{i \psi_+(x)/h}, \tag{7.14}$$

with

$$\widetilde{a}_+(x,h) = \sum_{m \geq 0} \widetilde{a}_{+,m}(x) h^{m+1}, \tag{7.15}$$

and

$$\widetilde{a}_{+,0}(x) = -i \{ \widetilde{\chi}_+, p \}(x, \nabla \psi_+(x)) a_{+,0}(x). \tag{7.16}$$

From [5, Section 5], the Lagrangian manifold

$$\{ (x, \nabla_x \varphi^k(t,x)); \ \partial_t \varphi^k(t,x) = 0 \},$$

coincides with Λ_ω^-. In particular, since $\operatorname{MS}(v_+) \subset \Lambda_\theta^+$ and since there is no curve $\gamma^\infty(z_j^\infty)$ sufficiently close to the critical point, the finite times in (6.5) give a contribution $\mathcal{O}(h^\infty)$ to the scattering amplitude (4.19). In view of the equations (6.5), (6.12) and (7.14), the principal contribution of $\mathcal{A}^{sing}$ will come from the intersection of the manifolds Λ_θ^+ and Λ_+. Recall that, from **(A5)**, the manifolds Λ_θ^+ and Λ_+ intersect transversely along γ_ℓ^+.

In particular, to compute $\mathcal{A}^{sing}$, we can apply the method of stationary phase in the directions that are transverse to γ_ℓ^+. For each ℓ, after a linear and orthonormal change of variables, we can assume that $g_{\boldsymbol{\ell}}^+(z_\ell^+)$ is collinear to the $x_{\boldsymbol{\ell}}$–direction, and that $V(x)$ satisfies **(A2)**. We denote $H_{x_{\boldsymbol{\ell}}}^\ell = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n;\ y_{\boldsymbol{\ell}} = x_{\boldsymbol{\ell}}\}$ the hyperplane orthogonal to $(0, \dots, 0, x_{\boldsymbol{\ell}}, 0, \dots, 0)$.

We shall compute $\mathcal{A}^{sing}$ in the case where there is only one incoming curve γ_k^- in Λ_ω^- and one outgoing curve γ_ℓ^+ in Λ_θ^+. In the case of several but finitely many trajectories, $\mathcal{A}^{sing}$ is simply given by the sum over k and ℓ of such contributions. Using (4.19), (6.5) and (7.14), we can write

$$\begin{aligned}\mathcal{A}^{sing} &= \frac{\widetilde{c}(E) h^{-(n+1)/2}}{\sqrt{2\pi h}} \iint e^{i(\varphi^k(t,x) - \psi_+(x))/h} \alpha^k(t,x,h) \overline{\widetilde{a}_+}(x,h) e^{-i\nu_\ell^+ \pi/2} dt\, dx \\ &= \frac{\widetilde{c}(E) h^{-(n+1)/2}}{\sqrt{2\pi h}} \int_{x_{\boldsymbol{\ell}}} \iint_{y \in H_{x_{\boldsymbol{\ell}}}^\ell} e^{i(\varphi^k(t,x) - \psi_+(x))/h} \alpha^k(t,x,h) \overline{\widetilde{a}_+}(x,h) e^{-i\nu_\ell^+ \pi/2} dt\, dy\, dx_{\boldsymbol{\ell}}. \qquad (7.17)\end{aligned}$$

Let $\Phi(y) = \varphi^k(t, x_{\boldsymbol{\ell}}, y) - \psi_+(x_{\boldsymbol{\ell}}, y)$ be the phase function in (7.17). From (6.10)–(6.13), we can write

$$\Phi(y) = S_k^- + (\varphi_+ - \psi_+)(x_{\boldsymbol{\ell}}, y) + \widetilde{\psi}(t, x_{\boldsymbol{\ell}}, y), \qquad (7.18)$$

where $\widetilde{\psi} = \mathcal{O}(e^{-\lambda_1 t})$ is an expandible function. Since the manifolds Λ_θ^+ and Λ_+ intersect transversely along γ_ℓ^+, the phase function $y \mapsto (\varphi_+ - \psi_+)(x_{\boldsymbol{\ell}}, y)$ has a non degenerate critical point $y^\ell(x_{\boldsymbol{\ell}}) \in H_{x_{\boldsymbol{\ell}}}^\ell \cap \Pi_x \gamma_\ell^+$, and $x_{\boldsymbol{\ell}} \mapsto y^\ell(x_{\boldsymbol{\ell}})$ is C^∞ for $x_{\boldsymbol{\ell}} \neq 0$. Then, from the implicit function theorem, the function Φ has a unique critical point $y^\ell(t, x_{\boldsymbol{\ell}}) \in H_{x_{\boldsymbol{\ell}}}^\ell$ for t large enough depending on $x_{\boldsymbol{\ell}}$. The function $(t, x_{\boldsymbol{\ell}}) \mapsto y^\ell(t, x_{\boldsymbol{\ell}})$ is expandible and we have

$$y^\ell(t, x_{\boldsymbol{\ell}}) = y^\ell(x_{\boldsymbol{\ell}}) - \operatorname{Hess}(\varphi_+ - \psi_+)^{-1}\big(y^\ell(x_{\boldsymbol{\ell}})\big) \nabla\varphi_1\big(y^\ell(x_{\boldsymbol{\ell}})\big) e^{-\mu_1 t} + \widetilde{\mathcal{O}}\big(e^{-\mu_2 t}\big). \qquad (7.19)$$

As a consequence, $\Phi\big(y^\ell(t, x_{\boldsymbol{\ell}})\big)$ is also expandible.

Since φ_+ and ψ_+ satisfy the same eikonal equation, we get (see (5.25))

$$\partial_t(\varphi_+ - \psi_+)(x_\ell^+(t)) = |\xi_\ell^+(t)|^2 - |\xi_\ell^+(t)|^2 = 0. \qquad (7.20)$$

Thus, $(\varphi_+ - \psi_+)(y^\ell(x_{\boldsymbol{\ell}}))$ does not depend of $x_{\boldsymbol{\ell}}$ and is equal to

$$\begin{aligned}(\varphi_+ - \psi_+)(y^\ell(x_{\boldsymbol{\ell}})) &= \lim_{t \to -\infty} (\varphi_+ - \psi_+)(x_\ell^+(t)) \\ &= \int_{-\infty}^{+\infty} |\xi_\ell^+(s)|^2 - 2E_0 1_{s>0}\, ds \\ &= \int_{-\infty}^{+\infty} \frac{1}{2}|\xi_\ell^+(s)|^2 - V(x_\ell^+(s)) - E_0 \operatorname{sgn}(s)\, ds \\ &= S_\ell^+, \qquad (7.21)\end{aligned}$$

where we have used (7.11). Therefore, the phase function Φ at the critical point $y^\ell(t, x_{\boldsymbol\ell})$ is equal to

$$\begin{aligned}\Phi\big(y^\ell(t,x_{\boldsymbol\ell})\big) =& S_k^- + S_\ell^+ + \sum_{\substack{m\in\mathbb{N}\\ \mu_m\leq 2\lambda_1}} \varphi_m\big(t, y^\ell(x_{\boldsymbol\ell})\big)e^{-\mu_m t}\\ &-\frac{1}{2}\big(\operatorname{Hess}(\varphi_+-\psi_+)^{-1}\big(y^\ell(x_{\boldsymbol\ell})\big)\nabla\varphi_1\big(y^\ell(x_{\boldsymbol\ell})\big)\cdot\nabla\varphi_1\big(y^\ell(x_{\boldsymbol\ell})\big)\big)e^{-2\mu_1 t} + \widetilde{\mathcal{O}}(e^{-\widetilde{\mu}t}),\end{aligned}\tag{7.22}$$

where $\widetilde{\mu}$ is the first of the μ_j's such that $\mu_j > 2\lambda_1$.

Using the method of the stationary phase for the integration with respect to $y \in H^\ell_{x_{\boldsymbol\ell}}$ in (7.17), we get

$$\mathcal{A}^{sing} = \frac{\widetilde{c}(E)h^{-(n+1)/2}}{\sqrt{2\pi h}}(2\pi h)^{(n-1)/2}\iint e^{i\Phi(y^\ell(t,x_{\boldsymbol\ell}))/h}f^\ell(t,x_{\boldsymbol\ell},h)\,dt\,dx_{\boldsymbol\ell} + \mathcal{O}(h^\infty).\tag{7.23}$$

The $\mathcal{O}(h^\infty)$ term follows from the fact that the error term stemming from the stationary phase method can be integrated with respect to time t, since $\alpha^k \in \mathcal{S}^{0,2\operatorname{Re}\Sigma(E)}$, with $\operatorname{Re}\Sigma(E) > 0$ (see the beginning of Section 6). The symbol $f^\ell(t, x_{\boldsymbol\ell}, h)$ is a classical expandible function of order $\mathcal{S}^{1,2\operatorname{Re}\Sigma(E)}$ in the sense of Definition 6.2:

$$f^\ell(t,x_{\boldsymbol\ell},h) \sim \sum_{m\geq 0} f^\ell_m(t,x_{\boldsymbol\ell},\ln h)h^{1+m},\tag{7.24}$$

where the f^ℓ_m are polynomials with respect to $\ln h$ and

$$f^\ell_0(t,x_{\boldsymbol\ell},\ln h) = \alpha^k_0\big(t,y^\ell(t,x_{\boldsymbol\ell})\big)\overline{\widetilde{a}_{+,0}}\big(y^\ell(t,x_{\boldsymbol\ell})\big)e^{-i\nu^+_\ell\pi/2}\frac{e^{i\operatorname{sgn}\Phi''_{|H^\ell_{x_{\boldsymbol\ell}}}(y^\ell(t,x_{\boldsymbol\ell}))\pi/4}}{\big|\det\Phi''_{|H^\ell_{x_{\boldsymbol\ell}}}\big(y^\ell(t,x_{\boldsymbol\ell})\big)\big|^{1/2}}.\tag{7.25}$$

Using Proposition C.1, we compute the Hessian of Φ, and we get

$$\begin{aligned}\psi''_+\big(y^\ell(x_{\boldsymbol\ell})\big) &= \operatorname{diag}(-\lambda_1,\dots,-\lambda_{\boldsymbol\ell-1},\lambda_{\boldsymbol\ell},-\lambda_{\boldsymbol\ell+1},\dots,-\lambda_n)+o(1),\\ \varphi''_+\big(y^\ell(x_{\boldsymbol\ell})\big) &= \operatorname{diag}(\lambda_1,\dots,\lambda_n)+o(1).\end{aligned}$$

Then, for $x_{\boldsymbol\ell}$ small enough and t large enough depending on $x_{\boldsymbol\ell}$, we have

$$\big|\det\Phi''_{|H^\ell_{x_{\boldsymbol\ell}}}\big(y^\ell(t,x_{\boldsymbol\ell})\big)\big|^{1/2} = \sqrt{\prod_{j\neq\boldsymbol\ell}2\lambda_j} + o(1),\tag{7.26}$$

$$\operatorname{sgn}\Phi''_{|H^\ell_{x_{\boldsymbol\ell}}}\big(y^\ell(t,x_{\boldsymbol\ell})\big) = n-1,\tag{7.27}$$

as $x_{\boldsymbol\ell}$ goes to 0.

7.2. **Behaviour of the phase function Φ.**

Suppose that $j \in \mathbb{N}$ is such that $j < \widehat{\jmath}$. From (6.40), we have

$$\varphi^k_j(x^+_\ell(s_0)) = e^{-\mu_j(s-s_0)}\varphi^k_j(x^+_\ell(s)).\tag{7.28}$$

Combining (6.41) with (6.109), we obtain

$$\begin{aligned}\varphi^k_j(x^+_\ell(s_0)) =& e^{\mu_j s_0}e^{-\mu_j s}\big(-2\mu_j\langle g^-_j(z^-_k)|g^+_j(z^+_\ell)\rangle e^{\mu_j s} + \mathcal{O}(e^{2\lambda_1 s})\big)\\ =& -2\mu_j\big\langle g^-_j(z^-_k)\big|g^+_j(z^+_\ell)\big\rangle e^{\mu_j s_0}.\end{aligned}\tag{7.29}$$

We suppose first that we are in the case **(a)** of assumption **(A7)**. Then, (7.22) becomes

$$\Phi\big(y^{\ell}(t,x_{\boldsymbol{\ell}})\big) = S_k^- + S_\ell^+ - 2\mu_{\mathbf{k}}\big\langle g_{\mathbf{k}}^-(z_k^-)\big|g_{\mathbf{k}}^+(z_\ell^+)\big\rangle e^{\mu_{\mathbf{k}} s(x_{\boldsymbol{\ell}})} e^{-\mu_{\mathbf{k}} t} + \widetilde{\mathcal{O}}(e^{-\mu_{\mathbf{k}+1}t}). \tag{7.30}$$

Here $s(x_{\boldsymbol{\ell}})$ is such that $x_\ell^+(s(x_{\boldsymbol{\ell}})) = x^\ell(x_{\boldsymbol{\ell}})$ and the $\widetilde{\mathcal{O}}(e^{-\mu_{\mathbf{k}+1}t})$ is in fact expandible, uniformly with respect to $x_{\boldsymbol{\ell}}$ when $x_{\boldsymbol{\ell}}$ varies in a compact set avoiding 0.

Suppose now that we are in the case **(b)** of assumption **(A7)**. Of course, from (7.29), we have $\varphi_j\big(y^\ell(x_{\boldsymbol{\ell}})\big) = 0$ for all $j < \widehat{\jmath}$. On the other hand, Corollary 6.8 and (6.111) imply

$$\varphi_{\widehat{\jmath},2}^k(x_\ell^+(s_0)) = e^{-2\lambda_1(s-s_0)}\varphi_{\widehat{\jmath},2}^k(x_\ell^+(s)). \tag{7.31}$$

Combining this with (6.126), we get

$$\begin{aligned}\varphi_{\widehat{\jmath},2}^k(x_\ell^+(s_0)) =& e^{2\lambda_1 s_0} e^{-2\lambda_1 s}\Big(- \frac{1}{8\lambda_1} \sum_{\substack{j\in\mathcal{I}_1(2\lambda_1)\\ \alpha,\beta\in\mathcal{I}_2(\lambda_1)}} \frac{\partial^{\alpha+1_j}V(0)}{\alpha!}\frac{\partial^{\beta+1_j}V(0)}{\beta!}\big(g_1^-(z_k^-)\big)^\alpha\big(g_1^+(z_\ell^+)\big)^\beta e^{2\lambda_1 s}\\ &+ \mathcal{O}(e^{3\lambda_1 s})\Big)\\ =& - \frac{1}{8\lambda_1} \sum_{\substack{j\in\mathcal{I}_1(2\lambda_1)\\ \alpha,\beta\in\mathcal{I}_2(\lambda_1)}} \frac{\partial^{\alpha+1_j}V(0)}{\alpha!}\frac{\partial^{\beta+1_j}V(0)}{\beta!}\big(g_1^-(z_k^-)\big)^\alpha\big(g_1^+(z_\ell^+)\big)^\beta e^{2\lambda_1 s_0}. \end{aligned} \tag{7.32}$$

In particular, (7.22) becomes, in that case,

$$\begin{aligned}\Phi\big(y^{\ell}(t,x_{\boldsymbol{\ell}})\big) =& S_k^- - S_\ell^+ - \frac{1}{8\lambda_1} \sum_{\substack{j\in\mathcal{I}_1(2\lambda_1)\\ \alpha,\beta\in\mathcal{I}_2(\lambda_1)}} \frac{\partial^{\alpha+1_j}V(0)}{\alpha!}\frac{\partial^{\beta+1_j}V(0)}{\beta!}\big(g_1^-(z_k^-)\big)^\alpha\big(g_1^+(z_\ell^+)\big)^\beta e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}\\ &\times t^2 e^{-2\lambda_1 t} + \mathcal{O}(te^{-2\lambda_1 t})\\ =& S_k^- + S_\ell^+ + \mathcal{M}_2(k,\ell)t^2 e^{-2\lambda_1 t} + \mathcal{O}(te^{-2\lambda_1 t}). \end{aligned} \tag{7.33}$$

As in (7.30), the term $\mathcal{O}(te^{-2\lambda_1 t})$ is in fact expandible uniformly with respect to $x_{\boldsymbol{\ell}}$ when $x_{\boldsymbol{\ell}}$ varies in a compact set avoiding 0.

Eventually, we suppose that we are in the case **(c)** of assumption **(A7)**. Then we obtain from (7.29) and (7.32) that $\varphi_j\big(y^\ell(x_{\boldsymbol{\ell}})\big) = 0$ for all $j < \widehat{\jmath}$ and $\varphi_{\widehat{\jmath},2}\big(y^\ell(x_{\boldsymbol{\ell}})\big) = 0$. With the last identity in mind, Equation (6.111) on $\varphi_{\widehat{\jmath},1}^k$ implies

$$\varphi_{\widehat{\jmath},1}^k(x_\ell^+(s_0)) = e^{-2\lambda_1(s-s_0)}\varphi_{\widehat{\jmath},1}^k(x_\ell^+(s)). \tag{7.34}$$

In order to compute $\varphi_{\widehat{\jmath},1}^k(x_\ell^+(s))$, we put the expansion (2.17) for $x_\ell^+(s)$ (with Proposition 6.11 in mind) into (6.127). The third term in (6.127) will be, at least, $\mathcal{O}(e^{(\mu_2+\mu_1)s}) = o(e^{2\lambda_1 s})$. Thank to (6.91) and thanks to the fact that $\mathcal{M}_2(k,\ell) = 0$, the first term in (6.127) will give no contribution of order $se^{2\lambda_1 s}$ and will be of the form

$$-4\lambda_1 g_{\widehat{\jmath},1}^-(z_k^-)\cdot x_\ell^+(s) = - \sum_{\substack{j\in\mathcal{I}_1\\ \alpha\in\mathcal{I}_2(\lambda_1)}} \frac{\partial_j\partial^\alpha V(0)}{\alpha!}(g_1^-(z_k^-))^\alpha (g_{\widehat{\jmath},0}^+(z_\ell^+))_j e^{2\lambda_1 s} + \widetilde{\mathcal{O}}(e^{\mu_{\widehat{\jmath}+1}s}) \tag{7.35}$$

It remains to study the contribution the second term in (6.127), as given in (6.143). As previously, the first term of the third line in (6.143) will give a term of order $o(e^{2\lambda_1 s})$. The other terms will contribute to the order $e^{2\lambda_1 s}$ for

$$
\begin{aligned}
- \sum_{\substack{j\in\mathcal{I}_1\\ \alpha\in\mathcal{I}_2(\lambda_1)}} & \frac{\partial_j\partial^\alpha V(0)}{\alpha!}(g^-_{\widehat{\jmath},0}(z^-_k))_j(g^+_1(z^+_\ell))^\alpha + \sum_{\alpha,\beta\in\mathcal{I}_2(\lambda_1)} \frac{(g^-_1(z^-_k))^\alpha}{\alpha!}\frac{(g^+_1(z^+_\ell))^\beta}{\beta!}\times\\
& \times\Big(-\partial^{\alpha+\beta}V(0) + \sum_{j\in\mathcal{I}_1\setminus\mathcal{I}_1(2\lambda_1)} \frac{4\lambda_1^2}{\lambda_j^2(4\lambda_1^2-\lambda_j^2)}\partial_j\partial^\alpha V(0)\partial_j\partial^\beta V(0)\\
& \qquad - \sum_{\substack{j\in\mathcal{I}_1\\ \gamma,\delta\in\mathcal{I}_2(\lambda_1)\\ \gamma+\delta=\alpha+\beta}} \frac{(\gamma+\delta)!}{\gamma!\,\delta!}\frac{1}{2\lambda_j^2}\partial_j\partial^\gamma V(0)\partial_j\partial^\delta V(0)\Big). \qquad (7.36)
\end{aligned}
$$

Thus, combining (7.35) and (7.36), the discussion above leads to

$$
\begin{aligned}
\varphi^k_{\widehat{\jmath},1}(x^+_\ell(s_0)) =& e^{2\lambda_1 s_0}e^{-2\lambda_1 s}\big(\mathcal{M}_1(k,\ell)e^{2\lambda_1 s} + o(e^{2\lambda_1 s})\big)\\
=& \mathcal{M}_1(k,\ell)e^{2\lambda_1 s_0}. \qquad (7.37)
\end{aligned}
$$

In particular, (7.22) becomes, in that case,

$$
\Phi\big(y^\ell(t,x_{\boldsymbol\ell})\big) = S^-_k + S^+_\ell + \mathcal{M}_1(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol\ell})}te^{-2\lambda_1 t} + \mathcal{O}(e^{-2\lambda_1 t}). \qquad (7.38)
$$

As above, the $\mathcal{O}(e^{-2\lambda_1 t})$ is expandible uniformly with respect to the variable $x_{\boldsymbol\ell}$ when $x_{\boldsymbol\ell}$ varies in a compact set avoiding 0.

7.3. **Integration with respect to time.**

Now we perform the integration with respect to time t in (7.23). We follow the ideas of [20, Section 5] and [5, Section 6]. Since $y^\ell(t,x_{\boldsymbol\ell})$ is expandible (see (7.19)), and since Φ is C^∞ outside of $x_{\boldsymbol\ell}=0$, the symbol $f^\ell(t,x_{\boldsymbol\ell},h)$ is expandible.

We compute only the contribution of the principal symbol (with respect to h) of f^ℓ, since the other terms can be treated the same way, and the remainder term will give a contribution $\mathcal{O}(h^\infty)$ to the scattering amplitude. In other word, we compute

$$
\mathcal{A}^{sing}_0 = \frac{\widetilde{c}(E)h^{-(n+1)/2}}{\sqrt{2\pi h}}(2\pi h)^{(n-1)/2}h\iint e^{i\Phi(y^\ell(t,x_{\boldsymbol\ell}))/h}f^\ell_0(t,x_{\boldsymbol\ell})\,dt\,dx_{\boldsymbol\ell} + \mathcal{O}(h^\infty). \qquad (7.39)
$$

First, we assume that we are in the case **(a)** of the assumption **(A7)**. In that case, Φ is given by (7.30). For $x_{\boldsymbol\ell}$ fixed in a compact set away from 0, we set

$$
\begin{aligned}
\tau =& \Phi\big(y^\ell(t,x_{\boldsymbol\ell})\big) - (S^-_k + S^+_\ell)\\
=& -2\mu_{\mathbf{k}}\big\langle g^-_{\mathbf{k}}(z^-_k)\big|g^+_{\mathbf{k}}(z^+_\ell)\big\rangle e^{\mu_{\mathbf{k}}s(x_{\boldsymbol\ell})}e^{-\mu_{\mathbf{k}}t} + R(t,x_{\boldsymbol\ell}), \qquad (7.40)
\end{aligned}
$$

and we perform the change of variable $t\mapsto\tau$ in (7.39). We assume for a moment that

$$
\big\langle g^-_{\mathbf{k}}(z^-_k)\big|g^+_{\mathbf{k}}(z^+_\ell)\big\rangle < 0. \qquad (7.41)
$$

Here $R(t,x_{\boldsymbol{\ell}}) = \widetilde{\mathcal{O}}(e^{-\mu_{\mathbf{k}+1}t})$ is expandible. As in [20, Section 5] and [5, Section 6], we get

$$e^{-t} \sim \big(-2\mu_{\mathbf{k}} \big\langle g_{\mathbf{k}}^{-}(z_k^{-}) \big| g_{\mathbf{k}}^{+}(z_\ell^{+}) \big\rangle e^{\mu_{\mathbf{k}} s(x_{\boldsymbol{\ell}})} \big)^{-1/\mu_{\mathbf{k}}} \tau^{1/\mu_{\mathbf{k}}} \Big(1 + \sum_{j=1}^{\infty} \tau^{\widehat{\mu}_j/\mu_{\mathbf{k}}} b_j(-\ln \tau, x_{\boldsymbol{\ell}}) \Big) \tag{7.42}$$

$$t \sim -\frac{1}{\mu_{\mathbf{k}}} \ln \tau + \frac{1}{\mu_{\mathbf{k}}} \ln \big(-2\mu_{\mathbf{k}} \big\langle g_{\mathbf{k}}^{-}(z_k^{-}) \big| g_{\mathbf{k}}^{+}(z_\ell^{+}) \big\rangle e^{\mu_{\mathbf{k}} s(x_{\boldsymbol{\ell}})} \big) + \sum_{j=1}^{\infty} \tau^{\widehat{\mu}_j/\mu_{\mathbf{k}}} b_j(-\ln \tau, x_{\boldsymbol{\ell}}) \tag{7.43}$$

$$\tau \frac{dt}{d\tau} \sim -\frac{1}{\mu_{\mathbf{k}}} + \sum_{j=1}^{\infty} \tau^{\widehat{\mu}_j/\mu_{\mathbf{k}}} b_j(-\ln \tau, x_{\boldsymbol{\ell}}), \tag{7.44}$$

where the b_j's change from line to line. These expansions are valid in the following sense:

Definition 7.1. *Let $f(\tau,y)$ be defined on $]0,\varepsilon[\times U$ where $U \subset \mathbb{R}^m$. We say that $f = \widehat{\mathcal{O}}(g(\tau))$ (resp. $f = \widehat{o}(g(\tau))$), where $g(\tau)$ is a non-negative function defined in $]0,\varepsilon[$ if and only if for all $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}^m$,*

$$(\tau\partial_\tau)^\alpha \partial_y^\beta f(\tau,y) = \mathcal{O}(g(\tau)), \tag{7.45}$$

(resp. $o(g(\tau))$) for all $(\tau,y) \in]0,\varepsilon[\times U$.

Thus, an expression like $f \sim \sum_{j=1}^{\infty} \tau^{\widehat{\mu}_j/\mu_{\mathbf{k}}} f_j(-\ln\tau, x_{\boldsymbol{\ell}})$, where $f_j(-\ln\tau, x_{\boldsymbol{\ell}})$ is a polynomial with respect to $\ln\tau$, as in (7.42)–(7.44), means that, for all $J \in \mathbb{N}$,

$$f(\tau,x) - \sum_{j=0}^{J} \tau^{\widehat{\mu}_j/\mu_{\mathbf{k}}} f_j(-\ln\tau, x_{\boldsymbol{\ell}}) = \widehat{\mathcal{O}}(\tau^{\widehat{\mu}_J/\mu_{\mathbf{k}}}). \tag{7.46}$$

We shall call that such symbols f expandible near 0.

Since $f_0^\ell(t,x_{\boldsymbol{\ell}})$ is expandible (see Definition 6.1) with respect to t, this symbol is also expandible near 0 with respect to τ in the sense of Definition 7.1. In particular, we get

$$\widetilde{f}_0^\ell(\tau,x_{\boldsymbol{\ell}}) = -f_0^\ell(t,x_{\boldsymbol{\ell}})\tau\frac{dt}{d\tau} \sim \sum_{j=0}^{\infty} \tau^{(\Sigma(E)+\widehat{\mu_j})/\mu_{\mathbf{k}}} \widetilde{f}_{0,j}^\ell(-\ln\tau, x_{\boldsymbol{\ell}}), \tag{7.47}$$

where the $\widetilde{f}_{0,j}^\ell$'s are polynomials with respect to $\ln\tau$. The principal symbol $\widetilde{f}_{0,0}^\ell$ is independent of $\ln\tau$ and we have

$$\widetilde{f}_{0,0}^\ell(x_{\boldsymbol{\ell}}) = \frac{1}{\mu_{\mathbf{k}}} \big(-2\mu_{\mathbf{k}} \big\langle g_{\mathbf{k}}^{-}(z_k^{-}) \big| g_{\mathbf{k}}^{+}(z_\ell^{+}) \big\rangle e^{\mu_{\mathbf{k}} s(x_{\boldsymbol{\ell}})} \big)^{-\Sigma(E)/\mu_{\mathbf{k}}} f_{0,0}^\ell(x_{\boldsymbol{\ell}}). \tag{7.48}$$

In that case, (7.39) becomes

$$\mathcal{A}_0^{sing} = \frac{\widetilde{c}(E)h^{-1/2}}{(2\pi)^{1-n/2}} e^{i(S_k^- + S_\ell^+)/h} \iint_0^{+\infty} e^{i\tau/h} \widetilde{f}_0^\ell(\tau,x_{\boldsymbol{\ell}}) \, \frac{d\tau}{\tau} \, dx_{\boldsymbol{\ell}} + \mathcal{O}(h^\infty). \tag{7.49}$$

Note that $\widetilde{f}_0^\ell(\tau,x_{\boldsymbol{\ell}})$ has in fact a compact support with respect to τ. Now, using Lemma D.1, we can perform the integration with respect to t of each term in the right hand side of (7.47), modulo a term $\mathcal{O}(h^\infty)$ (see (D.3)–(D.4) in Lemma D.1). Then, we get

$$\mathcal{A}_0^{sing} = \frac{\widetilde{c}(E)h^{-1/2}}{(2\pi)^{1-n/2}} e^{i(S_k^- + S_\ell^+)/h} \sum_{j=0}^{+\infty} \widehat{f}_j(\ln h) h^{(\Sigma(E)+\widehat{\mu}_j)/\mu_{\mathbf{k}}}, \tag{7.50}$$

where $\widehat{f}_j(\ln h)$ is a polynomial with respect to $\ln h$. The function $\widehat{f}_0$ does not depend on h and we have

$$\widehat{f}_0 = \Gamma(\Sigma(E)/\mu_{\mathbf{k}})(-i)^{-\Sigma(E)/\mu_{\mathbf{k}}} \int \widetilde{f}^{\ell}_{0,0}(x_{\boldsymbol{\ell}})\, dx_{\boldsymbol{\ell}}. \tag{7.51}$$

To finish the proof, it remains to perform the integration with respect to $x_{\boldsymbol{\ell}}$. From (7.25) and (7.48), (7.51) becomes

$$\begin{aligned}\widehat{f}_0 =&\Gamma(\Sigma(E)/\mu_{\mathbf{k}})\frac{1}{\mu_{\mathbf{k}}} \int \big(2i\mu_{\mathbf{k}} \langle g^-_{\mathbf{k}}(z^-_k) \big| g^+_{\mathbf{k}}(z^+_\ell) \rangle e^{\mu_{\mathbf{k}} s(x_{\boldsymbol{\ell}})}\big)^{-\Sigma(E)/\mu_{\mathbf{k}}} \\ &\times \alpha_{0,0}(y^\ell(x_{\boldsymbol{\ell}}))\overline{\widetilde{a}_{+,0}}(y^\ell(x_{\boldsymbol{\ell}}))e^{-i\nu^+_\ell \pi/2} \frac{e^{i \operatorname{sgn} \Phi''_{|H^\ell_{x_{\boldsymbol{\ell}}}}(y^\ell(x_{\boldsymbol{\ell}}))\pi/4}}{\big| \det \Phi''_{|H^\ell_{x_{\boldsymbol{\ell}}}}(y^\ell(x_{\boldsymbol{\ell}}))\big|^{1/2}}\, dx_{\boldsymbol{\ell}}.\end{aligned} \tag{7.52}$$

Now we make the change of variable $x_{\boldsymbol{\ell}} \mapsto s$ given by $y^\ell(x_{\boldsymbol{\ell}}) = x^+_\ell(s)$ (then $s(x_{\boldsymbol{\ell}}) = s$). In particular,

$$dx_{\boldsymbol{\ell}} = \partial_s(x^+_{\ell,\boldsymbol{\ell}}(s))ds = \lambda_{\boldsymbol{\ell}} |g^+_{\boldsymbol{\ell}}(z^+_\ell)| e^{\lambda_{\boldsymbol{\ell}} s}(1+o(1))ds, \tag{7.53}$$

as $s \to -\infty$. In this setting, we get

$$\alpha_{0,0}(x^+_\ell(s)) = \alpha_{0,0}(0)(1+o(1)), \tag{7.54}$$

as $s \to -\infty$, where $\alpha_{0,0}(0)$ is given in (6.8). We also have, from (7.12) and (7.16),

$$\overline{\widetilde{a}_{+,0}}(x^+_\ell(s)) = -i\partial_s\big(\widetilde{\chi}_+(\gamma^+_\ell(s))\big)(2E_0)^{1/4}(D^+_\ell(s))^{-1/2}e^{-isE_1}. \tag{7.55}$$

Then, substituting (7.26), (7.27), (7.53), (7.54) and (7.55) in (7.52), we obtain

$$\begin{aligned}\widehat{f}_0 =&\Gamma(\Sigma(E)/\mu_{\mathbf{k}})\frac{-i}{\mu_{\mathbf{k}}} \int \big(2i\mu_{\mathbf{k}} \langle g^-_{\mathbf{k}}(z^-_k) \big| g^+_{\mathbf{k}}(z^+_\ell) \rangle\big)^{-\Sigma(E)/\mu_{\mathbf{k}}} \alpha_{0,0}(0)\partial_s\big(\widetilde{\chi}_+(\gamma^+_\ell(s))\big)e^{-i\nu^+_\ell \pi/2} \\ &\times \frac{e^{i(n-1)\pi/4}}{\sqrt{\prod_{j\neq \boldsymbol{\ell}} 2\lambda_j}} \lambda_{\boldsymbol{\ell}}|g^+_{\boldsymbol{\ell}}(z^+_\ell)|(2E_0)^{1/4}(D^+_\ell(s))^{-1/2}e^{-isE_1}e^{-\Sigma(E)s}e^{\lambda_{\boldsymbol{\ell}} s}(1+o(1))\, ds \\ =& -\frac{e^{i(n+1)\pi/4}}{\mu_{\mathbf{k}}} \Big(\prod_{j\neq\boldsymbol{\ell}} 2\lambda_j\Big)^{-1/2} \lambda_{\boldsymbol{\ell}}|g^+_{\boldsymbol{\ell}}(z^+_\ell)|\Gamma(\Sigma(E)/\mu_{\mathbf{k}})\big(2i\mu_{\mathbf{k}} \langle g^-_{\mathbf{k}}(z^-_k) \big| g^+_{\mathbf{k}}(z^+_\ell) \rangle\big)^{-\Sigma(E)/\mu_{\mathbf{k}}} \\ &\times e^{-i\nu^+_\ell \pi/2}\alpha_{0,0}(0)(2E_0)^{1/4}(D^+_\ell)^{-1/2}\int \partial_s\big(\widetilde{\chi}_+(\gamma^+_\ell(s))\big)(1+o(1))\, ds.\end{aligned} \tag{7.56}$$

Here the $o(1)$ does not depend on $\widetilde{\chi}_+$. Now, we choose a family of cut-off functions $(\widetilde{\chi}^j_+)_{j\in\mathbb{N}}$ such that the support of $\partial_t\big(\widetilde{\chi}^j_+(\gamma^+_\ell(t))\big)$ goes to $-\infty$ as $j \to +\infty$. We also assume that $\partial_t\big(\widetilde{\chi}^j_+(\gamma^+_\ell(t))\big)$ is non-positive (see Figure 1). Then

$$\begin{aligned}\widehat{f}_0 =& -\frac{e^{i(n+1)\pi/4}}{\mu_{\mathbf{k}}} \Big(\prod_{j\neq\boldsymbol{\ell}} 2\lambda_j\Big)^{-1/2} \lambda_{\boldsymbol{\ell}}\Gamma(\Sigma(E)/\mu_{\mathbf{k}})e^{-i\nu^+_\ell \pi/2}e^{i\pi/4}(2\lambda_1)^{3/2}e^{-i\nu^-_k\pi/2} \\ &\times |g^-_1(z^-_k)|\; |g^+_{\boldsymbol{\ell}}(z^+_\ell)|\big(2i\mu_{\mathbf{k}} \langle g^-_{\mathbf{k}}(z^-_k) \big| g^+_{\mathbf{k}}(z^+_\ell) \rangle\big)^{-\Sigma(E)/\mu_{\mathbf{k}}} \\ &\times (2E_0)^{1/2}(D^-_k D^+_\ell)^{-1/2} \times (1+o(1)).\end{aligned} \tag{7.57}$$

as $j \to +\infty$. Since $\widehat{f}_0$ is also independent of $\widetilde{\chi}_+$, we obtain Theorem 2.6 from (7.50) and (7.51), in the case **(a)** and under the assumption (7.41). When $\langle g_{\mathbf{k}}^-(z_k^-)|g_{\mathbf{k}}^+(z_\ell^+)\rangle > 0$, we set τ as the opposite of the R.H.S. of (7.40), and we obtain the result along the same lines (see Remark D.2).

Now we assume that we are in the case **(b)** of the assumption **(A7)**. In that case, the phase function Φ is given by (7.33). For $x_{\boldsymbol{\ell}}$ fixed in a compact set outside from 0, we set, mimicking (7.40),

$$\begin{aligned} \tau =& \Phi\big(y^\ell(t,x_{\boldsymbol{\ell}})\big) - (S_k^- + S_\ell^+) \\ =& \mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}t^2 e^{-2\lambda_1 t} + R(t,x_{\boldsymbol{\ell}}) \end{aligned} \tag{7.58}$$

where $R(t,x_{\boldsymbol{\ell}}) = \mathcal{O}(te^{-2\lambda_1 t})$ is expandible with respect to t. As above, we assume that $\mathcal{M}_2(k,\ell)$ is positive (the other case can be studied the same way).

Following (7.42), we want to write $s := e^{-t}$ as a function of τ. Since $t \mapsto \tau(t)$ is expandible with respect to t, we have

$$\tau = \mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}(\ln s)^2 s^{2\lambda_1}(1 + r(s,x_{\boldsymbol{\ell}})), \tag{7.59}$$

where $r(s,x_{\boldsymbol{\ell}}) = \widehat{o}(1)$. In particular, $\partial_s\tau > 0$ for s positive small enough and then, for $\varepsilon > 0$ small enough, $s \mapsto \tau(s)$ is invertible for $0 < s < \varepsilon$. We denote by $s(\tau)$ the inverse of this function. We look for $s(\tau)$ of the form

$$s(\tau) = (2\lambda_1)^{1/\lambda_1}\Big(\frac{\tau}{\mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}}\Big)^{1/2\lambda_1}\frac{u(\tau,x_{\boldsymbol{\ell}})}{(-\ln\tau)^{1/\lambda_1}}, \tag{7.60}$$

where $u(\tau,x_{\boldsymbol{\ell}})$ has to be determined. Using (7.59), the equation for u is

$$\begin{aligned} \tau =& \mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}(\ln s)^2 s^{2\lambda_1}(1 + r(s,x_{\boldsymbol{\ell}})) \\ =& \tau u^{2\lambda_1}\Big(1 - \frac{\ln\big((2\lambda_1)^{-2}\mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}\big)}{\ln\tau} + 2\lambda_1\frac{\ln u}{\ln\tau} - 2\frac{\ln(-\ln\tau)}{\ln\tau}\Big)^2 \\ &\times\Big(1 + r\Big((2\lambda_1)^{1/\lambda_1}\Big(\frac{\tau}{\mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}}\Big)^{1/2\lambda_1}\frac{u}{(-\ln\tau)^{1/\lambda_1}},x_{\boldsymbol{\ell}}\Big)\Big) \\ =& \tau F(\tau,u,x_{\boldsymbol{\ell}}), \end{aligned} \tag{7.61}$$

where $F = u^{2\lambda_1}(1 + \widetilde{r}(\tau,u,x_{\boldsymbol{\ell}}))$ and $\widetilde{r} = \widehat{o}(1)$ for u close to 1 (here $(u,x_{\boldsymbol{\ell}})$ are the variables y in Definition 7.1). In other word, to find u, we have to solve $F(t,u,x_{\boldsymbol{\ell}}) = 1$.

First we remark that $u \mapsto F(\tau,u,x_{\boldsymbol{\ell}})$ is real-valued and continuous. Since, for $\delta > 0$ and τ small enough, $F(\tau,1-\delta,x_{\boldsymbol{\ell}}) < 1 < F(\tau,1+\delta,x_{\boldsymbol{\ell}})$, there exists $u \in [1-\delta,1+\delta]$ such that $F(\tau,1+\delta,x_{\boldsymbol{\ell}}) = 1$. Thanks to the discussion before (7.60), the function $s(\tau)$ is of the form (7.60) with $u(\tau,x_{\boldsymbol{\ell}}) \in [1-\delta,1+\delta]$, for τ small enough.

For $\tau > 0$, the function F is C^∞ and, since $\widetilde{r} = \widehat{o}(1)$, we have

$$\partial_u\big(F(\tau,u,x_{\boldsymbol{\ell}}) - 1\big)(u(\tau,x_{\boldsymbol{\ell}})) = 2\lambda_1 u^{2\lambda_1 - 1}(1 + o_\tau(1)) > \lambda_1, \tag{7.62}$$

for τ small enough. The notation $o_\tau(1)$ means a term which goes to 0 as τ goes to 0. Here we have used the fact that $u(\tau,x_{\boldsymbol{\ell}})$ is close to 1. In particular, the implicit function theorem implies that $u(\tau,x_{\boldsymbol{\ell}})$ is C^∞.

We write $u = 1 + v(\tau, x_{\boldsymbol{\ell}})$ and we known that $v \in C^\infty$ and $v = o_\tau(1)$. Differentiating the equality

$$1 = F(\tau, u(\tau, x_{\boldsymbol{\ell}}), x_{\boldsymbol{\ell}}) = \big(u(\tau, x_{\boldsymbol{\ell}})\big)^{2\lambda_1}\big(1 + \widetilde{r}(\tau, u(\tau, x_{\boldsymbol{\ell}}), x_{\boldsymbol{\ell}})\big), \tag{7.63}$$

one can show that $v = \widehat{o}(1)$. Thus we have

$$e^{-t} = s(\tau) = (2\lambda_1)^{1/\lambda_1}\Big(\frac{\tau}{\mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}}\Big)^{1/2\lambda_1}\frac{1 + \widehat{r}(\tau, x_{\boldsymbol{\ell}})}{(-\ln\tau)^{1/\lambda_1}}, \tag{7.64}$$

$$t = -\frac{\ln\tau}{2\lambda_1}(1 + \widehat{r}(\tau, x_{\boldsymbol{\ell}})), \tag{7.65}$$

$$\tau\frac{dt}{d\tau} = -\frac{1}{2\lambda_1} + \widehat{r}(\tau, x_{\boldsymbol{\ell}}), \tag{7.66}$$

where $\widehat{r}(\tau, x_{\boldsymbol{\ell}}) = \widehat{o}(1)$ change from line to line.

Since $f_0^\ell(t, x_{\boldsymbol{\ell}}, h)$ is expandible with respect to t, we get, from (7.64)–(7.66),

$$\widetilde{f}_0^\ell(\tau, x_{\boldsymbol{\ell}}) = -f_0^\ell(t, x_{\boldsymbol{\ell}})\tau\frac{dt}{d\tau} = \tau^{\Sigma(E)/2\lambda_1}(-\ln\tau)^{-\Sigma(E)/\lambda_1}\big(\widetilde{f}_{0,0}^\ell(x_{\boldsymbol{\ell}}) + \widehat{r}(\tau, x_{\boldsymbol{\ell}})\big), \tag{7.67}$$

where $\widehat{r} = \widehat{o}(1)$ and

$$\widetilde{f}_{0,0}^\ell(x_{\boldsymbol{\ell}}) = (2\lambda_1)^{\Sigma(E)/\lambda_1 - 1}\big(\mathcal{M}_2(k,\ell)e^{2\lambda_1 s(x_{\boldsymbol{\ell}})}\big)^{-\Sigma(E)/2\lambda_1}f_{0,0}^\ell(x_{\boldsymbol{\ell}}). \tag{7.68}$$

In that case, (7.39) becomes

$$\mathcal{A}_0^{sing} = \frac{\widetilde{c}(E)h^{-1/2}}{(2\pi)^{1-n/2}}e^{i(S_k^- + S_\ell^+)/h}\iint_0^{+\infty} e^{i\tau/h}\widetilde{f}_0^\ell(\tau, x_{\boldsymbol{\ell}})\,\frac{d\tau}{\tau}\,dx_{\boldsymbol{\ell}} + \mathcal{O}(h^\infty). \tag{7.69}$$

Note that $\widetilde{f}_0^\ell(\tau, x_{\boldsymbol{\ell}})$ has in fact a compact support with respect to τ. Now, using Lemma D.1, we can perform the integration with respect to t in (7.69), modulo an error term given by (D.3)–(D.4) in Lemma D.1. Then, we get

$$\begin{aligned}\mathcal{A}_0^{sing} =& \frac{\widetilde{c}(E)h^{-1/2}}{(2\pi)^{1-n/2}}e^{i(S_k^- + S_\ell^+)/h}\Gamma(\Sigma(E)/2\lambda_1)(-i)^{-\Sigma(E)/2\lambda_1}\\ &\times h^{\Sigma(E)/2\lambda_1}(-\ln h)^{-\Sigma(E)/\lambda_1}\Big(\int\widetilde{f}_{0,0}^\ell(x_{\boldsymbol{\ell}})\,dx_{\boldsymbol{\ell}} + o(1)\Big),\end{aligned} \tag{7.70}$$

as h goes to 0. The rest of the proof follows that of (7.57).

Lastly, the proof of Theorem 2.6 in the case **(c)** can be obtained along the same lines, and we omit it.

Appendix A. Proof of Proposition 2.5

We prove that $\Lambda_\theta^+ \cap \Lambda_+ \neq \emptyset$. From Assumption (A2), the Lagrangian manifold Λ_+ can be described, near $(0,0) \in T^*(\mathbb{R}^n)$, as

$$\Lambda_+ = \{(x,\xi);\ x = \nabla\widetilde{\varphi}_+(\xi)\}, \tag{A.1}$$

for $|\xi| < 2\varepsilon$, with $\varepsilon > 0$ small enough. For $\eta \in \mathbb{S}^{n-1}$, let $(x(t,\eta), \xi(t,\eta))$ be the bicharacteristic curve with initial condition $(\widetilde{\varphi}(\varepsilon\eta), \varepsilon\eta)$. We have

$$\Lambda_+ = \{(x(t,\eta), \xi(t,\eta));\ t \in \mathbb{R},\ \eta \in \mathbb{S}^{n-1}\} \cup \{(0,0)\}. \tag{A.2}$$

The function $\xi(t,\eta)$ is continuous on $\mathbb{R}\times\mathbb{S}^{n-1}$. From the classical scattering theory (see [13, Section 1.3]), we know that this function $\xi(t,\eta)$ converges uniformly to

$$\xi(\infty,\eta) := \lim_{t\to+\infty}\xi(t,\eta), \tag{A.3}$$

as $t\to+\infty$ and $\xi(\infty,\eta)\in\sqrt{2E}\,\mathbb{S}^{n-1}$.

Then, the function

$$F(t,\eta)=\frac{\xi(\frac{t}{1-t},\eta)}{|\xi(\frac{t}{1-t},\eta)|}, \tag{A.4}$$

is well defined for $0\le t\le 1$ with the convention $F(1,\eta)=\xi(\infty,\eta)/\sqrt{2E}$. Here we used that $|\xi(t,\eta)|\neq 0$ for each $t\in[0,+\infty]$, $\eta\in\mathbb{S}^{n-1}$. The previous properties of $\xi(t,\eta)$ imply the continuity of $F(t,\eta)$ on $[0,1]\times\mathbb{S}^{n-1}$.

From (A.2), to prove that $\Lambda_\theta^+\cap\Lambda_+\neq\emptyset$ for all $\theta\in\mathbb{S}^{n-1}$, it is enough (equivalent) to show the surjectivity of $\eta\mapsto F(1,\eta)$. But if $\eta\mapsto F(1,\eta)$ is not onto, then $\operatorname{Im}F(1,\cdot)\subset\mathbb{S}^{n-1}\setminus\{\text{a point}\}$. And since $\mathbb{S}^{n-1}\setminus\{\text{a point}\}$ is a contractible space, $F(1,\cdot)$ is homotopic to a constant map

$$f:\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}. \tag{A.5}$$

On the other hand, $F:[0,1]\times\mathbb{S}^{n-1}\longrightarrow\mathbb{S}^{n-1}$ gives a homotopy between $F(0,\cdot)=\mathrm{Id}_{\mathbb{S}^{n-1}}$ and $F(1,\cdot)$. In particular, we have

$$1=\deg(F(0,\cdot))=\deg(F(1,\cdot))=\deg(f(\cdot))=0, \tag{A.6}$$

which is impossible (see [16, Section 23] for more details).

Appendix B. A lower bound for the resolvent

Let $\chi\in C^\infty(]0,+\infty[)$ be a non-decreasing function such that

$$\chi(x)=\begin{cases}x & \text{for } 0<x<1\\ 2 & \text{for } 2<x,\end{cases} \tag{B.1}$$

Let also $\varphi\in C_0^\infty(\mathbb{R})$ be an even function such that $0\le\varphi\le 1$, $1_{[-1,1]}\prec\varphi$, and $\operatorname{supp}\varphi\subset[-2,2]$. We set

$$u(x)=\prod_{j=1}^n e^{i\lambda_j x_j^2/2h}\varphi\Big(\frac{x_j}{h^\alpha}\Big)\chi\Big(\frac{h^\beta}{|x_j|^{1/2}}\Big)=\prod_{j=1}^n u_j(x), \tag{B.2}$$

where $0<\alpha<2\beta$ will be fixed later on. The u_j's are of course C^∞ functions, and we have

$$(P-E_0)u=-\frac{h^2}{2}\Delta u(x)-\sum_{j=1}^n\frac{\lambda_j^2}{2}x_j^2u(x)+\mathcal{O}(x^3u(x)). \tag{B.3}$$

Lemma B.1. *For any h small enough, we have*

$$h^{\beta n}|\ln h|^{n/2}\lesssim\|u\|_{L^2(\mathbb{R}^n)}\lesssim h^{\beta n}|\ln h|^{n/2}, \tag{B.4}$$

$$\big\||x|^3u(x)\big\|_{L^2(\mathbb{R}^n)}\lesssim h^{3\alpha}h^{\beta n}|\ln h|^{n/2}. \tag{B.5}$$

Proof. First of all, the second estimate follow easily from the first one: we have

$$\big\| |x|^3 u(x) \big\|^2 = \int_{\mathbb{R}^n} |x|^6 |u(x)|^2 dx \lesssim h^{6\alpha} \|u\|^2,$$

since u vanishes if $|x| > 2h^\alpha$. Thanks to the fact that u is a product of n functions of one variable, it is enough to estimate

$$I = \int \varphi^2\Big(\frac{t}{h^\alpha}\Big)\chi^2\Big(\frac{h^\beta}{|t|^{1/2}}\Big)dt = 2\int_0^{2h^\alpha} \varphi^2\Big(\frac{t}{h^\alpha}\Big)\chi^2\Big(\frac{h^\beta}{t^{1/2}}\Big)dt.$$

We have

$$2\int_{h^{2\beta}}^{h^\alpha} \chi^2\Big(\frac{h^\beta}{t^{1/2}}\Big)dt \leq I \leq 2\int_{h^{2\beta}}^{2h^\alpha} \chi^2\Big(\frac{h^\beta}{t^{1/2}}\Big)dt + 2\int_0^{h^{2\beta}} \chi^2\Big(\frac{h^\beta}{t^{1/2}}\Big)dt,$$

so that

$$2\int_{h^{2\beta}}^{h^\alpha} \frac{h^{2\beta}}{t} dt \leq I \leq 2\int_{h^{2\beta}}^{2h^\alpha} \frac{h^{2\beta}}{t} dt + 2\int_0^{h^{2\beta}} 4\, dt.$$

The first estimate follows from the fact that $2\beta - \alpha > 0$, once we have noticed that

$$\int_{h^{2\beta}}^{Ah^\alpha} \frac{h^{2\beta}}{t} dt = h^{2\beta}\big((2\beta-\alpha)|\ln h| + \ln A\big).$$

$\square$

On the other hand, we have

$$-\frac{h^2}{2}\Delta u(x) - \sum_{j=1}^n \frac{\lambda_j^2}{2} x_j^2 u(x) = \sum_{k=1}^n \prod_{j\neq k} u_j(x_j)\Big(-\frac{h^2}{2}u_k''(x_k) - \frac{\lambda_k^2}{2}x_k^2 u_k(x_k)\Big).$$

From Lemma B.1, we get

$$\begin{aligned}\big\|(P-E_0)u\big\| &\lesssim h^{\beta(n-1)}|\ln h|^{(n-1)/2} \sup_{1\leq k\leq n} \big\|h^2 u_k''(t) + \lambda_k^2 t^2 u_k(t)\big\| + h^{3\alpha}h^{\beta n}|\ln h|^{n/2} \\ &\lesssim \Big(h^{-\beta}|\ln h|^{-1/2} \sup_{1\leq k\leq n}\big\|h^2 u_k''(t) + \lambda_k^2 t^2 u_k(t)\big\| + h^{3\alpha}\Big)\|u\|. \qquad \text{(B.6)}\end{aligned}$$

We also have

$$h^2 u_k''(t) + \lambda_k^2 t^2 u_k(t) = e^{i\lambda_k t^2/2h}\big(h^2 v_h''(t) + ih\lambda_k(2t\partial_t + 1)v_h(t)\big), \qquad \text{(B.7)}$$

where we have set $v_h(t) = \varphi\big(\frac{t}{h^\alpha}\big)\chi\big(\frac{h^\beta}{|t|^{1/2}}\big)$. Notice that the right hand side of (B.7) is an even function, so that we only have to consider $t > 0$. The point here, is that we have, for $t > 0$,

$$(2t\partial_t + 1)\Big(\chi\Big(\frac{h^\beta}{t^{1/2}}\Big)\Big) = -\frac{h^\beta}{t^{1/2}}\chi'\Big(\frac{h^\beta}{t^{1/2}}\Big) + \chi\Big(\frac{h^\beta}{t^{1/2}}\Big) = \begin{cases} 2 & \text{if } 0 < t < \dfrac{h^{2\beta}}{4}, \\ \mathcal{O}(1) & \text{if } \dfrac{h^{2\beta}}{4} < t < h^{2\beta}, \\ 0 & \text{if } h^{2\beta} < t. \end{cases} \qquad \text{(B.8)}$$

Therefore, we obtain

$$\begin{aligned}\big\|(2t\partial_t+1)v_h\big\|^2 =&2\int_0^{2h^\alpha}\Big(\varphi\Big(\frac{t}{h^\alpha}\Big)(2t\partial_t+1)\Big(\chi\Big(\frac{h^\beta}{|t|^{1/2}}\Big)\Big)\Big)^2 dt\\ &+2\int_0^{2h^\alpha}\Big(2t\partial_t\Big(\varphi\Big(\frac{t}{h^\alpha}\Big)\Big)\chi\Big(\frac{h^\beta}{|t|^{1/2}}\Big)\Big)^2 dt\\ &\lesssim\int_0^{h^{2\beta}}dt+\int_{h^\alpha}^{2h^\alpha}\frac{t^2}{h^{2\alpha}}\Big(\varphi'\Big(\frac{t}{h^\alpha}\Big)\chi\Big(\frac{h^\beta}{|t|^{1/2}}\Big)\Big)^2dt\lesssim h^{2\beta}.\end{aligned}\tag{B.9}$$

On the other hand, an easy computation gives, still for $t>0$,

$$\begin{aligned}v_h''(t)=&h^{-2\alpha}\varphi''\Big(\frac{t}{h^\alpha}\Big)\chi\Big(\frac{h^\beta}{t^{1/2}}\Big)-\frac{h^{\beta-\alpha}}{t^{3/2}}\varphi'\Big(\frac{t}{h^\alpha}\Big)\chi'\Big(\frac{h^\beta}{t^{1/2}}\Big)\\ &+\frac{3h^\beta}{4t^{5/2}}\varphi\Big(\frac{t}{h^\alpha}\Big)\chi'\Big(\frac{h^\beta}{t^{1/2}}\Big)+\frac{h^{2\beta}}{4t^3}\varphi\Big(\frac{t}{h^\alpha}\Big)\chi''\Big(\frac{h^\beta}{t^{1/2}}\Big).\end{aligned}\tag{B.10}$$

Computing the L^2–norm of each of these terms as in Lemma B.1 and (B.9), we obtain

$$\|h^2v_h''\|\lesssim h^{2+\beta-2\alpha}+h^{2+\beta-2\alpha}+h^{2-3\beta}+h^{2-3\beta},\tag{B.11}$$

and, eventually, from (B.6), (B.7), (B.9) and (B.11),

$$\big\|(P-E_0)u\big\|\lesssim\Big(h^{-\beta}|\ln h|^{-1/2}\big(h^{1+\beta}+h^{2+\beta-2\alpha}+h^{2-3\beta}\big)+h^{3\alpha}\Big)\|u\|.$$

Therefore we obtain Proposition 2.2 if we can find $\alpha>0$ and $\beta>0$ such that

$$2-2\alpha>1,\;\; 2-4\beta>1,\;\; 3\alpha>1 \text{ and } 2\beta>\alpha,$$

and one can check that $\alpha=5/12$ and $\beta=11/48$ satisfies these four inequalities.

Appendix C. Lagrangian manifolds which are transverse to $\Lambda_\pm$

Let $\Lambda\subset p^{-1}(E_0)$ be a Lagrangian manifold such that $\Lambda\cap\Lambda_-$ is transverse along a Hamiltonian curve $\gamma(t)=(x(t),\xi(t))$. Then, there exist $a\neq0$ and $\nu\in\{1,\dots,n\}$ such that

$$\gamma(t)=(a+\mathcal{O}(e^{-\varepsilon t}))e^{-\lambda_\nu t},\tag{C.1}$$

as $t\to+\infty$. The vector a is an eigenvector of

$$\begin{pmatrix}0 & Id\\ V''(0) & 0\end{pmatrix},\tag{C.2}$$

for the eigenvalue λ_ν. Thus, up to a linear change of variable in $\mathbb{R}^n$, we can always assume that $\Pi_x a$ is collinear to the x_ν–direction. The goal of this section is to prove the following geometric result.

Proposition C.1. *For t large enough, Λ projects diffeomorphically on $\mathbb{R}^n_x$ near $\gamma(t)$. In particular, there exists $\psi\in C^\infty(\mathbb{R}^n)$ defined near $\Pi_x\gamma$, unique up to a constant, such that*

$\Lambda = \Lambda_\psi := \{(x, \nabla\psi(x));\ x \in \mathbb{R}^n\}$. *Moreover, we have*

$$\psi''(x(t)) = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_{\nu-1} & & & & \\ & & & -\lambda_\nu & & & \\ & & & & \lambda_{\nu+1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n \end{pmatrix} + \mathcal{O}(e^{-\varepsilon t}), \tag{C.3}$$

as $t \to +\infty$.

Remark C.2. *The same result holds in the outgoing region: If* $\gamma = \Lambda \cap \Lambda_+$ *is transverse,* Λ *projects nicely on* $\mathbb{R}^n_x$ *near* $\gamma(t)$, $t \to -\infty$. *Then* $\Lambda = \Lambda_\psi$ *for some function* ψ *satisfying* $\psi''(x(t)) = \operatorname{diag}(-\lambda_1, \dots, -\lambda_{\nu-1}, \lambda_\nu, -\lambda_{\nu+1}, \dots, -\lambda_n) + \mathcal{O}(e^{\varepsilon t})$.

Proof. We follow the proof of [20, Lemma 2.1]. There exist symplectic local coordinates (y, η) centered at $(0, 0)$ such that Λ_- (resp. Λ_+) is given by $y = 0$ (resp. $\eta = 0$) and

$$y_j = \frac{1}{\sqrt{2\lambda_j}}(\xi_j + \lambda_j x_j) + \mathcal{O}((x, \xi)^2), \tag{C.4}$$

$$\eta_j = \frac{1}{\sqrt{2\lambda_j}}(\xi_j - \lambda_j x_j) + \mathcal{O}((x, \xi)^2). \tag{C.5}$$

Then, $p(x, \xi) = A(y, \eta)y \cdot \eta$ with $A_0 := A(0, 0) = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. In particular, the tangent vectors (δ_y, δ_η) to Λ at $\gamma(t)$ satisfy the following evolution equation

$$\frac{d}{dt}\begin{pmatrix} \delta_y \\ \delta_\eta \end{pmatrix} = \begin{pmatrix} A_0 + \mathcal{O}(e^{-\lambda_1 t}) & 0 \\ \mathcal{O}(e^{-\lambda_1 t}) & A_0 + \mathcal{O}(e^{-\lambda_1 t}) \end{pmatrix} \begin{pmatrix} \delta_y \\ \delta_\eta \end{pmatrix}. \tag{C.6}$$

We denote by $U(t, s)$ the linear operator such that $U(t, s)\delta$ solves (C.6) with $U(s, s) = Id$.

Since the intersection $\Lambda \cap \Lambda_- = \gamma$ is transverse, there exists $E_{n-1}(t_0) \subset T_{\gamma(t_0)}\Lambda$, a vector space of dimension $n-1$ disjoint from $T_{\gamma(t_0)}\Lambda_-$. For convenience, we set $E_n(t_0) = E_{n-1}(t_0) \oplus \mathbb{R}v$ for some $v \notin T_{\gamma(t_0)}\Lambda + T_{\gamma(t_0)}\Lambda_-$. Let $E_\bullet(t) = U(t, t_0)E_\bullet(t_0)$. From [20, Lemma 2.1], there exists a $n \times n$ matrix $B_t = \mathcal{O}(e^{-\lambda_1 t})$ such that $E_n(t)$ is given by $\delta_\eta = B_t\delta_y$. Now, if $\delta \in E_{n-1}(t)$, we have $\sigma(H_p, \delta) = 0$ since $E_{n-1}(t) \oplus \mathbb{R}H_p = T_{\gamma(t)}\Lambda$ and Λ is a Lagrangian manifold. From (C.1), we have

$$H_p(\gamma(t)) = \dot\gamma(t) = -\lambda_\nu(\widetilde{a}e_{\eta_\nu} + \mathcal{O}(e^{-\varepsilon t}))e^{-\lambda_\nu t}, \tag{C.7}$$

where e_{η_ν} is the basis vector corresponding to η_ν, $\widetilde{a} = \pm|a|$, and then

$$0 = \sigma(e^{\lambda_\nu t}H_p, \delta) = \lambda_\nu\widetilde{a}\delta_{y_\nu} + \mathcal{O}(e^{-\varepsilon t})|\delta|. \tag{C.8}$$

It follows that $\delta \in E_{n-1}(t)$ if and only if $(\delta_{y_\nu}, \delta_\eta) = \widetilde{B}_t\delta_{y'}$ where $\widetilde{B}_t = \mathcal{O}(e^{-\varepsilon t})$ is a $(n+1) \times (n-1)$ matrix. Using $T_{\gamma(t)}\Lambda = E_{n-1}(t) \oplus \mathbb{R}H_p$, we obtain that $T_{\gamma(t)}\Lambda$ has a basis formed of vector $f_j(t)$ such that

$$f_j = e_{y_j} + \mathcal{O}(e^{-\varepsilon t}) \qquad \text{for } j \neq \nu \tag{C.9}$$

$$f_\nu = e_{\eta_\nu} + \mathcal{O}(e^{-\varepsilon t}). \tag{C.10}$$

In the (x, ξ)-coordinates, $T_{\gamma(t)}\Lambda$ has a basis formed of vector $\widetilde{f}_j(t)$ of the form

$$\widetilde{f}_j = e_{\xi_j} + \lambda_j e_{x_j} + \mathcal{O}(e^{-\varepsilon t}) \qquad \text{for } j \neq \nu \tag{C.11}$$

$$\widetilde{f}_\nu = e_{\xi_\nu} - \lambda_j e_{x_\nu} + \mathcal{O}(e^{-\varepsilon t}), \tag{C.12}$$

and the lemma follows. □

Appendix D. Asymptotic behavior of certain integrals

Lemma D.1. *Let $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $\beta \in \mathbb{R}$ and $\chi \in C_0^\infty(]-\infty, 1/2[)$ be such that $\chi = 1$ near 0. As λ goes to $+\infty$, we have*

$$\int_0^\infty e^{i\lambda t} t^\alpha (-\ln t)^\beta \chi(t) \, \frac{dt}{t} = \Gamma(\alpha)(\ln \lambda)^\beta (-i\lambda)^{-\alpha} (1 + o(1)). \tag{D.1}$$

Moreover, if $\beta \in \mathbb{N}$, we get

$$\int_0^\infty e^{i\lambda t} t^\alpha (-\ln t)^\beta \chi(t) \, \frac{dt}{t} = (-i\lambda)^{-\alpha} \sum_{j=0}^{\beta} C_\beta^j \Gamma^{(j)}(\alpha)(-1)^j \big(\ln(-i\lambda)\big)^{\beta - j} + \mathcal{O}(\lambda^{-\infty}). \tag{D.2}$$

Finally, if $s(t) \in C^\infty(]0, +\infty[)$ satisfies

$$|\partial_t^j s(t)| = o\big(t^{\alpha - j}(-\ln t)^\beta\big), \tag{D.3}$$

for all $j \in \mathbb{N}$ and $t \to 0$, then

$$\int_0^\infty e^{i\lambda t} s(t) \chi(t) \, \frac{dt}{t} = o\big((\ln \lambda)^\beta \lambda^{-\alpha}\big). \tag{D.4}$$

Here $(-i\lambda)^{-\alpha} = e^{i\alpha\pi/2}\lambda^{-\alpha}$ and $\ln(-i\lambda) = \ln \lambda - i\pi/2$.

Remark D.2. *Notice that one obtains the behavior of these quantities as $\lambda \to -\infty$ by taking the complex conjugate in these expressions.*

Proof. We begin with (D.2) and assume first that $\beta = 0$. Then, we can write

$$\begin{aligned}\int_0^\infty e^{i\lambda t} t^\alpha \chi(t) \, \frac{dt}{t} &= \lim_{\varepsilon \to 0} \int_0^\infty e^{i(\lambda + i\varepsilon)t} t^\alpha \chi(t) \, \frac{dt}{t} \\ &= \lim_{\varepsilon \to 0} \big(I_1(\alpha, \varepsilon) - I_2(\alpha, \varepsilon)\big),\end{aligned} \tag{D.5}$$

where

$$I_1(\alpha, \varepsilon) = \int_0^\infty e^{-(\varepsilon - i\lambda)t} t^\alpha \, \frac{dt}{t} \,, \tag{D.6}$$

$$I_2(\alpha, \varepsilon) = \int_0^\infty e^{i(\lambda + i\varepsilon)t} t^\alpha (1 - \chi(t)) \, \frac{dt}{t} \cdot \tag{D.7}$$

It is clear that

$$I_1(\alpha, \varepsilon) = (\varepsilon - i\lambda)^{-\alpha} \Gamma(\alpha), \tag{D.8}$$

where $z^{-\alpha}$ is defined on $\mathbb{C} \backslash]-\infty, 0]$ and is real positive on $]0, +\infty[$. In particular

$$\lim_{\varepsilon \to 0} I_1(\alpha, \varepsilon) = (-i\lambda)^{-\alpha} \Gamma(\alpha). \tag{D.9}$$

Concerning $I_2(\alpha,\varepsilon)$, we remark that $r(t,\alpha)=t^{\alpha-1}(1-\chi(t))$ is a symbol which satisfies

$$|\partial_t^j\partial_\alpha^k r(t,\alpha)|\lesssim\langle t\rangle^{\operatorname{Re}\alpha-1-j}\langle\ln t\rangle^k, \tag{D.10}$$

for all $j,k\in\mathbb{N}$ uniformly for $t\in[0,+\infty[$ and α in a compact subset of $\{\operatorname{Re}z>0\}$. Then, performing integration by parts in (D.7), we obtain

$$I_2(\alpha,\varepsilon)=\frac{1}{(\varepsilon-i\lambda)^j}\int_0^{+\infty}e^{(i\lambda-\varepsilon)t}\partial_t^j r(t,\alpha)\,dt, \tag{D.11}$$

for all $j\in\mathbb{N}$. Now, if j is large enough ($j>\operatorname{Re}\alpha$), $\partial_t^j r(t,\alpha)$ is integrable in time and does not depend on ε. In particular, for such j,

$$\lim_{\varepsilon\to0}I_2(\alpha,\varepsilon)=e^{ij\pi/2}\lambda^{-j}\int_0^{+\infty}e^{i\lambda t}\partial_t^j r(t,\alpha)\,dt, \tag{D.12}$$

and then (see (D.10) or Cauchy's formula)

$$\begin{aligned}\partial_\alpha^k\lim_{\varepsilon\to0}I_2(\alpha,\varepsilon)=&e^{ij\pi/2}\lambda^{-j}\int_0^{+\infty}e^{i\lambda t}\partial_t^j\partial_\alpha^k r(t,\alpha)\,dt\\=&\mathcal{O}(\lambda^{-\infty}),\end{aligned} \tag{D.13}$$

for all $k\in\mathbb{N}$. Then we obtain (D.2) for $\beta=0$. To obtain the result for $\beta\in\mathbb{N}$, it is enough to observe that

$$\begin{aligned}\int_0^\infty e^{i\lambda t}t^\alpha(\ln t)^\beta\chi(t)\,\frac{dt}{t}=&\partial_\alpha^\beta\int_0^\infty e^{i\lambda t}t^\alpha\chi(t)\,\frac{dt}{t}\\=&\partial_\alpha^\beta\big((-i\lambda)^{-\alpha}\Gamma(\alpha)\big)+\partial_\alpha^\beta\lim_{\varepsilon\to0}I_2(\alpha,\varepsilon)\\=&(-i\lambda)^{-\alpha}\sum_{j=0}^\beta C_\beta^j\Gamma^{(j)}(\alpha)\big(-\ln(-i\lambda)\big)^{\beta-j}+\mathcal{O}(\lambda^{-\infty}),\end{aligned} \tag{D.14}$$

from (D.13). Thus, (D.2) is proved.

Let $u\in C^\infty(]0,+\infty[)$ be such that

$$|\partial_t^j u(t)|\lesssim t^{\operatorname{Re}\alpha-j}(-\ln t)^\beta, \tag{D.15}$$

near 0. Let $\varphi\in C^\infty(\mathbb{R})$ be such that $\varphi=1$ for $t<1$ and $\varphi=0$ for $t>2$. For $\delta>0$, we have

$$\int_0^{+\infty}e^{i\lambda t}u(t)\chi(t)\big(1-\varphi(t/\delta)\big)\,\frac{dt}{t}=(-i\lambda)^{-N}\int_0^\infty e^{i\lambda t}\partial_t^N\Big(u(t)\chi(t)\big(1-\varphi(t/\delta)\big)t^{-1}\Big)dt, \tag{D.16}$$

for all N.

If one of the derivatives falls on $1-\varphi(t/\delta)$, the support of this contribution is contained in $[\delta,2\delta]$. Therefore, the corresponding term will be bounded by $\delta^{\operatorname{Re}\alpha-N-1}(\ln\delta)^\beta$ and will contribute like $\delta^{\operatorname{Re}\alpha-N}(-\ln\delta)^\beta$ to the integral.

If one of the derivatives falls on $\chi(t)$, the support of the integrand will be a compact set away from 0 and then this function will be $\mathcal{O}(1)$. The contribution to the integral of such a term will be like 1.

If all the derivatives fall on $u(t)t^{-1}$, the corresponding term will satisfies

$$\begin{aligned}\int_0^{\infty} e^{i\lambda t}\partial_t^N\big(u(t)t^{-1}\big)\chi(t)\big(1-\varphi(t/\delta)\big)dt =&\mathcal{O}(1)\int_\delta^{+\infty} t^{\operatorname{Re}\alpha-1-N}(-\ln t)^\beta(1-\chi(t))dt\\ \lesssim&(-\ln\delta)^\beta\delta^{\operatorname{Re}\alpha-N},\end{aligned}\tag{D.17}$$

for N large enough ($N>\operatorname{Re}\alpha$).

From these three cases, we deduce

$$\int_0^{+\infty} e^{i\lambda t}u(t)\chi(t)\big(1-\varphi(t/\delta)\big)\frac{dt}{t}=\mathcal{O}\big((-\ln\delta)^\beta\delta^{\alpha-N}\lambda^{-N}\big).\tag{D.18}$$

Taking $\delta=(\varepsilon\lambda)^{-1}$, we get

$$\int_0^{+\infty} e^{i\lambda t}u(t)\chi(t)\big(1-\varphi(t/\delta)\big)\frac{dt}{t}=\mathcal{O}\big(\varepsilon(\ln\lambda)^\beta\lambda^{-\alpha}\big),\tag{D.19}$$

as $\lambda\to+\infty$.

We now assume (D.3), and we want to prove (D.4). Since, for t small enough

$$t^{\operatorname{Re}\alpha-1}(-\ln t)^\beta\lesssim\big(t^{\operatorname{Re}\alpha}(-\ln t)^\beta\big)',\tag{D.20}$$

we obtain

$$\begin{aligned}\Big|\int_0^{+\infty} e^{i\lambda t}s(t)\chi(t)\varphi(t/\delta)\frac{dt}{t}\Big|=&o_{\delta\to0}(1)\int_0^{2\delta}t^{\operatorname{Re}\alpha-1}(-\ln t)^\beta dt\\ =&o_{\delta\to0}(1)\delta^{\operatorname{Re}\alpha}(-\ln\delta)^\beta.\end{aligned}\tag{D.21}$$

Here $o_{\delta\to0}(1)$ stands for a term which goes to 0 as δ goes to 0. If $\delta=(\varepsilon\lambda)^{-1}$, we have

$$\Big|\int_0^{+\infty} e^{i\lambda t}s(t)\chi(t)\varphi(t/\delta)\frac{dt}{t}\Big|=o_{\lambda\to+\infty}(1)\lambda^{-\alpha}(\ln\lambda)^\beta,\tag{D.22}$$

when $\lambda\to+\infty$ and ε fixed. Taking ε small enough in (D.19), and then λ large enough in (D.22), we obtain (D.4).

It remains to prove (D.1). We need to compute

$$\mathcal{I}=\int_0^{+\infty} e^{i\lambda t}t^\alpha(-\ln t)^\beta\varphi(t/\delta)\frac{dt}{t}.\tag{D.23}$$

Performing the change of variable $s=\lambda t$, we get

$$\begin{aligned}\mathcal{I}=&\lambda^{-\alpha}\int_0^{2/\varepsilon}e^{is}s^\alpha(\ln\lambda-\ln s)^\beta\varphi(\varepsilon s)\frac{ds}{s}\\ =&(\ln\lambda)^\beta\lambda^{-\alpha}\int_0^{2/\varepsilon}e^{is}s^\alpha(1-\ln s/\ln\lambda)^\beta\varphi(\varepsilon s)\frac{ds}{s}.\end{aligned}\tag{D.24}$$

We remark that, in the previous equation, $-\ln s/\ln\lambda > -\ln(2/\varepsilon)/\ln\lambda > -1/2$ for λ large enough. Using $(1+u)^\beta = 1 + \mathcal{O}(|u| + |u|^{\max(1,\beta)})$ for $u > -1/2$, we get

$$\begin{aligned}\mathcal{I} =&(\ln\lambda)^\beta \lambda^{-\alpha} \int_0^{2/\varepsilon} e^{is} s^\alpha \varphi(\varepsilon s) \frac{ds}{s}\\ &+ (\ln\lambda)^\beta \lambda^{-\alpha} \int_0^{2/\varepsilon} s^{\operatorname{Re}\alpha} \mathcal{O}\Big(\frac{|\ln s|}{\ln\lambda} + \Big(\frac{|\ln s|}{\ln\lambda}\Big)^{\max(1,\beta)} \Big) \varphi(\varepsilon s) \frac{ds}{s}\\ =&(\ln\lambda)^\beta \int_0^{+\infty} e^{i\lambda t} t^\alpha (-\ln t)^\beta \varphi(t/\delta) \frac{dt}{t} + \mathcal{O}_\varepsilon\big((\ln\lambda)^{\beta-1}\lambda^{-\alpha}\big). \end{aligned} \tag{D.25}$$

Note that the $\mathcal{O}_\varepsilon$ in (D.25) depends on ε.

Then, using (D.19), (D.25) and (D.19) again, we get

$$\begin{aligned}&\int_0^\infty e^{i\lambda t} t^\alpha (-\ln t)^\beta \chi(t) \frac{dt}{t}\\ &\quad=\mathcal{I} + \mathcal{O}\big(\varepsilon(\ln\lambda)^\beta\lambda^{-\alpha}\big)\\ &\quad=(\ln\lambda)^\beta \int_0^{+\infty} e^{i\lambda t} t^\alpha (-\ln t)^\beta \varphi(t/\delta) \frac{dt}{t} + \mathcal{O}_\varepsilon\big((\ln\lambda)^{\beta-1}\lambda^{-\alpha}\big) + \mathcal{O}\big(\varepsilon(\ln\lambda)^\beta\lambda^{-\alpha}\big)\\ &\quad=(\ln\lambda)^\beta \int_0^{+\infty} e^{i\lambda t} t^\alpha (-\ln t)^\beta \chi(t) \frac{dt}{t} + \mathcal{O}\big(\varepsilon(\ln\lambda)^\beta\lambda^{-\alpha}\big) + \mathcal{O}_\varepsilon\big((\ln\lambda)^{\beta-1}\lambda^{-\alpha}\big)\\ &\qquad + \mathcal{O}\big(\varepsilon(\ln\lambda)^\beta\lambda^{-\alpha}\big). \end{aligned} \tag{D.26}$$

Choosing ε small enough, then λ large enough, and using (D.2) with $\beta = 0$ to compute the first term, we obtain

$$\int_0^\infty e^{i\lambda t} t^\alpha (-\ln t)^\beta \chi(t) \frac{dt}{t} = \Gamma(\alpha)(\ln\lambda)^\beta(-i\lambda)^{-\alpha}(1 + o(1)), \tag{D.27}$$

and this completes the proof of (D.1). □

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