EDGE CONNECTIVITY IN GRAPHS: AN EXPANSION THEOREM

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ABSTRACT. We show that if a graph is k-edge-connected, and we adjoin to it another graph satisfying a "contracted diameter ≤ 2" condition, with minimal degree $\geq k$, and some natural hypothesis on the edges connecting one graph to the other, the resulting graph is also k-edge-connected.

1. Introduction

Let G be a simple graph (*i.e.* a graph with no loops, no multiple edges) with vertex set $V(G)$ and edge set $E(G)$ (we follow in notation the book [Wes01]). Given $A, B \subset V(G)$, $[A, B]$ is the set of edges of the form ab, joining a vertex $a \in A$ to a vertex $b \in B$. As we consider edges without orientation, $[A, B] = [B, A]$. Abusing of notation, for $v \in V(G), A \subset V(G)$, we write $[v, A]$ instead of $[\{v\}, A]$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) \doteq |[v, V(G)]|$. The *neighbourhood* of a vertex v, $N(v)$, is the set of vertexes w such that $vw \in E(G)$. Given $A \subset V(G)$, $G(A)$ is the graph G' such that $V(G') = A$ and $E(G')$ is the set of edges in $E(G)$ having both endpoints in A. Given $v, w \in V(G)$, $d_G(v, w)$ is the distance in G from v to w, that is the minimum length of a path from v to w. If $v \in V(G), A \subset V(G)$ we set $d_G(v, A) \doteq \min_{w \in A} d_G(v, w)$.

An edge cut in G is a set of edges $[S, \bar{S}]$, where $S \subset V(G)$ is non void, and $\bar{S} \doteq V(G) \setminus S$ is also assumed to be non void.

The edge-connectivity of G, $k'(G)$, is the minimum cardinal of the cuts in G. We say that G is k-edge-connected if $k'(G) \geq k$. Menger's theorem has as a consequence that given two vertices v, w in $V(G)$, if G is k-edge-connected there are at least k-edge-disjoint paths joining v to w (see [Wes01], pp.153-169).

In this paper we address the following expansion problem: given a k-edge-connected graph G_2, give conditions under which the result of adjoining to G_2 a graph G_1 will be also k edge-connected (see Corollary 1 below).

2. An expansion theorem

Let G be a simple graph. Let $V_1 \subset V(G), V_2 \doteq V(G) \setminus V_1$, and set $G_1 = G(V_1), G_2 = G(V_2)$. We assume in the sequel that V_1 and V_2 are non void. We define, for $x, y \in V_1$, the *contracted distance*

$$\delta(x, y) \doteq \min\{d_{G_1}(x, y), d_G(x, V_2) + d_G(y, V_2)\}$$

and for $x \in V, y \in V_2$

$$\delta(x, y) = \delta(y, x) \doteq d_G(x, V_2)$$

If $x \in V$ and $A \subset V$, we set $\delta(x, A) \doteq \min_{a \in A} \delta(x, a)$.

Notice that with these definitions, if $\delta(x, y) = 2$ for some $x, y \in V$, then there exists $z \in V$ such that $\delta(x, z) = \delta(z, y) = 1$.

We shall also use the notations

$$\begin{aligned} \partial^j V_1 &\doteq \{x \in V_1 : |[x, V_2]| \geq j\} \\ i^j V_1 &\doteq \{x \in V_1 : |[x, V_2]| < j\} \end{aligned}$$

Under these settings, we consider also

$$\Phi \doteq \sum_{x \in V_1} \min\{\max\{1, |[x, i^2 V_1]|\}, |[x, V_2]|\}$$

In this general framework, we have

Theorem 1. *If* $\max_{x,y \in V} \delta(x, y) \leq 2$ *(*i.e. *the* contracted diameter *of* V *is* ≤ 2*),* $[S, \bar{S}]$ *is an edge cut in* G *such that* $V_2 \subset S$*, and* $k \doteq \min_{x \in V_1} \deg_G(v) > |[S, \bar{S}]|$*, then*

(1) $\exists \bar{s} \in \bar{S} : \delta(\bar{s}, S) = 2$.
(2) $\forall s \in S : \delta(s, \bar{S}) = 1$.
(3) $|S \cap V_1| < |[S, \bar{S}]| < k < |\bar{S}|$.
(4) $S \cap V_1 \subset \partial^2 V_1, \bar{S} \supset i^2 V_1$.
(5) $\Phi \leq |[S, \bar{S}]|$.

(See the examples in Figure 1.)
Proof.

(1) Arguing by contradiction, suppose that for any $\bar{s} \in \bar{S}$: $\delta(\bar{s}, S) = 1$. Let $\bar{s} \in \bar{S}$. Then we have k_1 edges $\bar{s}s_i, 1 \leq i \leq k_1$ with $s_i \in S$ and (eventually) k_2 edges $\bar{s}\bar{s}_j$, $\bar{s}_j \in \bar{S}$. But each $\bar{s}_j$ satisfies $\delta(\bar{s}_j, S) = 1$, thus we have k_2 new edges (here we used that G is simple, because we assumed that the vertices $\bar{s}_j$ are different) $\bar{s}_j s'_j$, with $s'_j \in S$, whence
$$|[S, \bar{S}]| \geq k_1 + k_2 = \deg_G(\bar{s}) \geq k$$
which contradicts our hypothesis.
(2) Let $\bar{s}_0 \in \bar{S}$ be such that $\delta(\bar{s}_0, S) = 2$. Then for each $s \in S$, as $\delta(\bar{s}_0, s) = 2$, there exists $\bar{s}'$ such that $\delta(\bar{s}_0, \bar{s}') = \delta(\bar{s}', s) = 1$. But, again, as $\delta(\bar{s}_0, S) = 2$, it follows that $\bar{s}' \in \bar{S}$, hence $\delta(s, \bar{S}) = 1$.
(3) By the previous point, we have for each $s \in S \cap V_1$ some edge in $[S, \bar{S}]$ incident in s, and for some $v \in V_2$ we have also some edge in $[S, \bar{S}]$ incident in v, thus
$$|S \cap V_1| + 1 \leq |[S, \bar{S}]|$$
On the other hand, if $\bar{s} \in \bar{S}$ satisfies $\delta(\bar{s}, S) = 2$ (such $\bar{s}$ exists by our first point), then $N(\bar{s}) \subset \bar{S}$ (recall that $N(\bar{s})$ is the neighbourhood of $\bar{s}$), whence
$$|\bar{S}| \geq 1 + |N(\bar{s})| \geq 1 + k$$
and our statement follows.
(4) Let $s \in S \cap V_1$. By our second point, and using again that there is at least one edge in $[\bar{S}, V_2]$, we have
$$\begin{aligned} |N(s) \cap V_1| + |[s, \bar{S}]| &\leq |[S, \bar{S}]| - 1 \\ &< \deg_G(s) - 1 \end{aligned}$$
and the first of our statements follows if we notice that
$$\deg_G(s) = |N(s) \cap V_1| + |[s, \bar{S}]| + |[s, V_2]|$$
Now, $\bar{S} = V_1 \setminus S \cap V_1$, and our second statement follows immediately.

(5) By our previous points, if $s \in S \cap V_1$ then
$$|[s, \bar{S}]| \geq \max\{1, |[s, i^2 V_1]|\}$$
and of course for $\bar{s} \in \bar{S}$, $|[\bar{s}, S]| \geq |[\bar{s}, V_2]|$, thus
$$\begin{aligned} |[S, \bar{S}]| &= |[S \cap V_1, \bar{S}]| + |[\bar{S}, V_2]| \\ &\geq \sum_{s \in S \cap V_1} \max\{1, |[s, i^2 V_1]|\} + \sum_{\bar{s} \in \bar{S}} |[\bar{s}, V_2]| \\ &\geq \Phi \end{aligned}$$

Corollary 1. *Assume that*
(1) $\deg_G(x) \geq k, x \in V(G)$
(2) G_2 *is* k*-edge connected*
(3) $\max_{x,y \in V} \delta(x, y) \leq 2$

Then any of the following
(1) $\Phi \geq k$
(2) $|\partial^1 V_1| \geq k$
(3) $V_1 = \partial^1 V_1$

implies that G *is* k*-edge-connected.*

(See the examples in Figure 2.)

Proof. Let $[S, \bar{S}]$ be any cut in G. We shall show that, under the listed hypotheses and any of the alternatives, $|[S, \bar{S}]| \geq k$.

If $S \cap V_2 \neq \emptyset$ and $\bar{S} \cap V_2 \neq \emptyset$, then, as
$$[S \cap V_2, \bar{S} \cap V_2] \subset [S, \bar{S}]$$
is a cut in G_2, which we assumed to be k-edge connected, we obtain $|[S, \bar{S}]| \geq k$.

Without loss of generality, we assume in the sequel that $V_2 \subset S$. We argue by contradiction assuming that there exists some S such that $|[S, \bar{S}]| < k$, so that we are under the hypothesis of Theorem 1.

The first of our alternative hypothesis contradicts the last of the conclusions of Theorem 1.

When $x \in \partial^1 V_1$,
$$\min\{\max\{1, |[x, i^2 V_1]|\}, |[x, V_2]|\} \geq 1$$
so that we have $|\partial^1 V_1| \leq \Phi$ *i.e.* the second of our alternative hypothesis implies the first one.

To finish our proof, notice that if $V_1 = \partial^1 V_1$, as $\bar{S} \subset V_1$, we have $\delta(\bar{s}, S) = 1$ for any $\bar{s} \in \bar{S}$, contradicting the first of the conclusions in Theorem 1.

3. Final remarks

Corollary 1 is related to a well known theorem of Plesník (see [Ple75], Theorem 6), which states that in a simple graph of diameter 2 the edge connectivity is equal to the minimum degree.

References

[Ple75] J. Plesník. Critical graphs of a given diameter. *Acta Fac. Rerum Natur. Univ. Comenian. Math.*, 30:71–93, 1975.

[Wes01] Douglas B. West. *Introduction to Graph Theory.* Prentice Hall, 2001.

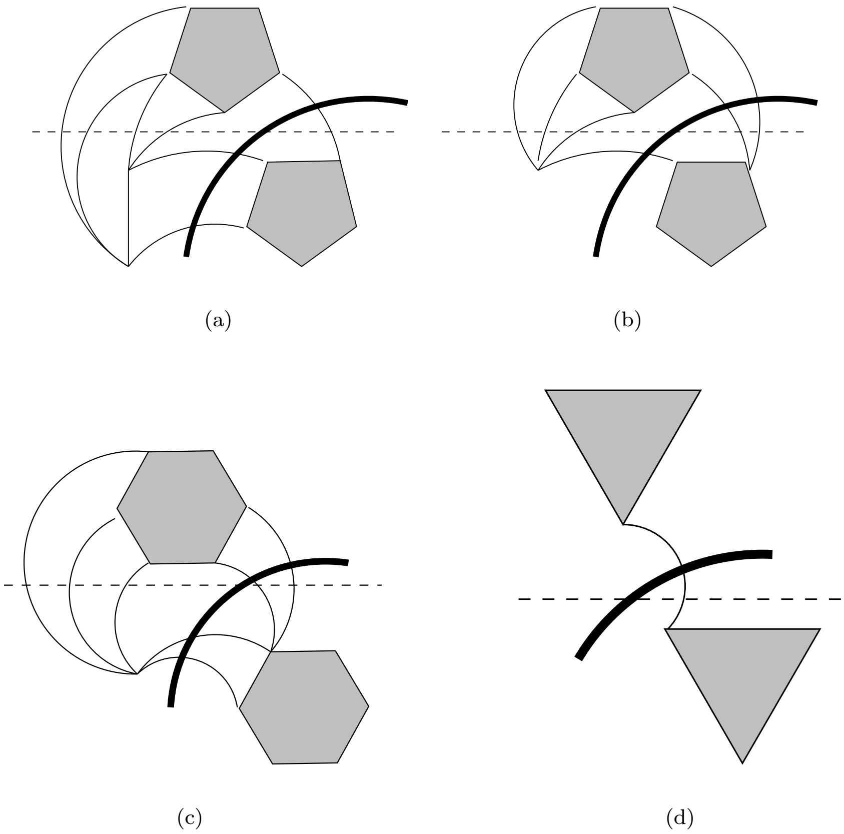

FIGURE 1. **Conventions:** 1.Filled polygons represent cliques, and curved arcs represent edges. 2.The dotted line separates G_2 (the upper graph) from G_1 (the lower graph). 3. The widest arc shows the cut $[S, \bar{S}]$. **Descriptions:** (a) Here $|[S, \bar{S}]| = 3 < k = 4$, $|S \cap V_1| = 2$, $|\bar{S}| = 5$, $\Phi = 3$, $S \cap V_1 = \partial^2 V_1$. (b) Here $|[S, \bar{S}]| = 3 < k = 4$, $|S \cap V_1| = 1$, $|\bar{S}| = 5$, $\Phi = 3$, $S \cap V_1 \neq \partial^2 V_1$. (c) Here $|[S, \bar{S}]| = 4 < k = 5$, $|S \cap V_1| = 1$, $|\bar{S}| = 6$, $\Phi = 3$, $S \cap V_1 \neq \partial^2 V_1$. (d) Here $|[S, \bar{S}]| = 1 < k = 2$, $|S \cap V_1| = 0$, $|\bar{S}| = 3$, $\Phi = 1$, $S \cap V_1 = \partial^2 V_1 = \emptyset$.

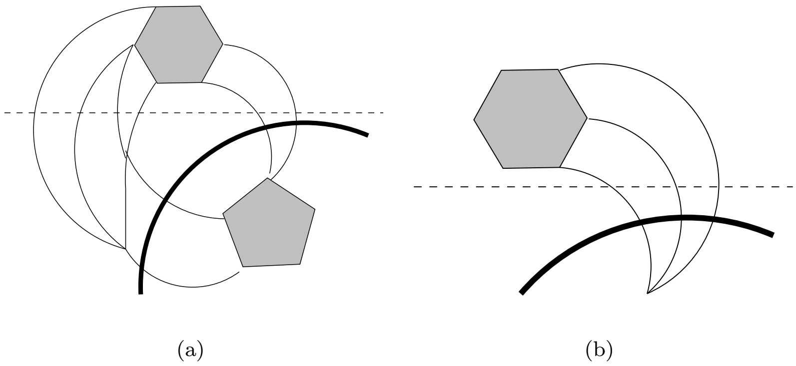

FIGURE 2. **Conventions:** 1.Filled polygons represent cliques, and curved arcs represent edges. 2.The dotted line separates G_2 (the upper graph) from G_1 (the lower graph). 3. The widest arc shows a minimal cut $[S, \bar{S}]$. **Descriptions:** (a) Here $|[S, \bar{S}]| = k = 4$, $\Phi = 4$, $|\partial^1 V_1| = 3$. (b) Here $|[S, \bar{S}]| = k = 3$, $\Phi = 1$, $|\partial^1 V_1| = 1$, $V_1 = \partial^1 V_1$. This example shows that Corollary 1 includes an edge-connectivity version of the Expansion Lemma in [Wes01], Lemma 4.2.3.