

CAPACITARY REPRESENTATION OF POSITIVE SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS

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ABSTRACT. We give a global bilateral estimate on the maximal solution $\bar{u}_F$ of $\partial_t u - \Delta u + u^q = 0$ in $\mathbb{R}^N \times (0, \infty)$, $q > 1$, $N \geq 1$, which vanishes at $t = 0$ on the complement of a closed subset $F \subset \mathbb{R}^N$. This estimate is expressed by a Wiener test involving the Bessel capacity $C_{2/q, q'}$. We deduce from this estimate that $\bar{u}_F$ is σ -moderate in Dynkin's sense.

Représentation capacitaire des solutions positives d'équations paraboliques semi-linéaires

RÉSUMÉ. Nous donnons une estimation bilatérale précise de la solution maximale $\bar{u}_F$ de $\partial_t u - \Delta u + u^q = 0$ dans $\mathbb{R}^N \times (0, \infty)$, $q > 1$, $N \geq 1$, qui s'annule en $t = 0$ sur le complémentaire d'un sous-ensemble fermé $F \subset \mathbb{R}^N$. Cette estimation s'exprime par un test de Wiener impliquant la capacité de Bessel $C_{2/q, q'}$. Nous déduisons de cette estimation que $\bar{u}_F$ est σ -modérée au sens de Dynkin.

VERSION FRANÇAISE ABRÉGÉE

Soit $q > 1$. Si u est une solution positive de

$$(1) \quad \partial_t u - \Delta u + |u|^{q-1} u = 0$$

dans $\mathbb{R}^N \times (0, \infty)$, nous avons démontré dans [6] qu'elle admet une trace initiale, notée $Tr(u)$, dans la classe des mesures de Borel positives et régulières, mais pas nécessairement localement bornées. Si F est un sous-ensemble fermé de $\mathbb{R}^N$, nous désignons par $\bar{u}_F$ la solution maximale de (1) dont le support de la trace initiale est inclus dans F . Si $1 < q < q_c := (N+2)/N$, il est montré dans [5] que les inégalités suivantes sont vérifiées

$$t^{-1/(q-1)} f(|x-a|/\sqrt{t}) \leq \bar{u}_F(x, t) \leq ((q-1)t)^{-1/(q-1)} \quad \forall a \in F,$$

où f est l'unique fonction positive vérifiant

$$\Delta f + \frac{1}{2} y \cdot Df + \frac{1}{q-1} f - |f|^{q-1} f = 0 \quad \text{dans } \mathbb{R}^N \quad \text{et} \quad \lim_{|y| \rightarrow \infty} |y|^{2/(q-1)} f(y) = 0.$$

Ces inégalités jouent un rôle fondamental dans la démonstration du caractère bi-univoque de la correspondance, par l'opérateur de trace initiale, entre l'ensemble des solutions positives de (1) et l'ensemble des mesures de Borel positives régulières. Quand $q \geq q_c$ la fonction f est identiquement nulle car les singularités isolées de (1) sont éliminables [4].

Définition. Soit $N \geq 1$, $q \geq q_c$ et F un sous-ensemble fermé de $\mathbb{R}^N$. On définit le potentiel $(2/q, q')$ -capacitaire W_F de F par

$$(2) \quad W_F(x, t) = t^{-1/(q-1)} \sum_{n=0}^{\infty} (n+1)^{N/2-1/(q-1)} e^{-n/4} C_{2/q, q'} \left(\frac{F_n}{\sqrt{(n+1)t}} \right),$$

$\forall (x, t) \in \mathbb{R}^N \times [0, \infty)$, où $F_n = F_n(x, t) = \left\{ y \in F : \sqrt{nt} \leq |x - y| < \sqrt{(n+1)t} \right\}$.

Notre résultat principal est l'estimation bilatérale.

Théorème 1. Soit $N \geq 1$, $q \geq q_c$. Il existe deux constantes $C_1 > C_2 > 0$, ne dépendant que de N et q , telles que pour tout sous-ensemble fermé F de $\mathbb{R}^N$

$$(3) \quad C_2 W_F(x, t) \leq \bar{u}_F(x, t) \leq C_1 W_F(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty).$$

Si $\mu \in \mathfrak{M}^b(\mathbb{R}^N)$ (l'espace des mesures de Radon bornées dans $\mathbb{R}^N$) appartient à $W^{-2/q, q}(\mathbb{R}^N)$, il existe une unique solution $u = u_\mu$ de (1) dont la trace initiale est μ [3]. On définit alors

$$(4) \quad \underline{u}_F = \sup \{ u_\mu : \mu \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap W^{-2/q, q}(\mathbb{R}^N) : \mu(F^c) = 0 \}.$$

Cette solution est σ -modérée au sens de Dynkin, c'est à dire qu'il existe une suite croissante de mesures positives $\mu_n \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap W^{-2/q, q}(\mathbb{R}^N)$ telles que $u_{\mu_n} \uparrow \underline{u}_F$.

La clef de la démonstration du Théorème 1 est l'estimation inférieure de $\underline{u}_F$.

Théorème 2. Soit $N \geq 1$, $q \geq q_c$. Il existe une constante $C = C(N, q) > 0$ telle que pour tout sous-ensemble fermé F de $\mathbb{R}^N$,

$$(5) \quad \underline{u}_F(x, t) \geq C W_F(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty).$$

Une conséquence importante des estimations précédentes est la suivante:

Théorème 3. Soit $N \geq 1$, $q > 1$. Pour tout sous-ensemble fermé F de $\mathbb{R}^N$, $\bar{u}_F = \underline{u}_F$.

1. MAIN RESULTS

Let u be a nonnegative solution of

$$(1) \quad \partial_t u - \Delta u + |u|^{q-1} u = 0, \quad q > 1$$

in $\mathbb{R}^N \times (0, \infty)$. It was proved in [6] that u admits an initial trace, denoted by $Tr(u)$, in the class of outer regular positive Borel measures, not necessarily locally bounded. If F is a closed subset of $\mathbb{R}^N$, we denote by $\bar{u}_F$ the maximal solution of (1) which belongs to $C(F^c \times [0, \infty))$ and vanishes on $F^c \times \{0\}$, where $F^c := \mathbb{R}^N \setminus F$.

If $1 < q < (N+2)/N$, the following inequalities are verified

$$(2) \quad t^{-1/(q-1)} f(|x - a|/\sqrt{t}) \leq \bar{u}_F(x, t) \leq ((q-1)t)^{-1/(q-1)} \quad \forall a \in F,$$

where f is the unique positive solution [5] of

$$(3) \quad \Delta f + \frac{1}{2} y \cdot Df + \frac{1}{q-1} f - |f|^{q-1} f = 0 \quad \text{in } \mathbb{R}^N \quad \text{s. t.} \quad \lim_{|y| \rightarrow \infty} |y|^{2/(q-1)} f(y) = 0.$$

These inequalities play a fundamental role in proving that (in the subcritical case) any positive solution is uniquely determined by its initial trace [6].

Definition 1. *Let*

$$(4) \quad \begin{aligned} B_{\sqrt{nt}}(x) &= \{y \in \mathbb{R}^N : |x - y| < \sqrt{nt}\} \\ T_n(x, t) &= \bar{B}_{\sqrt{(n+1)t}}(x) \setminus B_{\sqrt{nt}}(x). \end{aligned}$$

For every $q \geq q_c$ and every closed subset $F \subset \mathbb{R}^N$, we define the $(2/q, q')$ -capacitary potential W_F of F by

$$(5) \quad W_F(x, t) = t^{-1/(q-1)} \sum_{n=0}^{\infty} (n+1)^{N/2-1/(q-1)} e^{-n/4} C_{2/q, q'} \left(\frac{F \cap T_n(x, t)}{\sqrt{(n+1)t}} \right),$$

$$\forall (x, t) \in \mathbb{R}^N \times [0, \infty).$$

Our main result is the following bilateral estimate

Theorem 1. *Let $N \geq 1$, $q \geq q_c$. There exist two positive constants C_1 and C_2 , depending on N and q such that for any closed subset F of $\mathbb{R}^N$,*

$$(6) \quad C_2 W_F(x, t) \leq \bar{u}_F(x, t) \leq C_1 W_F(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty).$$

Remark. It is important to notice that although W_F is not a solution of (1), it is invariant with respect to the similarity transformation associated with this equation:

$$(7) \quad k^{1/(q-1)} W_F(\sqrt{k}x, kt) = W_{F/\sqrt{k}}(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty), \forall k > 0.$$

Clearly $\bar{u}_F$ is also self similar, i.e., invariant with respect to the above transformation.

If μ is a bounded Borel measure such that $|\mu| \in W^{-2/q, q}(\mathbb{R}^N)$ then, there exists a unique solution $u = u_\mu$ of (1) with initial trace μ . We define

$$\underline{u}_F = \sup\{u_\mu : \mu \in W_+^{-2/q, q}(\mathbb{R}^N), \mu(F^c) = 0\}.$$

This solution is σ -moderate in the sense of Dynkin, which means that there exists an increasing sequence of positive measures $\mu_n \in W_+^{-2/q, q}(\mathbb{R}^N)$ such that $u_{\mu_n} \uparrow \underline{u}_F$. Clearly $\underline{u}_F \leq \bar{u}_F$. Therefore the next result implies the lower estimate in Theorem 1.

Theorem 2. *Let $N \geq 1$, $q \geq q_c$. There exists a positive $C = C(N, q)$, such that for any closed subset F of $\mathbb{R}^N$,*

$$(8) \quad \underline{u}_F(x, t) \geq C W_F(x, t) \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty).$$

As a consequence of Theorems 1 and 2 we find that $\bar{u}_F \leq c \underline{u}_F$. Using this fact we obtain the following result (already known [6] in the case $1 < q < q_c$),

Theorem 3. *Let $N \geq 1$, $q > 1$. For any closed subset F of $\mathbb{R}^N$ one has $\bar{u}_F = \underline{u}_F$. In particular $\bar{u}_F$ is σ -moderate.*

2. PROOF OF THE UPPER ESTIMATE IN THEOREM 1

In the sequel we denote by c a positive constant which depends only on N and q ; its value may change from one occurrence to another. Without loss of generality, we assume that F is compact. Denote $B_r(x) = \{y : |x - y| < r\}$ and $B_r = B_r(0)$.

Let $r > 0$ be a positive number such that $F \subset B_r$. We start by deriving an upper estimate depending on r . Let $\rho > 0$ be a positive number, to be later determined

as a function of r, t . Let $\eta \in C_0^\infty(B_{r+\rho})$ be such that $\eta = 1$ on F and $0 \leq \eta \leq 1$. Put $\eta^* = 1 - \eta$ and choose $\zeta := (e^{t\Delta}[\eta^*])^{2q'}$ as a test function. Then

$$(9) \quad \int_0^1 \int_{\mathbb{R}^N} u(\partial_t - \Delta)\zeta dx dt + \int_0^1 \int_{\mathbb{R}^N} u^q \zeta dx dt = - \int_{\mathbb{R}^N} u(x, 1) dx.$$

A straightforward computation yields

$$\int_{Q_{r+\rho}} u^q dx dt + \int_{\mathbb{R}^N} u(x, 1) dx \leq \int_0^1 \int_{\mathbb{R}^N} (R(\eta))^{q'} dx dt,$$

$$R(\eta) := |De^{t\Delta}[\eta]|^2 + |\partial_t e^{t\Delta}[\eta] + \Delta e^{t\Delta}[\eta]|, \quad Q_r := \{(x, t) : t > 0, |x|^2 + t \geq r^2\}.$$

Using interpolation inequalities [8] one obtains,

$$(10) \quad \int_0^1 \int_{\mathbb{R}^N} (R(\eta))^{q'} dx dt \leq C_1 \|\eta\|_{W^{2/q, q'}}^{q'}.$$

This implies

$$\int_{\mathbb{R}^N} u(x, 1) dx + \int_{Q_{r+\rho}} u^q dx dt \leq c C_{2/q, q'}^{B_{r+\rho}}(F).$$

Further it can be shown that, for $0 < s < 1$,

$$(11) \quad \int_{\mathbb{R}^N} u(x, 1) dx + \int_s^1 \int_{\mathbb{R}^N} u^q dx dt = \int_{\mathbb{R}^N} u(x, s) dx.$$

Clearly $u(x, t+s) < w_s(x, t) := e^{t\Delta}[u(\cdot, s)]$ for $t > 0$. Therefore, by (10) and (11),

$$(12) \quad u(x, (r+2\rho)^2) \leq \frac{c}{(\rho^2 + r\rho)^{N/2}} C_{2/q, q'}^{B_{r+\rho}}(F).$$

Let v be the solution of the initial-boundary value problem

$$\begin{aligned} \partial_t v - \Delta v &= 0 \quad \text{in } Q_{r, \rho}^* := (\bar{B}_{r+\rho})^c \times (0, (r+2\rho)^2), \\ v(x, 0) &= 0 \quad \forall x \in B_{r+\rho}^c, \quad v = u \quad \forall (x, t) \in \partial B_{r+\rho} \times (0, (r+2\rho)^2). \end{aligned}$$

Using the fact that $u < v$ in $Q_{r, \rho}^*$ and (12) we obtain,

Lemma 4. *If $F \subset B_r$, there exists $c > 0$ such that*

$$(13) \quad \bar{u}_F(x, t) \leq c \left(1 + \frac{r}{\rho}\right)^{N/2} \frac{e^{-(|x|-r-3\rho)^2/4t}}{t^{N/2}} C_{2/q, q'}^{B_{r+\rho}}(F),$$

for any $(x, t) \in \mathbb{R}^N \times [(r+3\rho)^2, \infty)$.

The upper estimate in (6) is obtained by slicing F , relative to a given point $(x, t) \in \mathbb{R}^N \times (0, \infty)$, in such a way that each slice satisfies the assumption of Lemma 4 for an appropriate value of r depending on the point. Put

$$T_n(x, t) = B_{\sqrt{(n+1)t}}(x) \setminus B_{\sqrt{nt}}(x), \quad F_n(x, t) = F \cap T_n(x, t) \quad \forall n \in \mathbb{N}.$$

Since a sum of positive solutions of (1) is a super-solution,

$$\bar{u}_F \leq \sum_{n=0}^{\infty} \bar{u}_{F_n(x, t)}.$$

Using this fact and Lemma 4 we show that

$$(14) \quad \bar{u}_F(x, t) \leq c W_F(x, t)$$

for every $(x, t) \in \mathbb{R}^N \times (0, \infty)$. In view of the fact that both sides of this inequality are invariant with respect to the similarity transformation (7), it is sufficient to prove it in the case $(x, t) = (0, 1)$. We denote $F_n = F_n(0, 1)$ and $T_n = T_n(0, 1)$.

If $N = 1$, each of the sets F_n satisfies the condition of Lemma 4 with $r = \sqrt{n}$. But this is not the case when $N \geq 2$. Therefore, if $N \geq 2$, a secondary slicing is needed.

For every $n \in \mathbb{N}$ there exists a set of points $\Theta_n = \{a_{j,n}\}_{j=1}^{J_n}$ on the sphere $|y| = (\sqrt{n+1} + \sqrt{n})/2$ such that

$$|a_{n,j} - a_{n,k}| \geq 1/\sqrt{2(n+1)} \quad \text{for } j \neq k, \quad T_n \subset \bigcup_{1 \leq j \leq J_n} B_{\sqrt{1/(n+1)}}(a_{n,j}).$$

Clearly $J_n \leq (\sqrt{2}(n+1))^{N-1} < (4n)^{N-1}$. If $F_{n,j} := F_n \cap B_{\sqrt{1/(n+1)}}(a_{n,j})$,

$$(15) \quad \bar{u}_F(0, 1) \leq \sum_{n=0}^{\infty} \sum_{1 \leq j \leq J_n} \bar{u}_{F_{n,j}}(0, 1).$$

It is not difficult to verify that

$$C_{2/q,q'}^{B_{2/\sqrt{n+1}}(a_{n,j})}(F_{n,j}) \approx (n+1)^{N/2-1/(q-1)} C_{2/q,q'}(\sqrt{n+1} F_{n,j})$$

where the capacity on the right hand side (resp. left hand side) is the Bessel capacity relative to $\mathbb{R}^N$ (resp. relative to $B_{2/\sqrt{n+1}}(a_{n,j})$). The symbol $\approx$ stands for two-sided inequalities with constants $c = c(N, q)$. Further, the quasi-additivity of Bessel capacities [2] implies

$$(16) \quad \sum_{1 \leq j \leq J_n} C_{2/q,q'}(\sqrt{n+1} F_{n,j}) \leq c(N, q) C_{2/q,q'}(\sqrt{n+1} F_n).$$

Each set $F_{n,j}$ satisfies the condition of Lemma 4 with $r = \sqrt{n}$. Therefore, estimating $\bar{u}_{F_{n,j}}(0, 1)$ as in (13) and using (15) and (16) we obtain (14) for $(x, t) = (0, 1)$.

3. PROOF OF THEOREM 2

As in the elliptic case [7], for each $(x, t) \in \mathbb{R}^N \times (0, \infty)$, we construct a measure $\mu = \mu_{x,t} \in W_+^{-2/q,q}(\mathbb{R}^N)$, concentrated on F , such that

$$(17) \quad u_\mu(x, t) \geq cW_F(x, t).$$

Since $\underline{u}_F > u_\mu$, (17) implies (8).

For every measure $\mu \in W_+^{-2/q,q}(\mathbb{R}^N)$, $0 \leq u_\mu \leq e^{t\Delta}[\mu]$. Therefore

$$(18) \quad u_\mu \geq e^{t\Delta}[\mu] - \int_0^t e^{(t-s)\Delta}[(e^{s\Delta}[\mu])^q] ds.$$

Let ν_n be the capacity measure of $F_n(x, t)/\sqrt{t(n+1)}$ (see [1]) and define the measure μ_n by

$$\mu_n(A) = (t(n+1))^{N/2-1/(q-1)} \nu_n(A/\sqrt{t(n+1)}),$$

for every Borel set A . Thus μ_n is concentrated on F_n . Finally put $\mu := \mu_{x,t} = \sum_n \mu_n$. With this choice of μ it is not difficult to show that

$$(19) \quad e^{t\Delta}[\mu](x, t) \geq \frac{1}{(4\pi t)^{N/2}} \sum_{n=0}^{\infty} (\sqrt{(n+1)t})^{N-2/(q-1)} e^{-(n+1)/4} C_{2/q,q'}\left(\frac{F_n(x, t)}{\sqrt{(n+1)t}}\right).$$

We have to derive a corresponding upper estimate for the nonlinear term in (18). To this end we employ a partitioning $\{\mathcal{T}_n : n \in \mathbb{Z}\}$ of $\mathbb{R}^N \times (0, t)$ defined by,

$$\mathcal{T}_n = \begin{cases} \{(y, s) : tn \leq |x - y|^2 + t - s \leq t(n+1), 0 < s < t\}, & \text{if } n \in \mathbb{N}_*, \\ \{(y, s) : t\alpha^{-n} \leq |x - y|^2 + t - s \leq t\alpha^{-n-1}, 0 < s < t\}, & \text{if } n \leq 0, \end{cases}$$

where $\alpha \in (0, 1)$ must be appropriately chosen. If $G(\xi, \tau) := (4\pi|\tau|)^{-N/2} \exp(-|\xi|^2/4\tau)$ then

$$\int_0^t e^{-(t-s)\Delta} (e^{s\Delta}[\mu])^q ds = C \sum_{p \in \mathbb{Z}} \iint_{\mathcal{T}_p} G(y-x, s-t) \left(\sum_{n=0}^{\infty} \int_{\mathbb{R}^N} G(z-y, s) d\mu_n(z) \right)^q dy ds.$$

We denote

$$J_1 = \sum_{p \in \mathbb{Z}} \iint_{\mathcal{T}_p} G(y-x, s-t) \left(\sum_{n=0}^{p+2} \int_{\mathbb{R}^N} G(z-y, s) s^{N/2} d\mu_n(z) \right)^q dy ds,$$

$$J_2 = \sum_{p \in \mathbb{Z}} \iint_{\mathcal{T}_p} G(x-y, s-t) \left(\sum_{n=p+3}^{\infty} \int_{\mathbb{R}^N} G(z-y, s) s^{N/2} d\mu_n(z) \right)^q dy ds.$$

J_1 is estimated using Hölder's inequality and the following inequalities [8]

$$(20) \quad \frac{1}{C} \|\lambda\|_{W^{-2/q, q}} \leq \|e^{t\Delta}[\lambda]\|_{L^q(\mathbb{R}^N \times (0, 1))} \leq C \|\lambda\|_{W^{-2/q, q}},$$

valid for any $\lambda \in W^{-2/q, q}$. The estimate of J_2 , in the case $N > 1$, requires a more delicate argument using the secondary slicing introduced in the previous section and the quasi-additivity of Bessel capacities. For a suitable choice of α one obtains

$$(21) \quad J_1 + J_2 \leq \frac{C}{t^{N/2}} \sum_{n=0}^{\infty} (\sqrt{(n+1)t})^{N-2/(q-1)} e^{-(n+1)/4} C_{2/q, q'} \left(\frac{F_n}{\sqrt{(n+1)t}} \right).$$

The inequalities (18), (19) and (21) imply (17), with μ replaced by $\epsilon\mu$, provided that $\epsilon > 0$ is sufficiently small, depending on q and the constants in (19) and (21). This in turn implies (8).

It follows that

$$\underline{u}_F(x, t) \leq \bar{u}_F(x, t) \leq c\underline{u}_F(x, t).$$

By the uniqueness argument used in [6] we conclude that $\bar{u}_F(x, t) = \underline{u}_F(x, t)$.

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