On L-Functions of Cyclotomic Function Fields

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Abstract

We study two criterions of cyclicity for divisor class groups of function fields, the first one involves Artin L-functions and the second one involves "affine" class groups. We show that, in general, these two criterions are not linked.

Let P be a prime of $\mathbb{F}_q[T]$ of degree d and let K_P be the Pth cyclotomic function field. In this paper we study the relation between the p-part of $Cl^0(K_P)$ and the zeta function of K_P, where p is the characteristic of $\mathbb{F}_q$.

Let χ be an even character of the Galois group of $K_P/\mathbb{F}_q(T)$, $\chi \neq 1$. Let $g(X,\overline{\chi})$ be the "congruent to one modulo p" part of the L-function of $K_P/\mathbb{F}_q(T)$ associated to the character $\overline{\chi}$. We have two criterions of cyclicity ([2], chapter 8): if $\mathrm{deg_X g(X,\overline{\chi})} \leq 1$ then $Cl^0(K_P)_p(\chi)$ is a cyclic $\mathbb{Z}_p[\mu_{q^d-1}]$-module, and if $Cl(O_{K_P})_p(\chi) = \{0\}$ then $Cl^0(K_P)_p(\chi)$ is a cyclic $\mathbb{Z}_p[\mu_{q^d-1}]$-module. David Goss has obtained that if $Cl(O_{K_P})_p(\chi)$ is trivial then $g(X,\overline{\chi})$ is of degree at most one ([2], Theorem 8.21.2). Unfortunately, there is a gap in the proof of this result. In fact, we show that in general $Cl(O_{K_P})_p(\chi) = \{0\}$ does not imply $\mathrm{deg_X g(X,\overline{\chi})} \leq 1$ (Proposition 3.4). We also prove that if i is a q-magic number and if ω_P is the Teichmüller character at P, then $g(X,\omega_P^i)$ has simple roots when $i \equiv 0 \pmod{q-1}$ (Proposition 5.1).

Note that Goss conjectures that if i is a q-magic number then $\mathrm{deg_X g(X,\omega_P^i)} \leq 1$. This problem is still open and can be viewed as an analogue of Vandiver's Conjecture for function fields (see section 5).

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1 Notations

Let $\mathbb{F}_q$ be a finite field having q elements, $q = p^s$ where p is the characteristic of $\mathbb{F}_q$. Let T be an indeterminate over $\mathbb{F}_q$ and set $A = \mathbb{F}_q[T]$, $k = \mathbb{F}_q(T)$. We denote the set of monic elements of A by A^+. A prime of A is a monic irreducible polynomial in A. We fix $\overline{k}$ an algebraic closure of k. We denote the unique place of k which is a pole of T by ∞.

Let L/k be a finite geometric extension of k, $L \subset \overline{k}$. We set:
- O_L : the integral closure of A in L,
- O_L^* : the group of units of O_L,
- $S_\infty(L)$: the set of places of L above ∞,
- $Cl^0(L)$: the group of divisors of degree zero of L modulo the group of principal divisors,
- $Cl(O_L)$: the ideal class group of O_L,
- $R(L)$: the groupe of divisors of degree zero with supports in $S_\infty(L)$ modulo the group of principal divisors with supports in $S_\infty(L)$.

If d is the greatest common divisor of the degrees of the elements in $S_\infty(L)$, we have the following exact sequence:

$$0 \to R(L) \to Cl^0(L) \to Cl(O_L) \to \frac{\mathbb{Z}}{d\mathbb{Z}} \to 0\,.$$

Let P be a prime of A of degree d. We denote the Pth cyclotomic function field by K_P (see [2], chapter 7, and [4]). Recall that K_P/k is the maximal abelian extension of k contained in $\overline{k}$ such that:
- K_P/k is unramified outside of P, ∞,
- K_P/k is tamely ramified at P, ∞,
- for every place v of K_P above ∞, the completion of K_P at v is equal to $\mathbb{F}_q((\frac{1}{T}))(\sqrt[q-1]{-T})$.

We recall that $\mathrm{Gal(K_P/k)} \simeq \mathrm{(A/PA)}^*$, and that the decomposition group of ∞ in K_P/k is equal to its inertia group and is isomorphic to $\mathbb{F}_q^*$.

Let $E/\mathbb{F}_q$ be a global function field and let F/E be a finite geometric abelian extension. Set $G = \mathrm{Gal(F/E)}$ and $\widehat{G} = \mathrm{Hom(G, \mathbb{C}^*)}$.

Let $\chi \in \widehat{G}$, $\chi \neq 1$, we set:

$$L(X, \chi) = \prod_{v \text{ place of E}} (1 - \chi(v) X^{\mathrm{degv}})^{-1},$$

Where $\chi(v) = 0$ if v is ramified in $F^{\mathrm{Ker}(\chi)}/E$, and if v is unramified in $F^{\mathrm{Ker}(\chi)}/E$, $\chi(v) = \chi((v, F^{\mathrm{Ker}(\chi)}/E))$, where $(., F^{\mathrm{Ker}(\chi)}/E)$ is the global reciprocity map. If $\chi = 1$, we set $L(X, \chi) = L_E(X)$ where $L_E(X)$ is the numerator of the zeta function of E.

Therefore, if $L_F(X)$ is the numerator of the zeta function of F, we get:

$$L_F(X) = \prod_{\chi \in \widehat{G}} L(X, \chi)\,.$$

Let Δ be a finite abelian group and let M be a Δ-module. Let ℓ be a prime number such that $\mid \Delta \mid \not\equiv 0 \pmod{\ell}$. We fix an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}_\ell}$. Let $W = \mathbb{Z}_\ell[\mu_{\mid\Delta\mid}]$. For $\chi \in \widehat{\Delta}$, we set:

$$e_\chi = \frac{1}{\mid \Delta \mid} \sum_{\delta \in \Delta} \chi(\delta) \delta^{-1} \in W[\Delta],$$

and:

$$M_\ell(\chi) = e_\chi(M \otimes_{\mathbb{Z}} W).$$

Thus, we have:

$$M \otimes_{\mathbb{Z}} W = \bigoplus_{\chi \in \widehat{\Delta}} M_\ell(\chi)\,.$$

2 Cyclotomic Function Fields and Artin-Schreier Extensions

Let Q be a prime of A of degree n, write $Q(T) = T^n + \alpha T^{n-1} + \cdots$, $\alpha \in \mathbb{F}_q$. We set: $i(Q) = Tr_{\mathbb{F}_q/\mathbb{F}_p}(\alpha)$. Let $a \in A$, $a \neq 0$, we set:

$$i(a) = \sum_{Q \text{ prime of A}} v_Q(a) i(Q) \in \mathbb{F}_p,$$

where v_Q is the normalized Q-adic valuation on k.

Let $\theta \in \overline{k}$ such that $\theta^p - \theta = T$. Set $\widetilde{A} = \mathbb{F}_q[\theta]$, $\widetilde{k} = \mathbb{F}_q(\theta)$ and $G = \mathrm{Gal}(\widetilde{k}/k)$. Note that $\widetilde{k}/k$ is unramified outside ∞ and totally ramified at ∞. Let $\widetilde{\infty}$ be the unique place of $\widetilde{k}$ above ∞.

Lemma 2.1 *Let $(., \widetilde{k}/k)$ be the usual Artin symbol. For $a \in A \setminus \{0\}$:*

$$(a, \widetilde{k}/k)(\theta) = \theta - i(a).$$

Proof By the classical properties of the Artin symbol, it is enough to prove the Lemma when a is a prime of A. Thus, let P be a prime of A of degree d. We have:

$$(P, \widetilde{k}/k)(\theta) \equiv \theta^{q^d} \pmod{P}.$$

But, for $n \geq 0$, we have:

$$\theta^{p^n} = \theta + T + T^p + \cdots + T^{p^{n-1}}.$$

Therefore:

$$\theta^{q^d} \equiv \theta - i(P) \pmod{P}.$$

The Lemma follows. $\diamondsuit$

Lemma 2.2 *Let P be a prime of A of degree d such that $i(P) \neq 0$. Then P is a prime of $\widetilde{A}$ of degree pd. Let $\widetilde{K_P}$ be the Pth cyclotomic function field for the ring $\widetilde{A}$, then $K_P \subset \widetilde{K_P}$.*

Proof We have $-T = -\theta^p(1 - \theta^{1-p})$. Note that:

$$1 - \theta^{1-p} \in (F_q((\frac{1}{\theta}))^*)^{q-1}.$$

Therefore:

$$\sqrt[q-1]{-T} \in F_q((\frac{1}{\theta}))(\sqrt[q-1]{-\theta}).$$

Thus:
- $\widetilde{k}K_P/\widetilde{k}$ is unramified outside $P, \widetilde{\infty}$,
- $\widetilde{k}K_P/\widetilde{k}$ is tamely ramified at $P, \widetilde{\infty}$,
- for every place w of $\widetilde{k}K_P$ above $\widetilde{\infty}$, the completion of $\widetilde{k}K_P$ at w is contained in $F_q((\frac{1}{\theta}))(\sqrt[q-1]{-\theta})$.

The Lemma follows by class field theory. $\diamondsuit$

Let P be a prime of A, $\deg_{\mathrm{T}}\mathrm{P(T)} = \mathrm{d}$ and $i(P) \neq 0$. Let $L = \widetilde{k}K_P \subset \widetilde{K_P}$. Let $\Delta = \mathrm{Gal(K_P/k)} \simeq \mathrm{Gal(L/\widetilde{k})}$. We have an isomorphism compatible to class field theory: $\widehat{\Delta} \to \widehat{\mathrm{Gal(L/\widetilde{k})}}$, $\chi \mapsto \widetilde{\chi} = \chi \circ N_{\widetilde{k}/k}$. We fix $\zeta_p \in \overline{\mathbb{Q}}$ a primitive pth root of unity.

Lemma 2.3
(1) Let $\chi \in \widehat{\Delta}$, $\chi \neq 1$. Let $L(X, \widetilde{\chi})$ be the Artin L-function relative to $L/\widetilde{k}$ and to the character $\widetilde{\chi}$. We have:

$$L(X, \widetilde{\chi}) = \prod_{\phi \in \widehat{G}} L(X, \phi\chi),$$

where $L(X, \phi\chi)$ is the Artin L-function relative to L/k and the character $\phi\chi$.
(2) Let $\chi \in \widehat{\Delta}$, $\chi \neq 1$, χ even (i.e. $\chi(\mathbb{F}_q^*) = \{1\}$). Then:

$$\frac{L(X, \widetilde{\chi})}{L(X, \chi)} \equiv (1 - X)^{p-1} L(X, \chi)^{p-1} \pmod{(1 - \zeta_p)}.$$

Proof Te assertion (1) is a consequence of the usual properties of Artin L-functions. Now, let $\phi \in \widehat{G}$, $\phi \neq 1$. Since $\phi\chi$ is ramified at ∞, we get:

$$L(X, \phi\chi) = \sum_{n \geq 0} (\sum_{a \in A^+, \, \mathrm{deg(a)=n}} \phi(a)\chi(a))X^n.$$

Thus:

$$L(X, \phi\chi) \equiv \sum_{n \geq 0} (\sum_{a \in A^+, \, \mathrm{deg(a)=n}} \chi(a)))X^n \pmod{(1 - \zeta_p)}.$$

But, since χ is even, we have $\chi(\infty) = 1$. Therefore:

$$L(X, \phi\chi) \equiv (1 - X)L(X, \chi) \pmod{(1 - \zeta_p)}.$$

The Lemma follows. $\diamondsuit$

let $i \in \mathbb{F}_p$ and let $\sigma_i \in G$ such that $\sigma_i(\theta) = \theta - i$. Let $\psi \in \widehat{G}$ given by $\psi(\sigma_i) = \zeta_p^i$.

Lemma 2.4 *Let $\chi \in \widehat{\Delta}$, χ even and non-trivial.*
(1) Let $\phi \in \widehat{G}$, $\phi \neq 1$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{\mathrm{p}})/\mathbb{Q})$ such that $\phi = \psi^{\sigma}$. Then:

$$L(X, \phi\chi) = L(X, \psi\chi)^{\sigma}.$$

Furthermore $\mathrm{deg_X L(X, \phi\chi) = d}$.
(2) We have:

$$L(1, \psi\chi) \equiv (\sum_{a \in A^+, \mathrm{deg(a) \leq d}} i(a)\chi(a))(\zeta_p - 1) \pmod{(1-\zeta_p)^2}.$$

Proof Let $\mathbb{Q}(\chi)$ be the abelian extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ the values of χ. Let $\mathbb{Z}[\chi]$ be the ring of integers of $\mathbb{Q}(\chi)$. Note that p is unramified in $\mathbb{Q}(\chi)$ and:

$$\mathrm{Gal}(\mathbb{Q}(\chi)(\zeta_{\mathrm{p}})/\mathbb{Q}(\chi)) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_{\mathrm{p}})/\mathbb{Q}).$$

Since $L(X, \phi\chi)$ is a polynomial in $\mathbb{Z}[\chi][\zeta_p][X]$, we have:

$$L(X, \phi\chi) = L(X, \psi\chi)^{\sigma}.$$

Since χ and $\widetilde{\chi}$ are non-trivial even characters, we have:

$$\mathrm{deg_X L(X, \widetilde{\chi}) = pd - 2},$$

and:

$$\mathrm{deg_X L(X, \chi) = d - 2}.$$

Therefore $\mathrm{deg_X L(X, \phi\chi) = d}$.

Now, we have:

$$L(X, \psi\chi) = \sum_{n=0}^{d} (\sum_{a \in A^+ \, \mathrm{deg(a)=n}} \zeta_p^{i(a)} \chi(a)) X^n.$$

But recall that:

$$\zeta_p^{i(a)} \equiv 1 + i(a)(\zeta_p - 1) \pmod{(1-\zeta_p)^2}.$$

Thus, since χ is even and non-trivial, we get:

$$L(X, \psi\chi) \equiv L(X, \chi)(1-X) + (\zeta_p - 1)(\sum_{n=1}^{d} (\sum_{a \in A^+ \, \mathrm{deg(a)=n}} i(a)\chi(a)) X^n) \pmod{(1-\zeta_p)^2}.$$

The Lemma follows. $\diamondsuit$

We are now ready to prove the main result of this section:

Proposition 2.5 *Let $\chi \in \widehat{\Delta}$, $\chi \neq 1$, χ even. Let $W = \mathbb{Z}_p[\mu_{q^d-1}]$. We have:*

$$\mathrm{Long_W}(\frac{\mathrm{Cl(O_L)_p}(\widetilde{\chi})}{\mathrm{Cl(O_{K_P})_p}(\chi)}) \geq 1 \Leftrightarrow \sum_{\mathrm{a \in A^+ \ deg(a) \leq d}} \mathrm{i(a)}\overline{\chi}(\mathrm{a}) \equiv 0 \pmod{\mathrm{p}}.$$

Proof Fix τ a generator of $G \simeq \mathrm{Gal(L/K_P)}$. Let $\varepsilon \in O_L^*$. Since L/K_P is totally ramified at any prime above ∞, there exists $\zeta \in \mathbb{F}_q^*$ such that $\tau(\varepsilon) = \zeta\varepsilon$. But $\tau^p(\varepsilon) = \zeta^p\varepsilon = \varepsilon$. Since we are in characteristic p, we deduce that $\varepsilon \in O_{K_P}^*$. Therefore:

$$O_L^* = O_{K_P}^*.$$

Let I be an ideal of O_{K_P} such that $IO_L = \alpha O_L$ for some $\alpha \in O_L$. Then, there exists $\varepsilon \in O_L^*$ such that $\tau(\alpha) = \varepsilon\alpha$. Since $O_L^* = O_{K_P}^*$ and since τ is of order p, we deduce that $\alpha \in O_{K_P}$. This implies that:

$$Cl(O_{K_P}) \hookrightarrow Cl(O_L).$$

One can also show that:

$$Cl^0(K_P) \hookrightarrow Cl^0(L).$$

Set $\Delta^+ = \frac{\Delta}{\mathbb{F}_q^*}$. Let $\mathcal{I}$ be the augmentation ideal of $\mathbb{F}_p[\Delta^+]$. One sees that we have the following isomorphism of Δ-modules:

$$\frac{R(L)}{R(K_P)} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathcal{I}.$$

This implie that we have the following exact sequence of W-modules:

$$0 \to \frac{W}{pW} \to \frac{Cl^0(L)_p(\widetilde{\chi})}{Cl^0(K_P)_p(\chi)} \to \frac{Cl(O_L)_p(\widetilde{\chi})}{Cl(O_{K_P})_p(\chi)} \to 0\,.$$

Now, by the results of Goss and Sinnott ([3]):

$$\mathrm{Long_W Cl^0(L)_p}(\widetilde{\chi}) = \mathrm{v_p(L}(1, \overline{\widetilde{\chi}})),$$

and

$$\mathrm{Long_W Cl^0(K_P)_p}(\chi) = \mathrm{v_p(L}(1, \overline{\chi})).$$

Thus by Lemma 2.3:

$$\mathrm{Long_W}(\frac{\mathrm{Cl(O_L)_p}(\widetilde{\chi})}{\mathrm{Cl(O_{K_P})_p}(\chi)}) = (\mathrm{p}-1)\mathrm{v_p(L}(1, \psi\overline{\chi})) - 1.$$

It remains to apply Lemma 2.4. $\diamondsuit$

3 Derivatives of L-functions

Let P be a prime of A of degree d. We fix an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}_p}$. Set $\Delta = \mathrm{Gal(K_P/k)}$ anf $W = \mathbb{Z}_p[\mu_{q^d-1}]$. We fix an isomorphism $\Phi_P : A/PA \rightarrow W/pW$. Then Φ_P induces an isomorphism:

$$\omega_P : \Delta \rightarrow \mu_{q^d-1} \subset W^*.$$

The morphism ω_P is called "the" Teichmüller character at P. Note that $\widehat{\Delta}$ is a cyclic group and ω_P is a generator of this group.

Let $i \in \mathbb{N}$, set:
- $\beta(0) = 1$,
- $\beta(i) = \sum_{a \in A^+} a^i$ if $i \geq 1$, $i \not\equiv 0 \pmod{q-1}$,
- $\beta(i) = -\sum_{a \in A^+} \mathrm{deg(a)a^i}$if $i \geq 1$, $i \equiv 0 \pmod{q-1}$.

One can prove that for all $i \in \mathbb{N}$, $\beta(i) \in A$. We also see that:

$$\forall i \in \mathbb{N},\, 0 \leq i \leq q^d - 2,\, \Phi_P(\beta(i)) \equiv L(1, \omega_P^i) \pmod{p}.$$

Therefore, if $1 \leq i \leq q^d - 2$, by the results of Goss and Sinnott ([3]), we have:

$$\mathrm{Long_W Cl^0(K_P)_p(\omega_P^{-i}) \geq 1 \Leftrightarrow \beta(i) \equiv 0 \pmod{P}}.$$

The numbers $\beta(i)$ are called the Bernoulli-Goss polynomials.

Recall that we have a surjective morphism of Δ-modules:

$$W[\Delta^+] \rightarrow R(K_P) \otimes_{\mathbb{Z}} W,$$

where $\Delta^+ = \Delta/\mathbb{F}_q^*$. Thus for $\chi \in \widehat{\Delta}$, χ even, $R(K_P)_p(\chi)$ is a cyclic W-module. But, for such a character, we have the exact sequence of W-modules:

$$0 \rightarrow R(K_P)_p(\chi) \rightarrow Cl^0(K_P)_p(\chi) \rightarrow Cl(O_{K_P})_p(\chi) \rightarrow 0\,.$$

This implies that, if $Cl(O_{K_P})_p(\chi) = \{0\}$, $Cl^0(K_P)_p(\chi)$ is a cyclic W-module.

David Goss has shown ([2], Corollary 8.16.2) that for χ is even, $\chi \neq 1$, if $L'(1, \overline{\chi}) \not\equiv 0 \pmod{p}$ (here $L'(1, \overline{\chi})$ is the derivative of $L(X, \overline{\chi})$ taken at $X = 1$), then $Cl^0(K_P)_p(\chi)$ is a cyclic W-module.

Therefore a natural question arise. Let $\chi \in \widehat{\Delta}$, $\chi \neq 1$, χ even. Assume that $L(1, \overline{\chi}) \equiv 0 \pmod{p}$. Do we have:

$$Cl(O_{K_P})_p(\chi) = \{0\} \Rightarrow L'(1, \overline{\chi}) \not\equiv 0 \pmod{p}?$$

Our aim in this section is to show that in general the answer is no.

Let d be an integer, $d \geq 1$. For $i \in \{1, \cdots, q^d - 2\}$, we set:

$$\gamma(d, i) = \sum_{a \in A^+, \, \deg(a) \leq d} i(a) a^i.$$

Lemma 3.1 *Let* $\tau \in \mathrm{Gal}(\mathbb{F}_q(T)/\mathbb{F}_q(T^p - T))$ *such that* $\tau(T) = T + 1$. *Let* $i \in \{1, \cdots, q^d - 2\}$, $i \equiv 0 \pmod{q-1}$. *Recall that* $q = p^s$. *We have:*

$$\tau(\gamma(d, i)) = \gamma(d, i) + s\beta(i).$$

Proof Let Q be a prime of A of degree n. Write $Q = T^n + \alpha T^{n-1} + \cdots$, where $\alpha \in \mathbb{F}_q$. Then $\tau(Q) = T^n + (\alpha + n)T^{n-1} + \cdots$. Therefore $i(\tau(Q)) = i(Q) + s\mathrm{deg}(Q)$. This implies that:

$$\forall a \in A \setminus \{0\}, \, i(\tau(a)) = i(a) + s\mathrm{deg}(a).$$

Now:

$$\tau(\gamma(d, i)) = \sum_{a \in A^+, \, \deg(a) \leq d} i(a) \tau(a)^i.$$

Therefore:

$$\tau(\gamma(d, i)) = \sum_{a \in A^+, \, \deg(a) \leq d} (i(\tau(a)) - s\mathrm{deg}(a)) \tau(a)^{\mathrm{i}}.$$

Thus:

$$\tau(\gamma(d, i)) = \sum_{a \in A^+, \, \deg(a) \leq d} i(\tau(a)) \tau(a)^i - s \sum_{a \in A^+, \, \deg(a) \leq d} \deg(\tau(a)) \tau(a)^{\mathrm{i}}.$$

Observe that $\sum_{a \in A^+, \, \deg(a) \leq d} i(\tau(a)) \tau(a)^i = \gamma(d, i)$ and $-\sum_{a \in A^+, \, \deg(a) \leq d} \deg(\tau(a)) \tau(a)^{\mathrm{i}} = \beta(\mathrm{i})$. $\diamondsuit$

Proposition 3.2 *Let* P *be a prime of* A *of degree* d *such that* $i(P) \neq 0$. *Set* $Q(T) = P(T^p - T)$. *Then* Q *is a prime of* A *of degree* pd. *Let* i *be an integer such that* $1 \leq i \leq q^d - 2$, $i \equiv 0 \pmod{q-1}$ *and* $Cl(O_{K_P})_p(\omega_P^{-i}) = \{0\}$. *Then:*

$$\mathrm{Long_W Cl(O_{K_Q})_p}(\omega_{\mathrm{Q}}^{-\mathrm{i}(\mathrm{q}^{\mathrm{pd}}-1)/(\mathrm{q}^{\mathrm{d}}-1)}) \geq 1 \Leftrightarrow \gamma(\mathrm{d}, \mathrm{i}) \equiv 0 \pmod{\mathrm{P}}.$$

Proof We have:

$$\Phi_P(\gamma(d,i)) \equiv \sum_{a\in A^+,\,\mathrm{deg(a)\leq d}} i(a)\omega_P^i(a) \pmod p.$$

It remains to apply Proposition 2.5. $\diamondsuit$

Lemma 3.3 *Assume* $p\neq 2$. *Let* $d\geq 1$ *be an integer. There exists a prime* P *in* A, $\mathrm{deg(P)=d}$, *such that* $i(P(T))i(P(T+1))\neq 0$.

Proof Let Q be a prime of A of degree d such that $i(Q)\neq 0$. Such a prime exists by the normal basis Theorem. Fix $\overline{\mathbb{F}_q}$ an algebraix closure of $\mathbb{F}_q$. We assume that $i(Q(T+1))=0$. Write $Q=T^d+\alpha T^{d-1}+\cdots$. Then $Tr_{\mathbb{F}_q/\mathbb{F}_p}(\alpha)=-sd$. Therefore $sd\not\equiv 0 \pmod p$.Let $\theta\in\overline{\mathbb{F}_q}$ such that $Q(\theta)=0$. We observe that:

$$\forall\zeta\in\mathbb{F}_p,\ Tr_{\mathbb{F}_{q^d}/\mathbb{F}_p}(\zeta\theta)=-\zeta sd.$$

Since $p\geq 3$, we can find $\zeta\in\mathbb{F}_p^*$ such that $-\zeta sd\neq -sd$. Set $P(T)=\mathrm{Irr}(\zeta\theta,\mathrm{F_q};\mathrm{T})$. Then P is a prime of degree d such that $i(P)i(\tau(P))\neq 0$. $\diamondsuit$

Proposition 3.4 *Assume that* $p\neq 2$ *and* $s\not\equiv 0 \pmod p$. *Let* d *be an integer,* $d\geq 2$, *and let* P *be a prime of degree* d *such that* $i(P(T))i(P(T+1))\neq 0$. *Set* $Q(T)=P(T^p-T)$. *Then:*
- $L(1,\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)})\equiv 0 \pmod p$,
- $L'(1,\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)})\equiv 0 \pmod p$,
- $Cl(O_{K_Q})_p(\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)})=\{0\}$.

Proof Set $R=P(T+1)$ and $Z=R(T^p-T)$. We observe that we have an isomorphism:

$$Cl(O_{K_Q})_p(\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)})\simeq Cl(O_{K_Z})_p(\omega_Z^{-(q-1)(q^{pd}-1)/(q^d-1)}).$$

Not also that $\beta(q-1)=1$. Thus:

$$Cl(O_{K_P})_p(\omega_P^{-(q-1)})=Cl(O_{K_R})_p(\omega_R^{-(q-1)})=\{0\}.$$

We have:

$$L(1,\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)})\equiv L(1,\omega_Z^{-(q-1)(q^{pd}-1)/(q^d-1)})\equiv 0 \pmod p.$$

And, by Lemma 2.3, since $p \geq 3$:

$$L'(1, \omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}) \equiv L'(1, \omega_Z^{-(q-1)(q^{pd}-1)/(q^d-1)}) \equiv 0 \pmod{p}.$$

Suppose that we have $Cl(O_{K_Q})_p(\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}) \neq \{0\}$. Then by Proposition3.2:

$$\gamma(d, q-1) \equiv 0 \pmod{P},$$

and also:

$$\gamma(d, q-1) \equiv 0 \pmod{R}.$$

Thus:

$$\tau(\gamma(d, q-1)) \equiv 0 \pmod{\tau(P)}.$$

Now, by Lemma 3.1, and the fact that $\tau(P) = R$, we get:

$$\gamma(d, q-1) + s\beta(q-1) \equiv 0 \pmod{R}.$$

Therefore we get $s \equiv 0 \pmod{p}$ which is a contradiction. The Proposition follows. $\diamondsuit$

4 Cyclicity of Class Groups and L-Functions

Let $E/\mathbb{F}_q$ be a global function field and let F/E be a finite geometric abelian extension. Set $\Delta = \mathrm{Gal(F/E)}$. Let ℓ be a prime number. Let's recall some well-known facts about L-functions.

Set $T_\ell = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{J})$ where J is the inductive limit of the $Cl^0(\mathbb{F}_{q^n}F)$, $n \geq 1$. We fix an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}_\ell}$. Let γ be the Frobenius of $\mathbb{F}_q$. Then γ and Δ act on T_ℓ.

If $\ell \neq p$, we have (see [6],chapter 15):

$$\mathrm{Det}(1 - \gamma \mathrm{X} \mid_{\mathrm{T}_\ell}) = \mathrm{L_F(X)},$$

where $L_F(X)$ is the numerator of the zeta function of F.

If $\ell = p$, write $L_F(X) = \prod_i (1 - \alpha_i X)$ and set $L_F^{nr}(X) = \prod_{v_p(\alpha_i)=0}(1 - \alpha_i X)$. Then (see [1] and also [3]):

$$\mathrm{Det}(1 - \gamma \mathrm{X} \mid_{\mathrm{T_p}}) = \mathrm{L_F^{nr}(X)}.$$

Now assume that ℓ does not divide the cardinal of Δ, then the above results are also valid character by character. More precisely, if $\ell \neq p$, we have:

$$\forall \chi \in \widehat{\Delta},\ \mathrm{Det}(1-\gamma \mathrm{X}\mid_{\mathrm{T}_\ell(\chi)}) = \mathrm{L}(\mathrm{X}, \overline{\chi}).$$

If $\ell = p$, for $\chi \in \widehat{\Delta}$, write $L(X,\chi) = \prod_i (1-\alpha_i(\chi)X)$ and set $L^{nr}(X,\chi) = \prod_{v_p(\alpha_i(\chi)=0}(1-\alpha_i(\chi)X)$. Then:

$$\forall \chi \in \widehat{\Delta},\ \mathrm{Det}(1-\gamma \mathrm{X}\mid_{\mathrm{T_p}(\chi)}) = \mathrm{L}^{\mathrm{nr}}(\mathrm{X}, \overline{\chi}).$$

Now, let $\chi \in \widehat{\Delta}$, write:

$$L(X,\chi) = \prod_i (1-\alpha_i(\chi)X),$$

and set:

$$g(X,\chi) = \prod_{v_\ell(\alpha_i(\chi)-1)>0} (1-\alpha_i(\chi)X).$$

Set:

$$g(X) = \prod_{\chi \in \widehat{\Delta}} g(X,\chi).$$

We also set:

$$\forall \chi \in \widehat{\Delta},\ H(X,\chi) = (1+X)^{\deg_{\mathrm{X}} \mathrm{g(X,\chi)}} g((1+X)^{-1},\chi),$$

and:

$$H(X) = \prod_{\chi \in \widehat{\Delta}} H(X,\chi).$$

For $n \geq 0$, set $F_n = \mathbb{F}_{q^{\ell^n}}F$, and let A_n be the ℓ-Sylow subgroup of $Cl^0(F_n)$. Let $F_\infty = \cup_{n\geq 0} F_n$ and let A_∞ be the inductive limit of the A_n, $n \geq 0$. We set:

$$Y = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{A}_\infty).$$

Set $\Gamma = \mathrm{Gal(F_\infty/F)}$,then γ is a topological generator of $\Gamma \simeq \mathbb{Z}_\ell$.

Lemma 4.1
(1) For all $n \geq 0$, we have an isomorphism of Δ-modules:

$$\frac{Y}{(\gamma^{\ell^n}-1)Y} \simeq A_n.$$

(2) Assume $\mid \Delta \mid \not\equiv 0 \pmod{\ell}$*. Then,* $\forall \chi \in \widehat{\Delta}$, $\forall n \geq 0$, *we have:*

$$\frac{Y(\chi)}{(\gamma^{\ell^n} - 1)Y(\chi)} \simeq A_n(\chi).$$

Proof We prove assertion (1), and note that (2) is a consequence of (1). Recall that A_∞ is a divisible group (see [6], Proposition 11.16). We start with the following exact sequence:

$$0 \to A_n \to A_\infty \to A_\infty \to 0,$$

where the middle map is the multiplication by $\gamma^{\ell^n} - 1$. We apply $\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, .)$ to this sequence, we get:

$$0 \to Y \to Y \to \mathrm{Ext}^1(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{A_n}) \to 0.$$

we also have the following exact sequence:

$$0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \frac{\mathbb{Q}_\ell}{\mathbb{Z}_\ell} \to 0.$$

We apply $\mathrm{Hom}(., \mathrm{A_n})$ to this last sequence, using the fact that:

$$\mathrm{Ext}^1(\mathbb{Q}_\ell, \mathrm{A_n}) = \{0\},$$

we get:

$$\mathrm{Hom}(\mathbb{Z}_\ell, \mathrm{A_n}) \simeq \mathrm{Ext}^1(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathrm{A_n}).$$

The Lemma follows. ♢

Proposition 4.2
(1) Let $\Lambda = \mathbb{Z}_\ell[[X]]$ *be the Iwasawa algebra of* Γ *over* $\mathbb{Z}_\ell$ *where* X *acts like* $\gamma - 1$*. Then* Y *is a finitely generatyed* Λ*-module and a torsion* Λ*-module. The characteristic polynomial of the* Λ*-module* Y *is equal to* $H(X)$*.*
(2) Assume that ℓ *does not divide the cardinal of* Δ*. Let* $\Lambda = W[[X]]$ *be the Iwasawa algebra of* Γ *over* $W = \mathbb{Z}_\ell[\mu_{|\Delta|}]$ *where* X *acts like* $\gamma - 1$*. Then, for* $\chi \in \widehat{\Delta}$, $Y(\chi)$ *is a finitely generated* Λ*-module and a torsion* Λ*-module. The characteristic polynomial of tha* Λ*-module* Y *is equal to* $H(X, \overline{\chi})$*.*

Proof We prove (1), the proof of (2) is essentially similar. For all $n \geq 0$, we set $\omega_n(X) = (1+X)^{\ell^n} - 1$. By Lemma 4.1, we have:

$$\forall n \geq 0, \ \frac{Y}{\omega_n Y} \simeq A_n.$$

Therefore Y is a finitely generated Λ-module and a torsion Λ-module. Let $r \in \mathbb{N}$ such that we have an isomorphism of groups:

$$Y \simeq \mathbb{Z}_\ell^r.$$

Then, there exists a constant $\nu \in \mathbb{Z}$, such that, for all n sufficiently large:

$$\mid \frac{Y}{\omega_n Y} \mid = \ell^{rn+\nu}.$$

But, for all $n \geq 0$, we have:

$$\mid A_n \mid = \ell^{v_\ell(L_{F_n}(1))}.$$

Therefore, there exists a constant $\nu' \in \mathbb{Z}$ such that, for all n sufficiently large:

$$\mid A_n \mid = \ell^{\mathrm{deg_X H(X) n} + \nu'}.$$

Thus: $r = \mathrm{deg_X H(X)}$. But let $V(X)$ be the characteristic polynomial of the Λ-module Y. We know that $r = \mathrm{deg_X V(X)}$, and we also know that $V(X)$ divides $(1+X)^{\mathrm{deg L_F(X)}} L_F((1+X)^{-1})$. But $V(X)$ is a distinguished polynomial, thus $V(X)$ divides $H(X)$. The Proposition follows. ♢

Proposition 4.3
(1) If A_0 is a cyclic $\mathbb{Z}_\ell$-module then $g(X)$ has simple roots.
(2) Assume that $\mid \Delta \mid \not\equiv 0 \pmod{\ell}$. Let $\chi \in \widehat{\Delta}$. If $A_0(\chi)$ is a cyclic W-module then $g(X, \overline{\chi})$ has simple roots.

Proof We prove (1). By Nakayama's Lemma, Y is pseudo-isomorphic to $\Lambda/H(X)\Lambda$. But, by a result of Tate ([8]), we know that the action of γ on Y is semi-simple. This implies that $H(X)$ has simple roots. ♢

Let's give an application of this last Proposition.

Proposition 4.4 *We assume that $q \geq 5$. Let $E/\mathbb{F}_q(T)$ be a real quadratic field, i.e. $[E : \mathbb{F}_q(T)] = 2$ and ∞ splits completely in E. If O_E is a principal ideal domain then $L_E(X)$ has simple roots.*

Proof Let g be the genus of E and write:

$$L_E(X)=\prod_{i=1}^{2g}(1-\alpha_i X).$$

Let $K=\mathbb{Q}(\alpha_1,\cdots,\alpha_{2g})$, then K is a CM-field. Let $\alpha\in\{\alpha_1,\cdots\alpha_{2g}\}$. Then:

$$(1-\alpha)(1-\overline{\alpha})\geq q+1-2\sqrt{q}>1.$$

Therefore:

$$N_{K/\mathbb{Q}}(1-\alpha)>1.$$

Thus $1-\alpha$ is not a unit of K. Let ∞_1 and ∞_2 be tha places of E above ∞. Then $R(E)$ is a quotient of $\mathbb{Z}(\infty_1-\infty_2)$ and we have an exact sequence:

$$0\rightarrow R(E)\rightarrow Cl^0(E)\rightarrow Cl(O_E)\rightarrow 0\,.$$

Therefore, if O_E is a principal ideal domain then $Cl^0(E)$ is a cyclic group. It remains to apply Proposition 4.3. $\diamondsuit$

It is conjectured that there exists infinitely many real quadratic function fields $E/\mathbb{F}_q(T)$ such that O_E is a principal ideal domain. In view of this conjecture, it will be interesting to prove that there exists infinitely many real quadratic function fields $E/\mathbb{F}_q(T)$ such that $L_E(X)$ has simple roots.

5 A Conjecture of Goss

Set $D_0=1$ and for $i\geq 1$, $D_i=(T^{q^i}-T)D_{i-1}^q$. The Carlitz exponential is defined by:

$$Exp(X)=\sum_{i\geq 0}\frac{X^{q^i}}{D_i}\in k[[X]].$$

Let $n\in\mathbb{N}$, write $n=a_0+a_1q+\cdots+a_rq^r$, where $a_0,\cdots,a_r\in\{0,\cdots,q-1\}$. We set:

$$\Gamma_n=\prod_{i=0}^{r}D_i^{a_i}.$$

The ith Bernoulli-Carlitz number, $B(i)\in k$, is defined by:

$$\frac{X}{Exp(X)}=\sum_{i\geq 0}\frac{B(i)}{\Gamma_i}X^i.$$

Let P be a prime of A of degree d and let $i \in \{1, \cdots, q^d - 2\}$, $i \equiv 0 \pmod{q-1}$. We have the following result ([5]):

$$Cl(O_{K_P})_p(\omega_P^i) \neq \{0\} \Rightarrow B(i) \equiv 0 \pmod{P}.$$

We fix an embedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}_p}$. Let $i \in \{1, \cdots, q^d - 2\}$. Write:

$$L(X, \omega_P^i) = \prod_j (1 - \alpha_j(i)X),$$

and set:

$$g(X, \omega_P^i) = \prod_{v_p(\alpha_j(i)-1)>0} (1 - \alpha_j(i)X).$$

Let $i \in \mathbb{N}$. We say that i is a q-magic number if there exist $c \in \{0, \cdots, q-2\}$ and an integer $n \in \mathbb{N}$ such that $i = cq^n + q^n - 1$.

Proposition 5.1 *Let P be a prime of A of degree d. Let i be a q-magic number, $1 \leq i \leq q^d - 2$, $i \equiv 0 \pmod{q-1}$. Then $g(X, \omega_P^i)$ has simple roots.*

Proof We have $i = q^n - 1$ for some integer n, $1 \leq n \leq d-1$. By a result of Carlitz ([2], Lemma 8.22.4):

$$B(q^d - 1 - i) = \frac{(-1)^{d-n}}{L_{d-n}^{q^n}},$$

where $L_0 = 1$ and for $j \geq 1$, $L_j = (T^{q^j} - T)L_{j-1}$. Therefore:

$$Cl(O_{K_P})_p(\omega^{-i}) = \{0\}.$$

It remains to aplly Proposition 4.3. $\diamondsuit$

In [2], David Goss makes the following conjecture:
let P be a prime of degree d and let i be a q-magic number, $1 \leq i \leq q^d - 2$. Then $\mathrm{deg_X g(X, \omega_P^i)} \leq 1$.

It is natural to ask if there exist primes P and q-magic numbers i, $1 \leq i \leq q^{\mathrm{degP}} - 2$, such that $\mathrm{deg_X g(X, \omega_P^i)} \geq 1$. This is the case.

Proposition 5.2 *Let $c \in \{0, \cdots, q-2\}$. There exist infinitely many primes P such that:*

$$\prod_{n=1}^{\mathrm{degP}-1} \beta(cq^n + q^n - 1) \equiv 0 \pmod{P}.$$

Proof We prove this Proposition for $c \neq 0$. The proof for $c = 0$ is very similar. If we apply the results in [7], we get:

$$\forall n \geq 0, \, \mathrm{deg_T}\, \beta(\mathrm{cq^n} + \mathrm{q^n} - 1) = \mathrm{n(c+1)q^n} - \frac{\mathrm{q^{n+1}} - \mathrm{q}}{\mathrm{q} - 1}.$$

Let S be the set of primes P in A such that:

$$\prod_{i=1}^{\mathrm{degP}-1} \beta(cq^n + q^n - 1) \equiv 0 \pmod{P}.$$

Let's assume that S is a finite set. We set:

$$D = \prod_{P \in S} \mathrm{degP},$$

and $D = 1$ if $S = \emptyset$. Note that:

$$\forall P \in S, \, q^D \equiv 1 \pmod{q^{\mathrm{degP}} - 1}.$$

Therefore, since $\beta(c) = 1$, we have:

$$\forall P \in S, \, \beta(cq^D + q^D - 1) \equiv 1 \pmod{P}.$$

But $\mathrm{deg_T}\beta(\mathrm{cq^D} + \mathrm{q^D} - 1) \geq 1$, thus we can select a prime Q of A such that $\beta(cq^D + q^D - 1) \equiv 0 \pmod{Q}$. Note that $Q \notin S$. Set $d = \mathrm{degQ}$. Since d does note divide D, there exists an integer r, $1 \leq r \leq d-1$, such that $D \equiv r \pmod{d}$. Therefore:

$$\beta(cq^D + q^D - 1) \equiv \beta(cq^r + q^r - 1) \equiv 0 \pmod{Q}.$$

But this implies that $Q \in S$, which is a contradiction. $\diamondsuit$

Let P be a prime of A of degree d. Let J be the jacobian of K_P, i.e. J is the inductive limit of the $Cl^0(\mathbb{F}_{q^n} K_P)$, $n \geq 1$. Set $\mathbb{F}_{q^{p^\infty}} = \cup_{n \geq 0} \mathbb{F}_{q^{p^n}} \subset \overline{\mathbb{F}_q}$, where $\overline{\mathbb{F}_q}$ is the algebraic closure of $\mathbb{F}_q$ in $\overline{k}$. We consider the $\Delta = \mathrm{Gal(K_P/k)}$-module:

$$\mathcal{A}_P = \frac{J[p]^{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^{p^\infty}})}}{Cl^0(K_P)[p]}.$$

As a consequence of the results in section 4, we get:

Proposition 5.3 *Let $W = \mathbb{Z}_p[\mu_{q^d-1}]$ and let $\chi \in \widehat{\Delta}$. We have:*

$$\dim_{\frac{\mathrm{W}}{\mathrm{pW}}} \mathcal{A}_{\mathrm{P}}(\chi) = \mathrm{deg_X g(X,\overline{\chi})} - \dim_{\frac{\mathrm{W}}{\mathrm{pW}}} \mathrm{Cl^0(K_P)_p}(\chi).$$

Note that in general, by Proposition 3.4, we do not have $\mathcal{A}_P = \{0\}$. But Goss conjecture implies the following:
let P be a prime of A of degree d and let i be a q-magic number, $1 \leq i \leq q^d - 2$, then $\mathcal{A}_P(\omega_P^{-i}) = \{0\}$.

It would be interesting to prove (or find a counter-example) to this weak form of Goss conjecture.

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