Continuous Discrete Observer with Updated Sampling Period (long version)

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Abstract: This paper deals with the design of high gain observers for a class of continuous dynamical systems with discrete-time measurements. Indeed, different approaches based on high gain techniques have been followed in the literature to tackle this problem. Contrary to these works, the measurement sampling time is considered to be variable. Moreover, the new idea of the proposed work is to synthesize an observer requiring the less knowledge as possible from the output measurements. This is done by using an updated sampling time observer. Under the global Lipschitz assumption, the asymptotic convergence of the observation error is established. As an application of this approach, an estimation problem of state of an academic bioprocess is studied, and its simulation results are discussed. This paper is the long version of Andrieu et al. [2013] with detailed proofs.

Keywords: Nonlinear systems, sampled-data, continuous-discrete time observers, high gain observer, updated sampled-data.

1. INTRODUCTION

The observer design for continuous nonlinear systems with sampled measurements has been initiated by Jazwinski who introduced the continuous-discrete Kalman filter to solve a filtering problem for stochastic continuous-discrete time systems (see Jazwinski [1970]). Except a few other results, there are mainly two classes of nonlinear observers that have been studied.

First, continuous-discrete observers based on the popular high-gain observer introduced in Gauthier et al. [1992]. One of the well-known observers usually used for real applications is the continuous-discrete time extended Kalman filter (see for instance Deza et al. [1992]). In there work, the continuous-discrete observer is obtained in two steps: i) when no measurement is available, the estimate is obtained by integrating the model. ii) when a measure occurs, the observer makes an impulsive correction on the estimate. The correction gain of this impulsive observer is obtained by integrating a continuous-discrete time Riccati equation. This work has been extended to other classes of systems in Nadri et al. [2004] and Ahmed Ali et al. [2007]. Recently, it has been shown in Nadri et al. [2013] that a constant correction term can be used.

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Note also that a new continuous-time observer using an output predictor for the time interval between two consecutive measurements has been given in Karafyllis and Kravaris [2009].

We note that in all works cited above, the asymptotic convergence of the estimate to the state is obtained by dominating the Lipschitz nonlinearities with high-gain techniques. This can lead to restrictive design conditions on the sampling measurement time.

Secondly, hybrid observers which use linear matrix inequalities (LMI) techniques for the gain calculation have been developped. In this class we can cite Raff et al. [2008], where the authors designed an impulsive observer where the correction term is constant between two measurements. The observer gain is synthesized to guarantee that a particular error Lyapunov function (obtained from the work of Naghshtabrizi et al. [2008]) is strictly decreasing along the trajectories. To obtain the observer gain some LMI conditions have to be satisfied. Recently, a new continuous-discrete observer design methodology for Lipschitz nonlinear systems based on reachability analysis was presented in Andrieu and Nadri [2010]. In their work, the authors show that the system satisfied by the estimation error can be rewritten in terms of a linear parameter varying system (LPV). The gain ensuring the convergence toward zero of the estimation error can be obtained by solving specific linear matrix inequalities.

Contrary to these works, the problem we intend to solve in this paper, is to design a high-gain observer requiring the less knowledge as possible from the output measurements. Following what has been done in Nadri et al. [2013] and inspired by Andrieu et al. [2009]), we will design a continuous discrete observer. However in opposition to these results, we consider the case in which the sampling time is variable and used as a tuning parameter. More precisely, we consider that the quantity $t_{k+1} - t_k$ is a part of the design of the continuous discrete observer. In the proposed algorithm, the measurement time is computed online. In fact, the use of sensors follows an event based on an extended observer state component. This may be related to the event-triggered control methodology (see for instance Tabuada [2007], Seuret and Prieur [2011]).

2. PROBLEM STATEMENT

In this work we consider the problem of designing an observer for nonlinear systems that are diffeomorphic to the following form

$$\dot{x} = Ax + Bf_n(x,u) \tag{1}$$

where the state x is in $\mathbb{R}^n$; $u : \mathbb{R} \to \mathbb{R}^p$ is a known input. A is a matrix in $\mathbb{R}^{n\times n}$ and B is in $\mathbb{R}^n$ given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \qquad B = [0 \ \dots \ 0 \ 1]^{\mathrm{T}} .$$

The measured output is given as a sequence of values $(y_k)_{k\geqslant 0}$ in $\mathbb{R}$:

$$y_k = Cx(t_k), \tag{2}$$

where $(t_k)_{k\geqslant 0}$ is a sequence of times to be selected and C is in $\mathbb{R}^n$

$$C = [1 \ 0 \ \dots \ 0] .$$

In this paper, we shall denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^n$ and we shall use the same notation for the corresponding induced matrix norm.

We consider the case in which the function $f_n : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ satisfies the following assumption.

Assumption 1. The function f_n is such that the following incremental bound is satisfied for all $(x,e,u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$,

$$|f_n(x+e,u) - f_n(x,u)| \leqslant c(x,u)|e| \ , \tag{3}$$

where $c : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}_+$ is a continuous function which satisfies the following bound

$$c(x,u) \leqslant \Gamma(u) \ , \ \forall \ (x,u) \in \mathbb{R}^n \times \mathbb{R}^p \ , \tag{4}$$

where $\Gamma : \mathbb{R}^p \to \mathbb{R}_+$.

Note that in the case in which we know a bound on the input u then this would imply that we come back to the globally Lipschitz context. However, even in this case, we believe that employing a tighter bound in term of a state-dependent function c implies that the sensors are less used than it would be if we were considering directly the Lipschitz bound.

3. UPDATED SAMPLING TIME OBSERVER

3.1 Structure of the proposed observer

This section is devoted to the design of a high gain observer to asymptotically estimate the state of the systems of the form (1)-(2). To do so, we consider the continuous-discrete time observer with updated sampling period given by [1] [2]

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bf_n(\hat{x}(t),u(t)) \ , \ \forall \ t \in [t_k, t_{k+1}) \ , \\ \hat{x}(t_k) &= \hat{x}(t_k^-) + \delta_k \mathcal{L}(t_k^-)K(C\hat{x}(t_k^-) - y_k), \end{aligned} \tag{5}$$

where K is a gain matrix, the matrix function $\mathcal{L} : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is defined as $\mathcal{L}(t) = \mathtt{diag}(L(t), \dots, L(t)^n)$ with $L : \mathbb{R}_+ \to \mathbb{R}$ is given as a solution to the following system of continuous discrete differential equations

$$\begin{aligned} &\begin{cases} \dot{L}(t) = a_2 L(t)M(t)c(\hat{x}(t),u(t)) \\ \dot{M}(t) = a_3 M(t)c(\hat{x}(t),u(t)) \end{cases} , \forall t \in [t_k, t_{k+1}) \ , \\ &\begin{cases} L(t_k) = L(t_k^-)(1 - a_1\alpha) + a_1\alpha \\ M(t_k) = 1 \end{cases} . \end{aligned} \tag{6}$$

initiated from $L(0) \geqslant 1$ and with $a_1\alpha < 1$. We have for all k,

$$y_k = Cx(t_k),$$

where the t_k's, k in $\mathbb{N}$ are given with the following iteration,

$$t_0 = 0 \ , \ t_{k+1} = t_k + \delta_k \ ,$$

and,

$$\delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}, \tag{7}$$

where α, a_1, a_2 and a_3 are positive real numbers to be defined.

3.2 About the updating time period

Note that for all k, δ_k is well defined. Indeed, L is not decreasing in every time interval $[t_k, t_{k+1})$. Moreover, when there is a jump (when there exists k such that $t = t_k$), we see that $L(t_k) \geqslant 1$ if $L(t_k^-) \geqslant 1$. Hence, we get $L(t) \geqslant 1$ for every $t \geqslant 0$. The function $s \mapsto sL(t_k + s)$ being continuous, zero at zero and going to infinity (if there is no jump), the existence of δ_k is well defined by (7). Also, we have $\delta_k < \alpha$ for all k.

Moreover we have the following lemma which shows that if the input is bounded then the high-gain parameter L is bounded along solutions.

Lemma 1. (Boundedness of L). If u is in $L_\infty(\mathbb{R}_+)$ then L is bounded along any solution of System (1) and its observer (5)-(6).

The proof of this lemma is postponed in Appendix A.1. Note that since by definition we have $L(t_{k+1}^-)\delta_k = \alpha$ this implies that δ_k is lower bounded.

These comments implies that for all bounded input time function u, the sequence of sampling period $(\delta_k)_{k\in\mathbb{N}}$ is well defined, upper and lower bounded, for all k in $\mathbb{N}$ and that

$$\lim_{k\to+\infty} t_k = +\infty \ .$$

[1] The solution $\hat{x}$ is a right-continuous function.

[2] Given a right-continuous function $\phi : \mathbb{R} \to \mathbb{R}^n$, the notation $\phi(t^-)$ stands for

$$\phi(t^-) = \lim_{h\to 0, h<0} \phi(t+h)$$

Note that if we know a bound on u, the function $c(\hat{x}, u)$ in (6) could be simply replaced by a constant depending on the function Γ. Note however that in this case, $\dot{L}$ is larger and this reduces the size of each sampling period $(\delta_k)_{k\in\mathbb{N}}$. Consequently, the sensors are more frequently employed which is something we would like to avoid.

3.3 Observer convergence

With this property in hand, we are now able to state our main result.

Theorem 1. (Updating continuous-discrete time observer). There exists a gain matrix K and $\alpha_m > 0$ such that for all α in $[0, \alpha_m]$, there exist a_1, a_2, a_3 such that for all bounded input functions the estimation error obtained using the observer (5)-(6) converges asymptotically toward zero. More precisely, for all initial condition $(x(0), \hat{x}(0))$ in $\mathbb{R}^n \times \mathbb{R}^n$ and $L(0)) \geqslant 1$ for all input function u in $L_\infty(\mathbb{R}_+)$ the associated solution of System (1)-(5)-(6) satisfies

$$\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0 \ .$$

Proof. Let D be the diagonal matrix in $\mathbb{R}^{n\times n}$ defined by

$$D = \mathtt{diag}(1, 2, \dots, n) \ .$$

Let P be a positive definite matrix in $\mathbb{R}^{n\times n}$ and K a vector in $\mathbb{R}^n$ such that the following inequality is satisfied (see [Praly, 2003, equation (14)] or [Krishnamurthy et al., 2003, equation (18)] or Andrieu et al. [2008])

$$p_1 I \leqslant P \leqslant p_2 I \ , \tag{8}$$

I being the identity matrix, and

$$(A + KC)'P + P(A + KC) \leqslant -I \ , \tag{9}$$
$$p_3 P \leqslant PD + DP \leqslant p_4 P \ ,$$

with $p_1, \dots, p_4$ positive real numbers.

Let $e \triangleq \hat{x} - x$ be the estimation error; e satisfies the following differential equation (*cf.* equations (1)-(5))

$$\dot{e}(t) = Ae(t) + B\Delta(\hat{x}(t), e(t), u(t)), \quad \forall\, t \in [t_k, t_{k+1}), \tag{10}$$

where the function $\Delta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is defined as

$$\Delta(\hat{x}, e, u) = f_n(\hat{x}, u) - f_n(\hat{x} - e, u) \ , \ \forall\ (\hat{x}, e, u) \ .$$

From Assumption 1 (i.e., inequality (3)), this function satisfies

$$|\Delta(\hat{x}, e, u)| \leqslant c(\hat{x}, u)|e| \ , \ \forall\ (\hat{x}, e, u) \ .$$

If we integrate equation (10) on the interval $[t_k, t_k + \tau)$ with $\tau < \delta_k$, we get

$$e(t_k + \tau) = \exp(A\tau)e(t_k) + \tag{11}$$
$$\int_0^\tau \exp(A(\tau - s))B\Delta(\hat{x}(t_k + s), e(t_k + s), u(t_k + s))ds \ ,$$

and so,

$$e(t_{k+1}) = (I + \delta_k \mathcal{L}(t_{k+1}^-)KC)e((t_k + \delta_k)^-) \ . \tag{12}$$

In the sequel, and using the results presented in Khalil and Saberi [1987] (see also Andrieu et al. [2009]) we consider the scaled observation error defined for all t by $E(t) = \mathcal{L}(t)^{-1}e(t)$. In the remaining part of the proof, we shall show that the Lyapunov function

$$V(E(t_k)) = E(t_k)'PE(t_k),$$

is decreasing toward zero along the solution of the system; here the $'$ denotes the transposition.

To simplify the presentation, we introduce the notations $L_k^- = L(t_k^-)$, $\mathcal{L}_k^- = \mathcal{L}(t_k^-)$, $L_k = L(t_k)$, $\mathcal{L}_k = \mathcal{L}(t_k)$, $E_k = E(t_k)$.

In order to evaluate the Lyapunov function, let us first remark the following algebraic properties of the matrix function $\mathcal{L}_k$. Note that we have

$$\begin{aligned}\left(\mathcal{L}_{k+1}^-\right)^{-1}(I + \delta_k \mathcal{L}_{k+1}^- KC) &= (I + \delta_k L_{k+1}^- KC)\left(\mathcal{L}_{k+1}^-\right)^{-1} \\ &= (I + \alpha KC)\left(\mathcal{L}_{k+1}^-\right)^{-1}, \end{aligned} \tag{13}$$

where the last equality has been obtained from (7). Moreover, since we have for all k, $\left(\mathcal{L}_k^-\right)^{-1} A = L_k^- A \left(\mathcal{L}_k^-\right)^{-1}$, it yields for all k

$$\left(\mathcal{L}_k^-\right)^{-1} A^i = L_k^- A \left(\mathcal{L}_k^-\right)^{-1} A^{i-1} = (L_k^- A)^i \left(\mathcal{L}_k^-\right)^{-1} \ ,$$

and

$$\left(\mathcal{L}_k^-\right)^{-1} \exp(As) = \left(\mathcal{L}_k^-\right)^{-1} \sum_{i=0}^{+\infty} \frac{A^i s^i}{i!} = \exp(L_k^- As)\left(\mathcal{L}_k^-\right)^{-1} \ . \tag{14}$$

Hence employing the previous algebraic equalities (13) and (14), we get, when left multiplying (12) by $\left(\mathcal{L}_{k+1}^-\right)^{-1}$,

$$\left(\mathcal{L}_{k+1}^-\right)^{-1} e_{k+1} = Q(\alpha)\left(\mathcal{L}_{k+1}^-\right)^{-1} \mathcal{L}_k E_k + R,$$

with

$$Q(\alpha) = (I + \alpha KC)\exp(A\alpha) \ ,$$

and

$$R = (I + \alpha KC)\int_0^{\delta_k} \exp(AL_{k+1}^-(\delta_k - s))\left(\mathcal{L}_{k+1}^-\right)^{-1}$$
$$\cdot B\Delta(\hat{x}(t_k + s), e(t_k + s), u(t_k + s))ds \ .$$

Note that, since we have $E_{k+1} = \Psi\left(\mathcal{L}_{k+1}^-\right)^{-1} e_{k+1}$ with $\Psi = \left(\mathcal{L}_{k+1}\right)^{-1}\mathcal{L}_{k+1}^-$, it yields

$$V(E_{k+1}) = V(E_k) + T_1 + T_2 \ ,$$

with

$$T_1 = E_k'\,\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1} Q(\alpha)'\,\Psi P \Psi\, Q(\alpha)\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_k\,E_k - V(E_k),$$

and

$$T_2 = 2\,E_k'\,\left(\mathcal{L}_{k+1}\right)^{-1}\mathcal{L}_k Q(\alpha)'\Psi P \Psi R + R'\Psi P \Psi R.$$

The remaining part of the proof is divided into three parts. The first two ones are devoted to upper bound the two terms T_1 and T_2. The fact that the Lyapunov function is decreasing is due to the term T_1 which will be shown to be negative. The second term is handled by robustness. In the last part of the proof we show that the Lyapunov function is decreasing.

Upper bounding T_1 In this part of the proof we show that there exists $\alpha_1 > 0$ such that for all α in $[0, \alpha_1)$ there exists a_1 such that

$$T_1 \leqslant -\left(\frac{a_2 p_3 p_1}{a_3}\left[\exp\left(a_3 \int_0^{\delta_k} c(r)dr\right) - 1\right] + \frac{\alpha p_1}{4p_2}\right) \times \left|\left(\mathcal{L}_{k+1}^-\right)^{-1} e_k\right|^2, \tag{15}$$

where $c(r) = c(\hat{x}(r), u(r))$.

First of all, to show the previous inequality, note that we have the following lemma which proof is given in Appendix.

Lemma 2. Taking a_1 sufficiently small, there exists α_1 sufficiently small such that for all $\alpha < \alpha_1$ we have

$$Q(\alpha)'\Psi P\Psi Q(\alpha) \leqslant P - \alpha\frac{1}{4p_2}P. \qquad (16)$$

On another hand, we have, for all v in $\mathbb{R}^n$

$$\begin{aligned} &v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}P\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_kv - v'Pv \\ &= v'\Bigg(\int_{t_k}^{t_{k+1}}\mathcal{L}_k\frac{d}{ds}\left(\mathcal{L}(s)^{-1}\right)P\mathcal{L}(s)^{-1}\mathcal{L}_k \\ &\qquad + \mathcal{L}_k\mathcal{L}(s)^{-1}P\frac{d}{ds}\left(\mathcal{L}(s)^{-1}\right)\mathcal{L}_kds\Bigg)v. \end{aligned}$$

However, we have for all s in $[t_k, t_{k+1})$

$$\frac{d}{ds}\left(\mathcal{L}(s)^{-1}\right) = -\frac{\dot{L}(s)}{L(s)}D\mathcal{L}(s)^{-1}.$$

Consequently, it yields

$$\begin{aligned} &v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}P\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_kv - v'Pv \\ &= -v'\left(\int_{t_k}^{t_{k+1}}\frac{\dot{L}(s)}{L(s)}\mathcal{L}_k\mathcal{L}(s)^{-1}[PD+DP]\mathcal{L}(s)^{-1}\mathcal{L}_kds\right)v. \end{aligned}$$

Bearing in mind that $L \geqslant 1$ and $\dot{L} \geqslant 0$ and taking into account the bounds on $DP + PD$ in (9) and P in (8), we get

$$\begin{aligned} &v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}P\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_kv - v'Pv \\ &\leqslant -p_3v'\Bigg(\int_{t_k}^{t_{k+1}}a_2c(s)\exp\left(a_3\int_{t_k}^{s}c(r)dr\right) \\ &\qquad \mathcal{L}_k\mathcal{L}(s)^{-1}P\mathcal{L}(s)^{-1}\mathcal{L}_kds\Bigg)v \\ &\leqslant -p_3p_1v'\Bigg(\int_{t_k}^{t_{k+1}}a_2c(s)\exp\left(a_3\int_{t_k}^{s}c(r)dr\right) \\ &\qquad \mathcal{L}_k\mathcal{L}(s)^{-1}\mathcal{L}(s)^{-1}\mathcal{L}_kds\Bigg)v. \end{aligned}$$

Note that since $L_k \leqslant L(s) \leqslant L_{k+1}^-$, we finally get

$$\begin{aligned} &v'\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}P\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_kv - v'Pv \\ &\leqslant -\frac{a_2p_3p_1}{a_3}\left[\exp\left(a_3\int_{t_k}^{t_{k+1}}c(r)dr\right)-1\right]\left|\left(\mathcal{L}_{k+1}^-\right)^{-1}\mathcal{L}_kv\right|^2. \end{aligned}$$

Consequently, the bound (15) is obtained from the previous inequality with $v = E_k$, and from inequality (16) in Lemma 2 together with (8).

Upper bounding T_2 In this part of the proof we show that there exist two continuous functions N_1 and N_2 such that the following inequality holds

$$\begin{aligned} T_2 \leqslant \left|\left(\mathcal{L}_{k+1}^-\right)^{-1}e_k\right|^2\Bigg[&N_1(\alpha)\left[\exp\left(\int_0^{\delta_k}c(t_k+r)dr\right)-1\right] \\ &+ N_2(\alpha)\left[\exp\left(\int_0^{\delta_k}c(t_k+r)dr\right)-1\right]^2\Bigg]. \qquad (17) \end{aligned}$$

In order to show that (17) holds, let us first analyse the term R. Note that we have

$$\begin{aligned} |R| &\leqslant |I+\alpha KC|\int_0^{\delta_k}\exp(|A|L_{k+1}^-(\delta_k - s))\left(L_{k+1}^-\right)^{-n} \\ &\qquad |\Delta(\hat{x}(t_k+s), e(t_k+s), u(t_k+s))|ds \\ &\leqslant |I+\alpha KC|\int_0^{\delta_k}\exp(|A|L_{k+1}^-(\delta_k - s))\left(L_{k+1}^-\right)^{-n} \\ &\qquad c(t_k+s)|e(t_k+s))|ds. \qquad (18) \end{aligned}$$

On another hand, we have for all s in $[t_k, t_{k+1})$

$$|e(t_k+s)| \leqslant \exp\left(\int_0^s|A| + c(t_k+r)dr\right)|e(t_k)|. \qquad (19)$$

Let us prove equation (19). From (10), we get

$$|\dot{e}(t_k+s)| \leqslant |A|\,|e(t_k+s)| + c(t_k+s)|e(t_k+s)|, \qquad (20)$$

for every $s \in [0, \delta_k)$ (we have $|B| = 1$). On the other hand

$$\begin{aligned} \frac{d}{ds}(\log|e|) &= \frac{\langle e, \dot{e}\rangle}{|e|^2} \\ &\leqslant \frac{|\dot{e}|}{|e|} \\ &\leqslant |A| + c(t_k+s) \qquad \text{(according to (20))}, \end{aligned}$$

and the result follows easily from this last inequality.

Hence, according to (18) and (19),we get

$$\begin{aligned} |R| &\leqslant |I+\alpha KC|\int_0^{\delta_k}\exp\left(|A|L_{k+1}^-(\delta_k - s)\right)\left(L_{k+1}^-\right)^{-n} \\ &\qquad c(t_k+s)\exp\left(\int_0^s|A| + c(t_k+r)dr\right)|e_k|ds \\ &= |I+\alpha KC|\exp\left(|A|\alpha\right)\frac{|e_k|}{\left(L_{k+1}^-\right)^n}\int_0^{\delta_k}\exp\left(s|A|(1-L_{k+1}^-)\right) \\ &\qquad c(t_k+s)\exp\left(\int_0^s c(t_k+r)dr\right)ds \\ &\leqslant |I+\alpha KC|\exp\left(|A|\alpha\right)\frac{|e_k|}{\left(L_{k+1}^-\right)^n} \\ &\qquad \int_0^{\delta_k}c(t_k+s)\exp\left(\int_0^s c(t_k+r)dr\right)ds \\ &= |I+\alpha KC|\exp\left(|A|\alpha\right)\frac{|e_k|}{\left(L_{k+1}^-\right)^n} \\ &\qquad \left[\exp\left(\int_0^{\delta_k}c(t_k+r)dr\right)-1\right] \end{aligned}$$

Since $L_{k+1}^- \geqslant L_k$, we have

$$\frac{|e_k|}{\left(L_{k+1}^-\right)^n} \leqslant \left|\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}E_k\right|,$$

and we finally get the following inequality

$$\begin{aligned} |R| \leqslant |I+\alpha KC|\exp(|A|\alpha)\left|\mathcal{L}_k\left(\mathcal{L}_{k+1}^-\right)^{-1}E_k\right| \\ \left[\exp\left(\int_0^{\delta_k}c(t_k+r)dr\right)-1\right] \end{aligned}$$

Hence, employing Lemma 3 this gives the existence of two continuous function N_1 and N_2 such that $N_1(0) = N_2(0) = 1$ and

$$2E_k'(\mathcal{L}_{k+1}^-)^{-1}\mathcal{L}_k Q(\alpha)'\Psi P\Psi R$$
$$\leqslant \left|(\mathcal{L}_{k+1}^-)^{-1}\mathcal{L}_k E_k\right|^2 N_1(\alpha)\left[\exp\left(\int_0^{\delta_k} c(t_k+r)dr\right)-1\right].$$

Moreover,

$$R'\Psi P\Psi R \leqslant$$
$$\left|(\mathcal{L}_{k+1}^-)^{-1}\mathcal{L}_k E_k\right|^2 N_2(\alpha)\left[\exp\left(\int_0^{\delta_k} c(\hat{x}(t_k+r))dr\right)-1\right]^2.$$

The two previous inequalities imply that (17) holds.

Lyapunov analysis With the two bounds obtained for T_1 and T_2 in (15) and (17), we finally get

$$V(E_{k+1})-V(E_k)\leqslant \left|(\mathcal{L}_{k+1}^-)^{-1}e_k\right|^2\cdot\Big[N_1(\alpha)[e^\beta-1]$$
$$+N_2(\alpha)[e^\beta-1]^2-\frac{a_2p_3p_1}{a_3}\left[e^{a_3\beta}-1\right]-\frac{\alpha p_1}{4p_2}\Big],$$

where β denotes the integral $\beta=\int_0^{\delta_k}c(t_k+r)dr$. Note that for all α, thanks to a good choice of a_3 and a_2 it yields that the right-hand member in the previous inequality is negative for every β. For example, if we take $a_3=2$, the previous inequality becomes

$$V(E_{k+1})-V(E_k)\leqslant \left|(\mathcal{L}_{k+1}^-)^{-1}e_k\right|^2\cdot\Big[-\frac{\alpha p_1}{4p_2}$$
$$+\left[e^\beta-1\right]\left[-\frac{a_2p_3p_1}{2}\left[e^\beta+1\right]+N_1(\alpha)+N_2(\alpha)[e^\beta-1]\right]\Big],$$
$$\leqslant \left|(\mathcal{L}_{k+1}^-)^{-1}e_k\right|^2\cdot\Big[-\frac{\alpha p_1}{4p_2}$$
$$+\left[e^\beta-1\right]\left[e^\beta+1\right]\left[-\frac{a_2p_3p_1}{2}+N_1(\alpha)+N_2(\alpha)\right]\Big].$$

If $a_2\geqslant 2\frac{N_1(\alpha)+N_2(\alpha)}{p_3p_1}$ it yields

$$V(E_{k+1})-V(E_k)\leqslant -\frac{\alpha p_1}{4p_2}\left|(\mathcal{L}_{k+1}^-)^{-1}e_k\right|^2.$$

The function V being lower bounded, it yields that

$$\lim_{k\to+\infty}\left|(\mathcal{L}_{k+1}^-)^{-1}e_k\right|=0\ .$$

The function L being upper and lower bounded (by Lemma 1), this implies that the error $e(t_k)$ goes to zero. With (11), we get the result. □

4. ILLUSTRATION

In this section, the performance of the proposed observer is illustrated through a bioreactor. In most cases, a cheap and reliable instrumentation required for real-time measurement of key variables of such process (biomass, substrate) are not available. Nevertheless, biomass measurement can be obtained using off-line analysis (sampled measurements) which require time and staff investment. The proposed approach allows to reduce the measurements frequency and consequently, the monitoring cost is also reduced.

The bioprocess considered is an academic bioreactor which consists of a microbial culture which involves a biomass X growing on a substrate S. The bioprocess is supposed to be continuous with a scalar dilution rate D and an input substrate concentration S_{in}. We assume also that a filtration element is installed at the reactor output. Under these conditions and using the Contois model, the dynamical model of the process is

$$\begin{aligned}\dot{X}(t)&=\frac{K_1S(t)}{K_2X+S(t)}X(t)\\ \dot{S}(t)&=-K_3\frac{K_1S(t)}{K_2X(t)+S(t)}X(t)-D(t)\left(S(t)-S_{in}(t)\right),\end{aligned}\tag{21}$$

where $K_i;(i=1,2,3)$ are positive constants. Our objective is the on-line estimation of the substrate concentrations S through sampled biomass measurements. In the case where the output is assumed to be a time-continuous, the authors in Gauthier et al. [1992] gave a stationary high gain observer. In the sequel, the same hypothesis as in Gauthier et al. [1992] and the same notations are used. Set the state vector $z(t)=[X(t),S(t)]^T$, the input $u(t)=D(t)$ and the output $y(t_k)=X(t_k)$. Under the constraint $0<u_{\min}\leqslant u(t)\leqslant u_{\max}<K_1$, the authors in Gauthier et al. [1992], determined a compact domain $\mathcal{M}_z\in\mathbb{R}^2$ which is invariant under the normal form (1). Using normalized variables, and $K_i=1$ we have

$$\mathcal{M}_z=\{z\in\mathbb{R}^2: X\geqslant\epsilon_1, S\geqslant\epsilon_2, X+S\leqslant 1\},$$

where $\epsilon_1=\frac{(1-u_{\max})\epsilon_2}{S_{in}u_{\max}}$

Then, using the change of coordinates $\Phi(z)=x(z)=\left[X,\frac{SX}{X+S}\right]^T$ system (21) takes the normal form (1) with $n=2$ and

$$f_2(x,u)=S_{in}u-\left(1+u+\frac{2S_{in}u}{x_1}\right)x_2+\left(\frac{2}{x_1}+\frac{S_{in}u}{x_1^2}\right)x_2^2,$$

and x evolving in $\mathcal{M}_x=\Phi(\mathcal{M}_z)$.

Moreover, the function f_2 can be trivially extended to a global Lipschitz C^1 function on the whole domain $\mathbb{R}^2\times\mathcal{M}_u$. For all $(\hat{x},x,u)\in\mathbb{R}^2\times\mathcal{M}_z\times\mathcal{M}_u$, where $\mathcal{M}_u=[u_{\min},u_{\max}]$, we can write

$$\begin{aligned}|f_2(x,u)-f_2(\hat{x},u)|&\leqslant|f_2(x_1,x_2,u)-f_2(x_1,\hat{x}_2,u)|\\&\quad+|f_2(x_1,\hat{x}_2,u)-f_2(\hat{x}_1,\hat{x}_2,u)|\\&\leqslant c_1(x_1,x_2,\hat{x}_2,u)|x_2-\hat{x}_2|\\&\quad+c_2(x_1,\hat{x}_1,x_2,u)|x_1-\hat{x}_1|,\end{aligned}$$

where,

$$c_1(x_1,x_2,\hat{x}_2,u)=-\left(1+u+\frac{2S_{in}u}{x_1}\right)+\left(\frac{2}{x_1}+\frac{S_{in}u}{x_1^2}\right)(x_2+\hat{x}_2),$$

and,

$$c_2(x_1,\hat{x}_1,\hat{x}_2,u)=\frac{-2S_{in}u\hat{x}_2+2\hat{x}_2^2}{\hat{x}_1x_1}+S_{in}u\hat{x}_2^2\frac{\hat{x}_1+x_1}{\hat{x}_1^2x_1^2}.$$

Consequently, we obtain

$$|f_2(x,u)-f_2(\hat{x},u)|\leqslant c(\hat{x},u)|e|\ ,$$

where

$$c(\hat{x},u)=\max_{x\in\mathcal{M}_x}\left\{c_1(x_1,x_2,\hat{x}_2,u)+c_1(x_1,\hat{x}_1,\hat{x}_2,u)\right\}.$$

Now, it suffices to use (5)-(7) to give the updated sampling time observer.

4.1 Simulation results

For the simulation test [3], the output has been corrupted by an additive noisy signal as shown in Fig.1. The observer simulation was performed under similar operating conditions as the model ($K_i = 1$) and $S_{in} = .1$, and $u(t)$ is given in 1

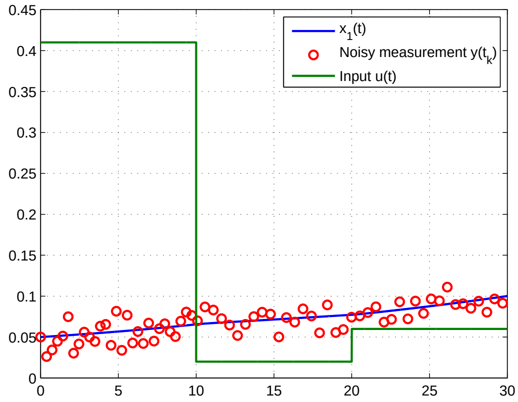

Fig. 1. Input $u(t) = D(t)$ and output $y = X(t_k)$ with measurement noise.

Fig. 2 displays displays the calculated values of the sampling-time δ_k. It may be noted that the sampling-time suggested by the proposed approach is relatively small when the estimation error is important and take a large value when the error is close to zero. In practice, where the initial error may be significant, the use an output sampling-time uniform can significantly increase the measurement cost. However, using this approach, we can significantly reduce the cost of measures.

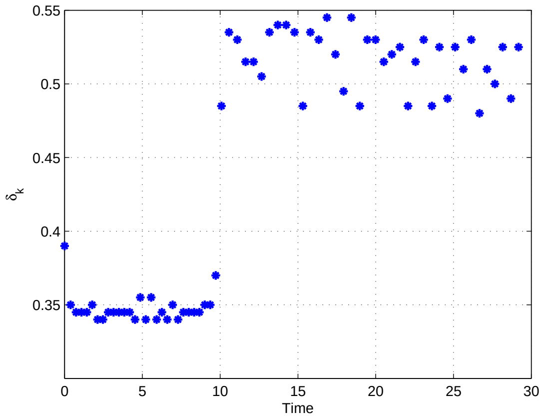

Fig. 2. Updated sampling-time δ_k.

Fig. 3 shows that, in spite of the measurement noise, the obtained estimation are not disturbed.

[3] The Matlab files can be downloaded from `https://sites.google.com/site/vincentandrieu/`

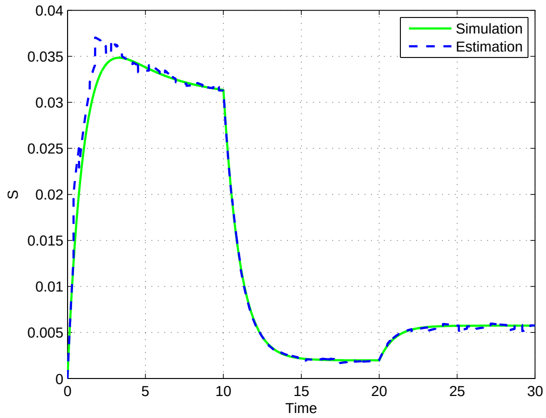

Fig. 3. S given by the model (21) compared to $\hat{S}$ given by system (5)-(7).

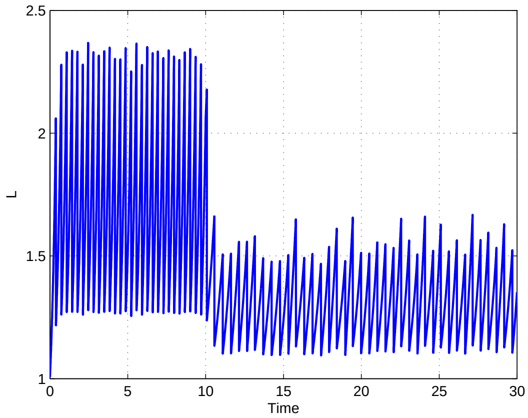

Fig. 4. Evolution of L .

5. CONCLUSION

In this paper, a high gain observer for continuous-discrete time systems in the observability normal form has been designed. The problem of observer synthesis for these systems is related to the sampling time of the output measurement which is always uniform and should be small to guarantee the observer convergence. To overcome this constraint which increases the control cost, a high gain updated sampling-time observer has been proposed. The principal advantage of this observer is that it requires the less knowledge as possible from the output measurement. The obtained results have been illustrated in the biological process and demonstrated good performances.

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Appendix A. PROOFS OF LEMMAS

A.1 Proof of Lemma 1

Assuming the input u is a bounded time function (with unknown bound), thanks to (4) we get that the function $t \mapsto c(\hat{x}(t), u(t))$ is bounded on the time of existence of the solution. Let c_m be an upper bound of $c(\hat{x}(t), u(t))$. Note that by integrating equation (6) with the previous upper bound on the interval $[t_k, t_{k+1})$, it yields for all k

$$L(t_{k+1}^-) \leqslant \kappa(\delta_k) L(t_k), \tag{A.1}$$

where κ is a strictly increasing function such that $\kappa(0) = 1$ defined as

$$\kappa(\delta_k) = \exp(a_2 \delta_k c_m \exp(a_3 \delta_k c_m)).$$

Hence, we have

$$L(t_{k+1}) \leqslant (1 - a_1\alpha)\kappa(\delta_k) L(t_k) + a_1\alpha \ .$$

Note moreover that we have $\dot{L}(s) \geqslant 0$ for all s in $[t_k, t_{k+1}^-)$, and so

$$L(t_{k+1}^-) \geqslant L(t_k).$$

Hence, since we have $L(t_{k+1}^-)\delta_k = \alpha$ we get $\delta_k \leqslant \frac{\alpha}{L(t_k)}$ which gives

$$\frac{L(t_{k+1})}{L(t_k)} \leqslant (1 - a_1\alpha)\kappa\left(\frac{\alpha}{L(t_k)}\right) + \frac{a_1\alpha}{L(t_k)}. \tag{A.2}$$

Note that for $L(t_k)$ sufficiently large, it yields that the left hand side of the previous inequality is smaller than 1. Hence this implies that the sequence $(L(t_k))_{k\in\mathbb{N}}$ is bounded. Consequently, it yields that δ_k is lower bounded on the time of existence of the solutions. To see this, denote by φ the function defined on the interval $(0, +\infty)$ as

$$\varphi(\ell) = (1 - a_1\alpha)\kappa\left(\frac{\alpha}{\ell}\right) + \frac{a_1\alpha}{\ell}.$$

Notice that φ is decreasing on this interval, that $\lim_{\ell\to 0}\varphi(\ell) = +\infty$ and that $\lim_{\ell\to+\infty}\varphi(\ell) = 1 - a_1\alpha < 1$; so there exists a unique $\ell_1 \in (0, +\infty)$ such that $\varphi(\ell_1) = 1$. Assume now that $L(t_k) \leqslant \ell_1$ for every $k \geqslant 0$, then we can say that the sequence $(L(t_k))_{k\geqslant 0}$ is bounded. If $L(t_k) \geqslant \ell_1$ for every $k \geqslant 0$, the inequality (A.2) implies that

$$\begin{aligned} L(t_{k+1}) &\leqslant L(t_k)\varphi(L(t_k)) \\ &\leqslant L(t_k)\varphi(\ell_1) \qquad \text{(because } L(t_k) \geqslant \ell_1) \\ &= L(t_k) \end{aligned}$$

and, arguing by induction, we see easily that $L(t_k) \leqslant L(t_0)$ for every k. The last situation is when some $L(t_k)$ are less than ℓ_1 while some others are greater than ℓ_1. So assume that we have, for some index k_0,

$$L(t_{k_0}) \leqslant \ell_1 \quad L(t_{k_0+1}) > \ell_1, \quad \dots \quad L(t_{k_0+i}) > \ell_1.$$

As above, we can prove that $L(t_{k_0+i}) \leqslant \ldots \leqslant L(t_{k_0+1})$. Now, from (A.1) and using the fact that $\delta_{k_0} \leqslant \frac{\alpha}{L(t_{k_0})}$ we get

$$\begin{aligned} L(t_{k_0+1}) &= (1 - a_1\alpha)L(t_{k_0+1}^-) + a_1\alpha \\ &\leqslant (1 - a_1\alpha)\kappa(\delta_{k_0})L(t_{k_0}) + a_1\alpha \\ &\leqslant (1 - a_1\alpha)\kappa\left(\frac{\alpha}{L(t_{k_0})}\right)L(t_{k_0}) + a_1\alpha \\ &\leqslant (1 - a_1\alpha)\kappa(\alpha)\ell_1 + a_1\alpha, \text{because } 1 \leqslant L(t_{k_0}) \leqslant \ell_1. \end{aligned}$$

Thus, we proved that $L(t_k) \leqslant \max(\ell_1, (1-a_1\alpha)\kappa(\alpha)\ell_1 + a_1\alpha)$ for every index k.

A.2 Proof of Lemma 2

In order to prove Lemma 2, we need the following lemma which will be proved in the next section.

Lemma 3. The matrix P satisfies the following property for all a_1 and α such that $a_1\alpha < 1$

$$\Psi P \Psi \leqslant \psi_0(\alpha) P \psi_0(\alpha) \ , \tag{A.3}$$

where

$$\psi_0(\alpha) = \texttt{diag}\left(\frac{1}{1-a_1\alpha}, \dots, \frac{1}{(1-a_1\alpha)^n}\right)$$

Given v in $S^{n-1} = \{v \in \mathbb{R}^n \mid |v| = 1\}$, consider the function

$$\nu(\alpha, v) = v' Q(\alpha)' \psi_0(\alpha) P \psi_0(\alpha) Q(\alpha) v \ .$$

We have

$$\nu(0, v) = v' P v,$$
$$\frac{\partial \nu}{\partial \alpha}(0, v) = v' \left[P[A + KC + a_1 D] + [A + KC + a_1 D]' P\right] v,$$

so using the inequalities in (9) and setting $a_1 = \frac{1}{2p_2p_4}$, we get

$$\begin{aligned}\frac{\partial \nu}{\partial \alpha}(0, v) &\leqslant v' \left(a_1 p_4 P - \frac{1}{p_2} P\right) v \\ &= -\frac{1}{2p_2} v' P v. \end{aligned} \tag{A.4}$$

Now, we can write

$$\nu(\alpha, v) = v' P v + \alpha \frac{\partial \nu}{\partial \alpha}(0, v) + \rho(\alpha, v)$$

with $\lim_{\alpha \to 0} \frac{\rho(\alpha, v)}{\alpha} = 0$. This equality together with (A.4) imply that

$$\nu(\alpha, v) \leqslant v' P v \left[1 - \alpha \frac{1}{2p_2}\right] + r(\alpha, v).$$

The vector v being in a compact set and the function r being continuous, there exists α_m such that for all α in $[0, \alpha_m)$ we have $r(\alpha, v) \leqslant \alpha \frac{1}{4p_2} v' P v$ for all v. This gives

$$\nu(\alpha, v) \leqslant v' P v \left[1 - \alpha \frac{1}{4p_2}\right] \ , \forall \ \alpha \in [0, \alpha_m) \ , \forall \ v \in S^{n-1} \ .$$

This property being true for every v, this ends the proof of Lemma 2.

A.3 Proof of Lemma 3

Consider the matrix function defined as

$$\mathcal{P}(s) = \texttt{diag}(s, \dots, s^n) P \texttt{diag}(s, \dots, s^n) \ .$$

Note that for all v in $\mathbb{R}^n$

$$\begin{aligned}&\tfrac{d}{ds} v' \mathcal{P}(s) v \\ &\quad = \tfrac{1}{s} v' \texttt{diag}(s, \dots, s^n)(D'P + PD)\texttt{diag}(s, \dots, s^n) v > 0.\end{aligned}$$

Hence $\mathcal{P}$ is a strictly increasing function. Furthermore, we have

$$\begin{aligned}\Psi P \Psi &= \mathcal{L}_{k+1}^{-1} \mathcal{L}_{k+1}^{-} P \mathcal{L}_{k+1}^{-} \mathcal{L}_{k+1}^{-1} \\ &= \mathcal{P}\left(\frac{L_{k+1}^{-}}{L_{k+1}^{-}(1 - a_1\alpha) + a_1\alpha}\right),\end{aligned}$$

so as

$$\frac{L_{k+1}^{-}}{L_{k+1}^{-}(1 - a_1\alpha) + a_1\alpha} \leqslant \frac{1}{1 - a_1\alpha},$$

we get the inequality of Lemma 3

$$\Psi P \Psi \leqslant \mathcal{P}\left(\frac{1}{1 - a_1\alpha}\right) \ .$$