INFINITE PRODUCTS OF NONNEGATIVE 2×2 MATRICES BY NONNEGATIVE VECTORS

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Abstract. Given a finite set $\{M_0, \dots, M_{d-1}\}$ of nonnegative 2×2 matrices and a nonnegative column-vector V, we associate to each $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$ the sequence of the column-vectors $\dfrac{M_{\omega_1} \dots M_{\omega_n} V}{\|M_{\omega_1} \dots M_{\omega_n} V\|}$. We give the necessary and sufficient condition on the matrices M_k and the vector V for this sequence to converge for all $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$ such that $\forall n,\ M_{\omega_1} \dots M_{\omega_n} V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

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Introduction

Let $\mathcal{M} = \{M_0, \dots, M_{d-1}\}$ be a finite set of nonnegative 2×2 matrices and $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ a nonnegative column-vector. We use the notation $Y_n = Y_n^\omega := M_{\omega_1} \dots M_{\omega_n}$ and give the necessary and sufficient condition for the pointwise convergence of $\dfrac{Y_n V}{\|Y_n V\|}$, $(\omega_n) \in \{0, \dots, d-1\}^{\mathbb{N}}$ such that $Y_n V \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any n, where $\|\cdot\|$ is the norm-sum. The idea of the proof is that, if the conditions are satisfied, either both columns of Y_n tends to the same limit, or they tend to different limits with different orders of growth, so in case V is positive the limit points of $\dfrac{Y_n V}{\|Y_n V\|}$ only depend on the limit of the dominant column. This problem is obviously very different from the one of the convergence of $\dfrac{Y_n}{\|Y_n\|}$, or the convegence of the Y_n itselves, see the intoduction of [5] for some counterexamples and [8, Proposition 1.2] for the infinite products of 2×2 stochastic matrices.
The conditions for the pointwise convergence of $\dfrac{Y_n V}{\|Y_n V\|}$ also differ from the conditions for its uniform convergence, see [2]. The uniform convergence can be used for the multifractal analysis of some continuous singular measures called Bernoulli convolutions (see [6] for the

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Bernoulli convolutions and [1] for their multifractal analysis). We study such measures in [2], [3] and [4]. The Birkhoff's contraction coefficient [7, Chapter 3] that we use in [3] and [5] but not here, is really not of great help to solve the main difficulties. Moreover the theorem that gives the value of this coefficient is difficult to prove (see [7, §3.4]) even in the case of 2×2 matrices. In [2] we use some other contraction coefficient quite more easy to compute ([2, Proposition 1.3]).

1. Condition for the pointwise convergence of $\frac{Y_nV}{\|Y_nV\|}$

Proposition 1.1. *The sequence* $\frac{Y_nV}{\|Y_nV\|}$ *converges for any* $\omega\in\{0,\dots,d-1\}^{\mathbb{N}}$ *such that* $\forall n,\ Y_nV\neq\begin{pmatrix}0\\0\end{pmatrix}$*, if and only if at least one of the following conditions holds:*
(i) V *has positive entries and it is an eigenvector of any invertible matrix of the form* $\begin{pmatrix}a&0\\0&d\end{pmatrix}$ *or* $\begin{pmatrix}0&b\\c&0\end{pmatrix}$ *that belongs to* $\mathcal{M}$*.*
(ii) Any invertible matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathcal{M}$ *satisfies* $a>0$ *and, if* $b=c=0$*,* $a\geq d$*.*
(iii) Any invertible matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathcal{M}$ *satisfies* $d>0$ *and, if* $b=c=0$*,* $d\geq a$*.*
(iv) V *has a null entry and all the invertible matrices* $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathcal{M}$ *satisfy* $ad>0$*.*

Proof. Let $\omega\in\{0,\dots,d-1\}^{\mathbb{N}}$. If there exists N such that $\det M_{\omega_N}=0$, the column-vectors $Y_NV,\ Y_{N+1}V,\ \dots$ are collinear and $\frac{Y_nV}{\|Y_nV\|}$ is constant for $n\geq N$. So we look only at the $\omega\in\{0,\dots,d-1\}^{\mathbb{N}}$ such that $\forall n,\ \det M_{\omega_n}\neq 0$. In order to use only matrices with positive determinant we set $\Delta:=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and

$$A_n=A_n^\omega:=\begin{cases}M_{\omega_n} & \text{if }\det Y_{n-1}>0\text{ (or }n=1)\text{ and }\det M_{\omega_n}>0\\ M_{\omega_n}\Delta & \text{if }\det Y_{n-1}>0\text{ (or }n=1)\text{ and }\det M_{\omega_n}<0\\ \Delta M_{\omega_n} & \text{if }\det Y_{n-1}<0\text{ and }\det M_{\omega_n}<0\\ \Delta M_{\omega_n}\Delta & \text{if }\det Y_{n-1}<0\text{ and }\det M_{\omega_n}>0.\end{cases}$$

We set also

$$\begin{pmatrix}a_n&b_n\\c_n&d_n\end{pmatrix}:=A_n\quad\text{and}\quad\begin{pmatrix}p_n&q_n\\r_n&s_n\end{pmatrix}:=A_1\dots A_n=\begin{cases}Y_n&\text{if }\det Y_n>0\\ Y_n\Delta&\text{if }\det Y_n<0.\end{cases}$$

The matrices A_n belong to the set

$$\mathcal{M}^+:=\{M\ ;\ \exists i,j,k,\ M=\Delta^iM_k\Delta^j\text{ and }\det M>0\}.$$

Since $\det A_n > 0$ we have $a_nd_np_ns_n \neq 0$. If $\{n;\ A_n \text{ not diagonal}\}$ is infinite we index this set by an increasing sequence $n_1 < n_2 < \dots$. We have $b_{n_{\mathbf{1}}} \neq 0$ or $c_{n_{\mathbf{1}}} \neq 0$; both cases are equivalent because, using the set of matrices $\mathcal{M}' = \Delta\mathcal{M}\Delta$ and defining similarly Y'_n and $A'_n = \begin{pmatrix} a'_n & b'_n \\ c'_n & d'_n \end{pmatrix}$ from this set, we have $Y'_n = \Delta Y_n \Delta$, $A'_n = \Delta A_n \Delta$ and $b'_{n_{\mathbf{1}}} = c_{n_{\mathbf{1}}}$.
So we can suppose $b_{n_{\mathbf{1}}} \neq 0$; we deduce $q_n \neq 0$ by induction on $n \geq n_1$. The sequences defined for any $n \geq n_1$ by

$$u_n = \frac{r_n}{p_n},\quad v_n = \frac{s_n}{q_n},\quad w_n = \frac{q_n}{p_n},\quad x_n = \begin{cases} v_2/v_1 & \text{if } \det Y_n > 0 \\ v_1/v_2 & \text{if not} \end{cases} \quad \text{and} \quad \lambda_n = (1 + w_n x_n)^{-1}$$

satisfy $0 \leq u_n < v_n < \infty$, $0 < w_n < \infty$ and

if the entries of V are positive, $0 < x_n < \infty$ and $0 < \lambda_n < 1$
if not, $x_n \in \{0, \infty\}$ and $\lambda_n \in \{0, 1\}$ according to the sign of $\det Y_n$.

Since we have assumed that $Y_nV \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the ratio $\dfrac{(Y_nV)_2}{(Y_nV)_1}$ exists in $[0, \infty]$ and we have to prove that it has a finite or infinite limit when $n \to \infty$. If A_n is not eventually diagonal we have for $n \geq n_1$

(1) $$\frac{(Y_nV)_2}{(Y_nV)_1} = \lambda_n u_n + (1 - \lambda_n)v_n \in I_n := [u_n, v_n] \text{ and } I_n \supseteq I_{n+1}.$$

An immediate consequence is the following lemma:

Lemma 1.1. *Suppose A_n is not eventually diagonal, then*
(i) the sequences (u_n) and (v_n) converge in $\mathbb{R}$ and the sequence $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ is bounded;
(ii) $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ converges if $\lim\limits_{n\to\infty} |I_n| = 0$;
(iii) if V has positive entries, $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ converges if w_n has limit 0 or ∞;
(iv) if V has a null entry, the necessary and sufficient condition for the convergence of $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ is that $\lim\limits_{n\to\infty} |I_n| = 0$ or the sign of $\det Y_n$ is eventually constant.

We also define for $n > n_1$

$$\alpha_n = \left(1 + \frac{c_n}{a_n} w_{n-1}\right)^{-1},\quad \beta_n = \left(1 + \frac{b_n}{d_n}(w_{n-1})^{-1}\right)^{-1},\quad \gamma_n = 1 - \frac{c_n}{a_n}\frac{b_n}{d_n}$$

that belong to $]0, 1]$ and satisfy

(2) $$\begin{aligned} |I_n| &= \alpha_n\beta_n\gamma_n|I_{n-1}| \\ w_n &= \frac{d_n}{a_n}\frac{\alpha_n}{\beta_n}w_{n-1} \end{aligned}$$

so $\prod_{n>n_{\mathbf{1}}} \alpha_n\beta_n\gamma_n = \lim_{n\to\infty} \frac{|I_n|}{|I_{n_{\mathbf{1}}}|}$ is positive if and only if $\lim_{n\to\infty} |I_n| > 0$. Using the equivalents of $\log \alpha_n$, $\log \beta_n$ and $\log \gamma_n$,

$$\text{(3)} \qquad \lim_{n\to\infty} |I_n| > 0 \Leftrightarrow \sum \frac{c_n}{a_n} w_{n-1} < \infty, \quad \sum \frac{b_n}{d_n}(w_{n-1})^{-1} < \infty \quad \text{and} \quad \sum \frac{c_n}{a_n}\frac{b_n}{d_n} < \infty.$$

The set of indexes $\{n;\ A_n \text{ not diagonal}\}$ is the union of

$$L^\omega = \{n;\ c_n \neq 0\} \quad \text{and} \quad U^\omega = \{n;\ b_n \neq 0\}.$$

Moreover, since A_n belongs to the finite set $\mathcal{M}^+$ there exists $K > 0$ such that

$$L^\omega = \left\{n;\ \frac{1}{K} \le \frac{c_n}{a_n} \le K\right\} \quad \text{and} \quad U^\omega = \left\{n;\ \frac{1}{K} \le \frac{b_n}{d_n} \le K\right\}.$$

We deduce a simpler formulation of (3):

$$\text{(4)} \qquad \lim_{n\to\infty} |I_n| > 0 \Leftrightarrow \sum_{n\in L^\omega} w_{n-1} < \infty, \quad \sum_{n\in U^\omega} (w_{n-1})^{-1} < \infty \quad \text{and} \quad L^\omega \cap U^\omega \text{ is finite.}$$

In view of Lemma 1.1 we may suppose from now that $\lim_{n\to\infty} |I_n| > 0$. Since $L^\omega \cap U^\omega = \{n\ ;\ A_n \text{ positive}\}$ is finite, for n large enough the matrix A_n is lower triangular if $n \in L^\omega$, upper triangular if $n \in U^\omega$, diagonal if $n \notin L^\omega \cup U^\omega$. When A_n is diagonal the second relation in (2) becomes $w_n = \frac{d_n}{a_n} w_{n-1}$; consequently any integer n in an interval $]n_i, n_{i+1}[$ with i large enough satisfies

$$\text{(5)} \qquad \frac{w_n}{w_{n_i}} = \prod_{n_i<j\le n} \frac{d_j}{a_j}.$$

Moreover if L^ω is infinite, (4) implies that w_{n-1} has limit to 0 when $L^\omega \ni n \to \infty$, and w_n also has limit 0 because $w_n = \frac{d_n w_{n-1}}{a_n + c_n w_{n-1}}$ for any $n \in L^\omega \setminus U^\omega$. We have a similar property if U^ω is infinite, so

$$\text{(6)} \qquad \begin{array}{ll} \text{if } L^\omega \text{ is infinite}, & w_{n-1} \to 0 \text{ and } w_n \to 0 \text{ for } L^\omega \ni n \to \infty; \\ \text{if } U^\omega \text{ is infinite}, & w_{n-1} \to \infty \text{ and } w_n \to \infty \text{ for } U^\omega \ni n \to \infty. \end{array}$$

First case: Suppose that (i) holds. Then the diagonal matrices of $\mathcal{M}$ are collinear to the unit matrix. If at least one matrix of $\mathcal{M}$ has the form $M_k = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$, its nonnegative eigenvalue – namely $\begin{pmatrix} \sqrt{b} \\ \sqrt{c} \end{pmatrix}$ – is collinear to $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ hence there exists $\lambda \in \mathbb{R}$ such that $M_k = \lambda \begin{pmatrix} 0 & v_1^2 \\ v_2^2 & 0 \end{pmatrix}$.

Notice that if A_n is diagonal from a rank N, the matrix M_{ω_n} has the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ hence it has V as eigenvector; consequently $\left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ converges because it is $\frac{(Y_NV)_2}{(Y_NV)_1}$ for any $n \geq N$.

Suppose now A_n is non-diagonal for infinitely many n. We apply (5) on each interval $]n_i, n_{i+1}[$ (if non empty), for i large enough. Among the integers $n \in]n_i, n_{i+1}[$ we consider the ones for which $\det M_{\omega_n} < 0$. For such n the matrix A_n is alternately $M_{\omega_n}\Delta$ and ΔM_{ω_n}, hence alternately proportional to $\begin{pmatrix} v_1^2 & 0 \\ 0 & v_2^2 \end{pmatrix}$ and to $\begin{pmatrix} v_2^2 & 0 \\ 0 & v_1^2 \end{pmatrix}$ and, according to (5),

$$n_i \leq n < n_{i+1} \Rightarrow \frac{w_n}{w_{n_i}} \in \left\{\frac{v_1^2}{v_2^2}, \frac{v_2^2}{v_1^2}, 1\right\}. \tag{7}$$

In particular this relation holds for $n = n_{i+1} - 1$. One deduce – according to (6) – that there do not exist infinitely many i such that $n_i \in L^\omega$ and $n_{i+1} \in U^\omega$. Thus $n_i \in L^\omega$ for i large enough (resp. $n_i \in U^\omega$ for i large enough) and, according to (6) and (7), $\lim\limits_{n\to\infty} w_n = 0$ (resp. $\lim\limits_{n\to\infty} w_n = \infty$). In view of Lemma 1.1(iii), $\left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ converges.

Second case: Suppose that (ii) holds (if (iii) holds the proof is similar).

Suppose first the M_{ω_n} are diagonal from a rank N. From the hypothesis (ii) there exists δ_n, δ'_n such that $M_{\omega_N} \dots M_{\omega_n} V = \begin{pmatrix} \delta_n v_1 \\ \delta'_n v_2 \end{pmatrix}$ and $\delta_n \geq \delta'_n$. Since the M_{ω_i} belong to a finite set we have $\lim\limits_{n\to\infty} \frac{\delta_n}{\delta'_n} = \infty$, or $\frac{\delta_n}{\delta'_n}$ is eventually constant in case M_{ω_n} is eventually the unit matrix, or $\delta'_n = 0 \neq \delta_n$ for n large enough. Denoting by $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ the matrix $M_{\omega_{\mathbf{1}}} \dots M_{\omega_{N-\mathbf{1}}}$, we have $\frac{(Y_nV)_2}{(Y_nV)_1} = \frac{r\delta_n v_1 + s\delta'_n v_2}{p\delta_n v_1 + q\delta'_n v_2}$ hence $\left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ converges in all the cases.

Suppose now M_{ω_n} is non-diagonal for infinitely many n. There exists from (6) an integer κ such that

$$i \geq \kappa \Rightarrow \begin{cases} w_{n_i-1} < 1 \text{ and } w_{n_i} < 1 & \text{if } n_i \in L^\omega \\ w_{n_i-1} > 1 \text{ and } w_{n_i} > 1 & \text{if } n_i \in U^\omega \end{cases} \tag{8}$$

and such that the A_n are diagonal for $n \in]n_i, n_{i+1}[$, $i \geq \kappa$. According to (ii), for such values of n the matrix M_{ω_n} is diagonal and $A_n = M_{\omega_n}$ with $a_n \geq d_n$, or $A_n = \Delta M_{\omega_n}\Delta$ with $a_n \leq d_n$.

If there exists $i \geq \kappa$ such that $n_i \in L^\omega$ and $n_{i+1} \in U^\omega$, $\det Y_{n_i}$ is necessarily negative: otherwise A_n should be equal to M_{ω_n} for $n \in]n_i, n_{i+1}[$, $\dfrac{d_n}{a_n} \leq 1$ and, by (5), $w_{n_{i+1}-1} \leq w_{n_i} < 1$ in contradiction with (8).

Now $M_{\omega_{n_{i+1}}}$ has positive determinant, otherwise it should have the form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and $A_{n_{i+1}} = \Delta M_{\omega_{n_{i+1}}} = \begin{pmatrix} c & 0 \\ a & b \end{pmatrix}$ in contradiction with $n_{i+1} \in U^\omega$.

We have again $\dfrac{d_n}{a_n} \geq 1$ for $n \in]n_{i+1}, n_{i+2}[$ and consequently $w_{n_{i+2}-1} \geq w_{n_{i+1}} > 1$; so, by induction, $n_j \in U^\omega$ and $\det Y_{n_j} < 0$ for any $j \geq i+1$. From (5) w_n lies between w_{n_j} and $w_{n_{j+1}-1}$ for any $n \in]n_j, n_{j+1}[$ and j large enough, and from (6) its limit is infinite. Distinguishing the cases where V has positive entries or V has a null entry, $\dfrac{(Y_nV)_2}{(Y_nV)_1}$ converges by Lemma 1.1.

The conclusion is the same if there do not exist $i \geq \kappa$ such that $n_i \in L^\omega$ and $n_{i+1} \in U^\omega$, because in this case $n_i \in L^\omega$ for i large enough, or $n_i \in U^\omega$ for any $i \geq \kappa$.

Third case: Suppose (iv) holds. As we have seen, from (4) A_n is eventually triangular or diagonal, and M_{ω_n} also is because – by (iv) – $\mathcal{M}$ do not contain invertible matrices of the form $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ or $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$. We deduce that the sign of $\det Y_n$ is eventually constant. If A_n is not eventually diagonal the sequence $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ converges by Lemma 1.1(iv) and, if A_n is, the sequence $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ is eventually constant.

Fourth case: Suppose that the set $\mathcal{M}$ do not satisfy (i), (ii), (iii) nor (iv), and that at least one matrix of this set, let M_k, has the form $M_k = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$; let us prove that $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ diverges.

- Suppose first there exists a matrix M_k of this form that do not have V as eigenvector; we chose as counterexample the constant sequence defined by $\omega_n = k$ for any n: Y_{2n} is collinear to the unit matrix, hence $Y_{2n}V$ is collinear to V and $Y_{2n+1}V$ to M_kV, so $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ diverges.

- Suppose now that all the matrices of $\mathcal{M}$ of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$ have V as eigenvector that is, V is collinear to $\begin{pmatrix} \sqrt{b} \\ \sqrt{c} \end{pmatrix}$ for all such matrix. Since (i) do not hold, at least one matrix M_h of $\mathcal{M}$ is diagonal with nonnull and distinct diagonal entries. In this case M_hM_k has the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ but do not have V as eigenvector. We recover the previous case; more precisely the counterexample is defined by $\omega_{2n-1} = h$ and $\omega_{2n} = k$ for any $n \in \mathbb{N}$.

Fifth case: Suppose that $\mathcal{M}$ do not satisfy (i), (ii), (iii) nor (iv), and that no matrix of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$ belongs to $\mathcal{M}$. Since (i) do not hold, at least one matrix of this set has the form $M_k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $ad \neq 0$ and $a \neq d$. We suppose that $a > d$ and we use the negation of (ii) (in case $a < d$ we use similarly the negation of (iii)). According to the negation of (ii) there exists in $\mathcal{M}$ at least one matrix of the form $M_h = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ with $\beta\gamma\delta \neq 0$, or one of the form $M_\ell = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ with $0 < \alpha < \delta$.

‗ Consider first the case where $\mathcal{M}$ contains some matrices M_k and M_h as above. Let $(n_i)_{i\in\mathbb{N}}$ be an increasing sequence of positive integers with $n_1 = 1$, and ω the sequence defined by $\omega_n = h$ for $n \in \{n_1, n_2, \ldots\}$ and $\omega_n = k$ otherwise.

For i odd, A_{n_i} is lower-triangular and $\forall n \in]n_i, n_{i+1}[,\ A_n = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$, $a_n = d$ and $d_n = a$.

For i even, A_{n_i} is upper-triangular and $\forall n \in]n_i, n_{i+1}[,\ A_n = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a_n = a$ and $d_n = d$.

Using (5) for $n = n_{i+1} - 1$ and choosing $n_{i+1} - n_i$ large enough one has $w_{n_{i+1}-1} \geq 2^i$ if i is odd, $w_{n_{i+1}-1} \leq 2^{-i}$ if i is even, so the three conditions in (4) are satisfied and the interval $\cap I_n$ is not reduced to one point. If the entries of V are positive, the first relation in (1) and the definition of λ_n imply that $\liminf_{n\to\infty} \left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ is the lower bound of this interval and $\limsup_{n\to\infty} \left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ its upper bound, so the sequence $\left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ diverges. If V has a null entry, the divergence of $\left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ results from Lemma 1.1(iv).

‗ In case $\mathcal{M}$ contains some matrices M_k and M_ℓ as above, one defines ω from a sequence $i_1 = 1 < i_2 < i_3 < \ldots$ by setting, for $j \geq 1$ and $i_j \leq n < i_{j+1}$,

$$\omega_n = \begin{cases} k & \text{if } j \text{ even} \\ \ell & \text{if } j \text{ odd.} \end{cases}$$

The diagonal matrix Y_n can be easily computed, and $\left(\frac{(Y_nV)_2}{(Y_nV)_1}\right)$ obviously diverges if one choose the $i_{j+1} - i_j$ large enough.

If V has a null entry, since (iv) do not hold $\mathcal{M}$ contains at least one matrix of the form $M_h = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$, $\beta\gamma\delta \neq 0$ or $M_{h'} = \begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}$, $\alpha\beta\gamma \neq 0$, or $M_{h''} = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$, $\beta\gamma \neq 0$. We already know that $\left(\dfrac{(Y_nV)_2}{(Y_nV)_1}\right)$ diverges if $\mathcal{M}$ contains M_k and M_h. Similarly it diverges if $\mathcal{M}$ contains M_ℓ and $M_{h'}$. If $\mathcal{M}$ contains M_k and $M_{h''}$ the counterexample is given – from a sequence $i_1 = 1 < i_2 < i_3 < \dots$ – by $\omega_{i_j} = h''$ and $\omega_n = k$ for $n \in]i_j, i_{j+1}[$, $j \in \mathbb{N}$: $\dfrac{(Y_nV)_2}{(Y_nV)_1}$ is alternately 0 and ∞ because Y_{i_j} has the form $\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ for j odd and $\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}$ for j even. □

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