A Unified Framework for the Study of the 2-microlocal and Large Deviation Multifractal Spectra

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Abstract

The large deviation multifractal spectrum is a function of central importance in multifractal analysis. It allows a fine description of the distribution of the singularities of a function over a given domain. The 2-microlocal spectrum, on the other hand, provides an extremely precise picture of the regularity of a distribution at a point. These two spectra display a number of similarities: their definitions use the same kind of ingredients; both functions are semi-continuous; the Legendre transform of the two spectra yields a function of independent interest: the 2-microlocal frontier in 2-microlocal analysis, and the "τ" function in multifractal analysis. This paper investigates further these similarities by providing a common framework for the definition and study of the spectra. As an application, we obtain slightly generalized versions of the 2-microlocal and weak multifractal formalisms (with simpler proofs), as well as results on the inverse problems for both spectra.

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1 Introduction and Background

The analysis of global regularity, classically based on the global Hölder exponent, is adapted for the study of homogeneous signals. However, the global Hölder exponent yields insufficient information when the regularity of a function evolves in “time”. Studying such functions requires tools that allow to characterize their behaviour at or around any point. One such tool is the pointwise Hölder exponent. We shall denote $\alpha_p(f, x_0)$ the exponent of the function f at the point x_0. Multifractal analysis [FP85, AP96, BMP92, CLP87, EM92, Fan97, HJK+86, KP76, Jaf97a, VT04, VV98, Man74, Ols95] studies the structure of the pointwise Hölder function, *i.e.* the function $x_0 \to \alpha_p(f, x_0)$: more precisely it aims at obtaining the *multifractal spectrum*, a function which measures the "size" of the level lines of $\alpha_p(f, x)$. Both the theoretical and the numerical computations of this spectrum are difficult. This is why physicists and mathematicians have investigated a "multifractal formalism", which allows, in certain situations, to obtain the spectrum as the Legendre transform of a function that can be computed more easily.

Instead of focusing on the pointwise Hölder exponent and the fine structure of $\alpha_p(f, x_0)$, one may follow a different approach and try to obtain a richer description of the local regularity at any fixed point by means of other exponents, such as the local Hölder exponent [GL98], the chirp exponent [Mey98], the oscillation exponent [ABJM98] or the "weak scaling" exponent [Mey98]. A powerful way to do so is to study the 2-microlocal frontier, defined in [GL98, Mey98] based on the local version of the 2-microlocal spaces introduced by J.M. Bony in [Bon83]. The main interest of these spaces is that they allow to describe completely the evolution of the pointwise Hölder exponent at any given point under integro-differentiation. The 2-microlocal frontier is a curve in an abstract space that is associated to each point, and that allows to predict this evolution. The 2-microlocal spaces were originally defined through a Littlewood-Paley decomposition. They were then characterized by conditions on the wavelet coefficients [Jaf91]. Time domain characterizations of increasing generality have been provided in [KL02, LS04, Ech07]. See also [Mey98] for related results.

The computation of the 2-microlocal frontier is somewhat delicate. A 2-microlocal formalism has been studied in [GL98, LS04, Ech07], with an approach that is analogous, in many respects, to the one of the multifractal formalism: at any fixed point, the 2-microlocal frontier is the Legendre transform of a certain function called the 2-microlocal spectrum.

Thus, for a function f, the multifractal spectrum characterizes the level sets of the pointwise Hölder function, while the 2-microlocal spectrum allows to predict the change of regularity by integro-differentiation at any point in the domain of f. These two descriptions yield a rather rich picture of the regularity, and they may be approached through related formalisms, which are essentially based on a Legendre transform.

In this work, we elaborate on the similitudes between the two formalisms. We also study the problem of prescribing both the 2-microlocal and multifractal spectra. In the next section, we expose some general notions that are useful in both settings. In Section 3, we provide an abstract (weak) formalism. This formalism is applied to various versions of the multifractal spectra in Section 4, and to the 2-microlocal spectrum in Section 5. Finally, Section 6 presents results on the prescription of the spectra.

Sections 2 and 3 stay at a very general level. As a consequence, the definitions and results they present might appear rather abstract to the reader. However, as will be apparent in Sections 4 and 5, they contain the essence of what is common to the multifractal and 2-microlocal formalisms. In particular, propositions 4.1, 4.6, 5.2 and 5.3 may be seen as concrete examples of applications of this abstract formalism. In fact, the results in Sections 2 and 3 elucidate the very mechanisms relating the spectra, and might have applications in other settings.

2 Recalls: basic properties of functions defined on $\mathcal{P}(\mathbb{R})$

We recall in subsections 2.1, 2.2 and 2.3 some known definitions and results on set functions. In subsection 2.4, we specialize to a case which will be relevant for both 2-microlocal and multifractal analysis.

2.1 Common frame

We consider a function $F : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. F is called *non-decreasing* if for all real sets E_1, E_2,

$$E_1 \subset E_2 \Rightarrow F(E_1) \leq F(E_2).$$

It is called *stable* (with respect to union) if

$$F(E_1 \cup E_2) = \max\{F(E_1), F(E_2)\}.$$

Given a real k, F is *k-stable* if

$$F(E_1 \cup E_2) - k \leq \max\{F(E_1), F(E_2)\} \leq F(E_1 \cup E_2) + k.$$

It is easily checked that any function $G : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$ that is non-decreasing and sub-additive (*i.e.* $G(E_1 \cup E_2) \leq G(E_1) + G(E_2)$) is such that $F := \log G$ is k-stable with $k = \log 2$.

2.2 Non-decreasing F

Assume that F is non-decreasing. For all real α, the function $F([\alpha - \epsilon, \alpha + \epsilon])$ has a limit when ϵ tends to 0. One may then define the *localized function* (also called *max-plus limit*).

$$F^{loc}(\alpha) = \lim_{\epsilon \to 0} F([\alpha - \epsilon, \alpha + \epsilon]).$$

Lemma 2.1. *Let F be non-decreasing. For all open set O in $\mathbb{R}$,*

$$\sup_{\alpha \in O} F^{loc}(\alpha) \leq F(O). \qquad (1)$$

Proof. Indeed, if $\alpha \in O$, there exists ϵ such that $[\alpha-\epsilon,\alpha+\epsilon] \subset O$, and thus $F([\alpha-\epsilon,\alpha+\epsilon]) \leq F(O)$. □

Lemma 2.2. *If F is non-decreasing, then F^{loc} is upper-semi-continuous.*

Proof. The preceding lemma implies that, for $\epsilon > 0$, $\sup_{\beta\in(\alpha-\epsilon,\alpha+\epsilon)} F^{loc}(\beta) \leq F((\alpha-\epsilon,\alpha+\epsilon))$. Letting ϵ tend to 0, one gets the semi-continuity of F^{loc} in α. □

Lemma 2.3. *If F is non-decreasing, then F^{loc} reaches its supremum on any compact set of $\mathbb{R}$.*

Proof. This is a direct consequence of the semi-continuity. □

2.3 Stability of F

If F is stable, then it is also non-decreasing: indeed, if $E_1 \subset E_2$,

$$F(E_1) \leq \max\{F(E_1), F(E_2)\} = F(E_1 \cup E_2) = F(E_2).$$

Lemma 2.4. *Let F be stable. For any compact set K in $\mathbb{R}$,*

$$F(K) \leq \max_{\alpha\in K} F^{loc}(\alpha). \tag{2}$$

Proof. The proof uses the closed dyadic intervals of $\mathbb{R}$. Since K is bounded, it may be covered by a finite number of dyadic interval of rank 0 (that is, of length 1). Since F is stable, one of these intervals, denoted J_0, is such that $F(J_0 \cap K) = F(K)$. However, J_0 is covered by two intervals of rank 1, say J' and J''. One of these intervals, denoted J_1, is also such that $F(J_1 \cap K) = F(K)$. By recurrence, one may construct a nested sequence (J_n) of dyadic intervals such that, for all n, $F(J_n \cap K) = F(K)$.

Let α_* denote the limit of the J_n. Since K is closed, α_* is a point of K. For any $\epsilon > 0$, there exists an integer n such that $J_n \subset [\alpha_*-\epsilon,\alpha_*+\epsilon]$. Using the fact that F is non-decreasing, one gets that: $F(J_n) \leq F([\alpha_*-\epsilon,\alpha_*+\epsilon])$. Thus $F(K) \leq F([\alpha_*-\epsilon,\alpha_*+\epsilon])$, and, letting ϵ tend to 0: $F(K) \leq F^{loc}(\alpha_*)$. As a consequence, $F(K) \leq \sup_{\alpha\in K} F^{loc}(\alpha)$.

Finally, since F is non-decreasing, F^{loc} reaches its supremum on K. □

Corollary 2.1. *Let F be stable and K be a compact set with non-empty interior $\mathring{K}$. Then*

$$\sup\{F^{loc}(\alpha)/\alpha \in \mathring{K}\} \leq F(K) \leq \max\{F^{loc}(\alpha)/\alpha \in K\}. \tag{3}$$

Proof. This is a consequence of lemmas 2.1 and 2.4. □

The relations (3) entail at once that, for any α and ϵ,

$$\sup_{|\beta-\alpha|<\epsilon} F^{loc}(\beta) \leq F([\alpha-\epsilon,\alpha+\epsilon]) \leq \max_{|\beta-\alpha|\leq\epsilon} F^{loc}(\beta). \tag{4}$$

2.4 A simple example

The following example aims at giving a straightforward application of the previous results, as well as at introducing some notations that will be used in the sequel.

Given a function $g : \mathbb{R} \to \mathbb{R}$, one derives a set function by setting:

$$M(g, E) = \sup_{t \in E} g(t). \tag{5}$$

It is straightforward to check that M is stable. One may thus apply the preceding analysis. The corresponding localized function, denoted $\overline{g}$, is well-defined, and it is upper-semi-continuous:

$$\overline{g}(\alpha) = \lim_{\epsilon \to 0} M(g, [\alpha - \epsilon, \alpha + \epsilon]). \tag{6}$$

From (6), one easily checks that this is the smallest upper-semi-continuous function that is larger than g, *i.e.* its upper-semi-continuous envelope.

Finally, the inequalities (3) yield that, for any compact set K with non-empty interior:

$$\sup\{\overline{g}(\alpha)/\alpha \in \overset{\circ}{K}\} \leq \sup_{t \in K} g(t) \leq \sup\{\overline{g}(\alpha)/\alpha \in K\}.$$

Note that symmetrical results may be obtained by setting:

$$m(g, E) = \inf_{t \in E} g(t). \tag{7}$$

The lower-semi-continuous envelope of g is $\underline{g}(\alpha) = \lim_{\epsilon \to 0} m(g, [\alpha - \epsilon, \alpha + \epsilon])$. Since $(-m)$ is stable, one gets the lower-semi-continuity of $\underline{g}$ and the bounds:

$$\inf\{\underline{g}(\alpha)/\alpha \in K\} \leq \inf_{t \in K} g(t) \leq \inf\{\underline{g}(\alpha)/\alpha \in \overset{\circ}{K}\}.$$

3 A weak formalism

In this section, we consider a particular form for the function F. This form allows to write a general (weak) formalism, stated in lemma 3.1 and equality (12). Specializing this form allows to treat the cases of both the multifractal (Section 4) and 2-microlocal (Section 5) spectra. Indeed, all the applications in the sequel will deal with set functions $\mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$ which have a particular form that we now proceed to describe.

We assume given:

- A subset L of $\mathbb{R}^n$ whose closure contains the origin 0.

  For instance, one may consider $L = (0, 1]$, $L = \{k2^{-j}, (j, k) \in \mathbb{Z}^2\}$ or $L = (0, 1] \times (0, 1]$.

- A function $h : L \to \mathbb{R}_+^*$ such that $\lim_{t\to 0} h(t) = 0$.

For a function

$$U : \left| \begin{array}{ccc} L \times \mathcal{P}(\mathbb{R}) & \to & \overline{\mathbb{R}} \\ (t, E) & \to & U_t(E) \end{array} \right.$$

we set:

$$F_U(E) = \limsup_{t\to 0} \quad \big\{ \ h(t)U_t(E) \ : \quad t \in L \big\}. \tag{8}$$

Proposition 3.1. *Assume that, for a given real number k and for all $t \in L$, U_t is k-stable. Let F_U be defined by (8). Then F_U is stable.*

Proof. By assumption, for all $t \in L$ and all (E_1, E_2) in $\mathcal{P}(\mathbb{R})^2$,

$$U(t, E_1 \cup E_2) - k \le \max\{U_t(E_1), U_t(E_2)\} \le U(t, E_1 \cup E_2) + k.$$

Multiplying by $h(t)$ and passing to the upper-limit gives the result. □

The following result may be considered as an abstract weak formalism. It will be used to prove propositions 4.1, 4.6, 5.2 and 5.3:

Lemma 3.1. *Let k be a real number and U, V be such that U_t, V_t are k-stable for all t.*

Assume there exist a function $g : \mathbb{R} \to \mathbb{R}$ and two positive constants c_1 and c_2 such that, for all t and all compact intervals I,

$$c_1 + U_t(I) + \frac{m(g, I)}{h(t)} \le V_t(I) \le c_2 + U_t(I) + \frac{M(g, I)}{h(t)}. \tag{9}$$

Then, for all $t \in \mathbb{R}$,

$$F_U^{loc}(t) + \underline{g}(t) \le F_V^{loc}(t) \le F_U^{loc}(t) + \overline{g}(t). \tag{10}$$

Furthermore, for all compact sets K with non-empty interior,

$$\sup_{\alpha \in \overset{\circ}{K}} \{F_U^{loc}(t) + \underline{g}(t)\} \le F_V(K) \le \sup_{t \in K} \{F_U^{loc}(t) + \overline{g}(t)\}. \tag{11}$$

Finally, if g is continuous, then:

$$\sup_{\alpha \in \mathbb{R}} \{g(\alpha) + F_U^{loc}(\alpha)\} = \lim_{R\to\infty} F_V([-R, R]). \tag{12}$$

Proof. Multiplying (9) by $h(t)$ and passing to the upper limit when t tends to 0, one gets:

$$F_U(I) + m(g, I) \le F_V(I) \le F_U(I) + M(g, I). \tag{13}$$

It then suffices to replace I by $[t - \epsilon, t + \epsilon]$ and to let ϵ tend to 0 to obtain (10).

Since F_V is stable (by proposition 3.1), one may use (2) and (10) to get (11).

Assume now that g is continuous. Set $I = [-R, R]$ and let R tend to infinity. Then the inequalities (11) reduce to equality (12)

□

We end this section with a remark concerning equality (12): the non-decreasing property ensures that $F_V(\mathbb{R}) \geq \lim_R F_V([-R,R])$. However, these two terms do not coincide in general.

4 Multifractal Analysis

4.1 Introduction

Multifractal analysis is a very active domain. Research focuses, among other topics, on checking the validity of the multifractal formalism, both in a deterministic and stochastic frame [AP96, BL04, BMP92, RM95], studying the analysis of function [Jaf97b] and capacities [LV98], introducing more refined spectra [VT04]. There is also a huge amount of work about estimating the multifractal spectra. A very partial list of reference is [Can98, CJ92, Lév96, GL05, GL07, GH10, TPVG06].

As we recall below, there exists a variety of multifractal spectra. They contain different information, and some are easier to compute than others. The so-called strong multifractal formalism, when valid, asserts that the Hausdorff multifractal spectrum is equal to the Legendre one (see below for definitions). This formalism will not be our concern here. Rather, we will concentrate on the weak multifractal formalism, which studies conditions under which the large deviation multifractal spectrum coincide with the Legendre one.

First introduced as a means to estimate the Hausdorff spectrum, the large deviation multifractal spectrum has then proved a useful tool on its own, in particular in applications [LR97, VV98]). A disadvantage of the original definition of this spectrum is that it is based on an arbitrary partition of the support of the function to be analyzed. Thus, different partitions may in general lead to different spectra. To overcome this drawback, a "continuous" large deviation spectrum has been introduced in [VT04]. We will in this work study both large deviation spectra.

The multifractal formalisms we will study will assert that, under some conditions, the discrete Legendre spectrum is the concave envelope of the original large deviation spectrum. Likewise, the continuous Legendre spectrum will be the concave envelope of the continuous large deviation spectrum

There is no added complication in considering an abstract set function A as a basis for defining the spectra instead of the usual coarse-grained exponents of a measure or real function. The correspondence between A and the quantities classically considered is as follows:

- For the analysis of a Borel measure μ, set $A(u) = \log \mu(u)/\log |u|$.
- For the analysis of a real function z, set $A(u) = \log v_z(u)/\log |u|$, where $v_z(u)$ is a measure of the "variation" of z in the interval u. Typical choices

include the increments $|z(u_{max}) - z(u_{min})|$ (where $u = [u_{min}, u_{max}]$), the oscillation $\sup_{t\in u} z(t) - \inf_{t\in u} z(t)$, or, when u is the dyadic interval $[k2^{-n}, (k+1)2^{-n}]$, the modulus of the wavelet coefficient of z at scale n and location k (note that, for applications in signal processing, one uses slight modifications known as the wavelet maxima or wavelet leader methods).

In this section, we shall apply the results of Section 2 to recover known results on the weak multifractal formalism, both in the discrete and continuous frames. We shall also obtain some new results in the case where $A(u) = \log \mu(u)/\log |u|$, with μ an absolutely continuous measure.

4.2 Notations

$\mathcal{X}$ will denote the set of closed intervals of $[0,1]$. We let $\mathcal{X}_n$ be the set of intervals of the form $[k2^{-n}, (k+1)2^{-n}]$ where $k \in [0..2^n]$. $|E|$ denotes the Lebesgue measure of the (measurable) set E. Finally, if $\mathcal{R}$ is a set of intervals, we shall write $\cup\mathcal{R} = \cup\{I : I \in \mathcal{R}\}$

4.3 Hausdorff Spectrum

Although we will not use it in this work, we briefly recall for completeness the definition of the Hausdorff multifractal spectrum, denoted f_h, in the case of a real function z. Let

$$E(\alpha) = \{x : \alpha_p(z, x) = \alpha\}$$

denote the set of points where z has pointwise Hölder exponent α.

The Hausdorff spectrum is the function which associates to each $\alpha \in \mathbb{R}$ the Hausdorff dimension of $E(\alpha)$ [AP96, Fal86, Fal03, Fan97, Jaf97a, VT04, LV98, Ols95]:

$$f_h(\alpha) = \dim_{\mathcal{H}}(E(\alpha)).$$

Of course, it is in general difficult to compute these Hausdorff dimensions. Most other multifractal spectra have been introduced as means to provide an indirect access to f_h.

4.4 Discrete weak multifractal formalism

4.4.1 Discrete large deviation spectrum

From a heuristic point of view, the large deviation spectrum describes the asymptotic behaviour of the density of the intervals of $\mathcal{X}_n$ having a coarse-grained Hölder exponent “close” to given value. More precisely, it is defined as follows.

Let $A : \mathcal{X} \to \overline{\mathbb{R}}^+$ be a measurable function.

For any measurable subset E of $\mathbb{R}$, let $N(n,E)$ denote the number of intervals I of $\mathcal{X}_n$ such that $A(I)$ belongs to E:

$$N(n,E) = \#\{I \in \mathcal{X}_n \text{ such that } A(I) \in E\}.$$

In order to assess the behaviour of $N(n,E)$ when n tends to infinity, we consider the function

$$F_g^d(E) = \limsup_{n\to\infty} \frac{\log N(n,E)}{n \log 2}$$

and its "localized" version

$$f_g^d(\alpha) = \lim_{\epsilon\to 0} F_g^d([\alpha-\epsilon, \alpha+\epsilon]).$$

This function is called the discrete large deviation spectrum associated to A [Fal03, LV98].

Roughly speaking, if for all $\epsilon > 0$, the number of intervals I of $\mathcal{X}_n$ for which $A(I)$ belongs to $[\alpha_0-\epsilon, \alpha_0+\epsilon]$ is proportional to $2^{r.n}$, then $f_g^d(\alpha_0) = r$. Another interpretation, that provides a link with the continuous spectrum to be introduced below, is as follows: if, for all $\epsilon > 0$, the union of dyadic intervals I of length 2^{-n} for which $A(I)$ belongs to $[\alpha_0-\epsilon, \alpha_0+\epsilon]$ has measure proportional to $2^{-n}.2^{r.n}$, then $f_g^d(\alpha_0) = r$.

4.4.2 Discrete Legendre spectrum

Define, for all $q \in \mathbb{R}$:

$$S^q(n) = \sum \{2^{-nqA(I)} : I \in \mathcal{X}_n \text{ and } A(I) < \infty\}$$

and

$$\tau^d(q) = \liminf_{n\to\infty} \frac{\log S^q(n)}{-n \log 2}.$$

We shall use in the sequel the following quantities:

$$S^q(n,E) = \sum \{2^{-nqA(I)} : I \in \mathcal{X}_n \text{ and } A(I) \in E\}.$$

The discrete Legendre spectrum [AP96, BMP92, CLP87, EM92, Fal03, FP85, HJK$^+$86, Jaf91, LV98, Ols95] associated to A is the Legendre transform of τ^d:

$$f_l^d(\alpha) = \sup_{q\in\mathbb{R}} \{q\alpha - \tau^d(q)\}.$$

4.4.3 Application of Section 3

We apply the results of Section 3 to the quantities defined in Sections 4.4.1 and 4.4.2. This will yield the weak multifractal formalism. More precisely, we set:

$$L = \{2^{-n}, n \in \mathbb{N}\},$$
$$h(2^{-n}) = 1/(n \log 2),$$
$$U_{2^{-n}}(E) = \log N(n, E).$$

Clearly, $E \to N(n, E)$ is non-decreasing and sub-additive. $U_{2^{-n}}$ is thus k-stable for all n, with $k = \log(2)$. It is easy to check that, with the notations of Section 3:

$$F_U = F_g^d \text{ and } F_U^{loc} = f_g^d.$$

Set:

$$V_{2^{-n}}(E) = \log \sum\{2^{-nqA(I)} : I \in \mathcal{X}_n \text{ and } A(I) \in E\}.$$

Now, $E \to \sum\{2^{-nqA(I)} : I \in \mathcal{X}_n \text{ and } A(I) \in E\}$ is non-decreasing and sub-additive, thus $V_{2^{-n}}$ is k-stable for all n with $k = \log(2)$.

Furthermore, $V_{2^{-n}}$ may be written as:

$$V_{2^{-n}}(E) = \log S^q(n, E),$$

and thus:

$$\tau^d(q) = -F_V(\mathbb{R}^+).$$

Set for convenience:

$$\tau_R^d(q) = -F_V([0, R]).$$

We wish to apply lemma 3.1. In that view, remark that:

$$N(n, E) \inf_{\alpha \in E} 2^{-nq\alpha} \leq \sum\{2^{-nqA(I)} : I \in \mathcal{X}_n \text{ and } A(I) \in E\} \leq N(n, E) \sup_{\alpha \in E} 2^{-nq\alpha}. \tag{14}$$

Set $g(\alpha) = -q\,\alpha$. Taking logarithms, the inequalities (14) read

$$U_{2^{-n}}(E) + \frac{\inf_{\alpha \in E} g(t)}{h(2^{-n})} \leq V_{2^{-n}}(E) \leq U_{2^{-n}}(E) + \frac{\sup_{\alpha \in E} g(t)}{h(2^{-n})}.$$

Lemma 3.1 applies. Writing relation (12) in this frame, one finds:

Proposition 4.1. *For all function $A : \mathcal{X} \to \overline{\mathbb{R}}^+$ and $q \in \mathbb{R}$,*

$$\inf_{\alpha \in \mathbb{R}} \{\alpha q - f_g^d(\alpha)\} = \lim_{R \to \infty} \tau_R^d(q). \tag{15}$$

Recall that, since F_V is non-decreasing, $\lim_{R \to \infty} \tau_R^d(q) \geq \tau^d(q)$. Let us now find a condition implying equality of these two terms. First, note that the equality is always true for $q > 0$. Indeed:

Proposition 4.2. *For all function $A : \mathcal{X} \to \overline{\mathbb{R}}^+$ and all $q > 0$*

$$\inf_{\alpha \in \mathbb{R}} \{\alpha q - f_g^d(\alpha)\} = \tau^d(q). \tag{16}$$

Proof. For any natural number n, there exists 2^n intervals of order n, thus at most 2^n dyadic intervals I of order n such that $A(I) > R$. As a consequence,

$$S^q(n) \leq S^q(n, [0, R]) + 2^n\, 2^{-qR}. \tag{17}$$

For any sequences (a_n) and (b_n) of positive reals,

$$\limsup \frac{\log(a_n + b_n)}{n} = \max\{\limsup \frac{\log a_n}{n}, \limsup \frac{\log b_n}{n}\}.$$

Set $a_n = S^q(n, [0, R])$ and $b_n = 2^{n(1-qR)}$. One deduces:

$$\limsup_{n\to\infty} \frac{\log S^q(n)}{n \log 2} \leq \max \left\{ \limsup_{n\to\infty} \frac{\log S^q(n, [0, R])}{n \log 2}, 1 - qR \right\}$$

and thus:

$$\min\{\tau_R^d(q), qR - 1\} \leq \tau^d(q)$$

for all R. When R tends to $+\infty$, so does $qR - 1$ (this is where the assumption $q > 0$ is used), which entails $\tau^d(q) = \lim_{R\to\infty} \tau_R^d(q)$. □

An additional assumption is necessary for the equality to hold when $q \leq 0$ (see Section 4.6 for a counter-example). For instance, the result is valid if $A(I)$ is bounded for $|I|$ small enough:

Proposition 4.3. *Assume there exist $\eta > 0$ and $M \in \mathbb{R}$ such that, for all dyadic interval I, $|I| \leq \eta$ implies that $A(I) \leq M$ or $A(I) = +\infty$. Then, for all $q \in \mathbb{R}$:*

$$\inf_{\alpha \in \mathbb{R}} \{\alpha q - f_g^d(\alpha)\} = \tau^d(q).$$

Proof. For all $R \geq M$: $S^q(n, [0, R]) = S^q(n)$ as soon as $2^{-n} \leq \eta$; thus $\tau_R^d(q) = \tau^d(q)$. Proposition 4.1 allows to conclude. □

It is straightforward to check that Proposition 4.3 applies in the paradigmatic cases of multinomial measures ([EM92, BMP92]) and fractal interpolation functions [LS04], yielding the well-known fact that the weak multifractal formalism indeed holds in these situations.

When the assumption of the previous proposition does not hold, the value of $\tau^d(q)$ for $q < 0$ is known. Indeed:

Proposition 4.4. *If, for all $\eta > 0$ and $M \in \mathbb{R}$, there exists a dyadic interval I such that $|I| \leq \eta$ and $M < A(I) < +\infty$, then for all $q < 0$:*

$$\tau^d(q) = -\infty.$$

Proof. For all $M > 0$, there exists a subsequence $I_{\sigma(n)}$ such that $M < A_{\sigma(n)} < +\infty$. This implies that $S^q(\sigma(n)) \geq 2^{-q\sigma(n)M}$, and thus that $\frac{\log S^q(\sigma(n))}{-\sigma(n)\log 2} \leq qM$. Taking lim inf, one gets $\tau^d(q) \leq qM$. □

Cases where A is not bounded in the neighbourhood of $|I| = 0$ and where $\tau^d_R(q)$ does not tend to $-\infty$ for $q < 0$ do exist. We shall provide an example in the next section, in the continuous frame.

An important particular case is when A may be written as $A = \log \mu(I)/\log |I|$, where μ is a positive measure. Note that, in this case, by definition of τ, $\tau^d(0) = -\dim_B(\text{support}(\mu))$, where $\dim_B$ stands for box dimension.

Theorem 4.1. *Let $A(I) = \frac{\log \mu(I)}{\log |I|}$, where μ is a positive measure non identically zero. Then, for all $q \neq 0$:*

$$\tau^d(q) = \inf_{\alpha \in \mathbb{R}} \{q\alpha - f^d_g(\alpha)\}.$$

The proof of the theorem will use the following lemma:

Lemma 4.1. *Let $q < 0$. If, for all $r > 0$, there exists $R_0 > 0$ such that: for all $N \in \mathbb{N}$, there exist $n \geq N$ and $I \in \mathcal{X}_n$ such that $A(I) \in [r, R_0]$, then:*

$$\lim_{R \to \infty} \tau^d_R(q) = -\infty$$

Proof. For all $R > R_0$ and all $N \in \mathbb{N}$, there exist $n \geq N$ and $I \in \mathcal{X}_n$ such that $A(I) \in [r, R]$. As a consequence, $S^q(n, [0, R]) \geq 2^{-nqr}$, and thus:

$$\frac{\log S^q(n, [0, R])}{-n \log 2} \leq qr$$

Taking lim inf on n, one gets:

$$\tau^d_R(q) \leq qr,$$

and the result follows. □

Proof of theorem 4.1. In view of propositions 4.1, 4.3 and 4.4, it is enough to prove that, when A is not bounded in the neighbourhood of 0, one has, for all q<0:

$$\lim_{R \to \infty} \tau^d_R(q) = -\infty.$$

Assume then that A is not bounded in the neighbourhood of $|I| = 0$: for all $\eta > 0$ and $M \in \mathbb{R}$, there exists a dyadic I with $|I| \leq \eta$ such that $M < A(I) < +\infty$.

We shall show that this implies that the assumptions of lemma 4.1 are verified. Assume the opposite: then there exists $r > 0$ such that, for all $R_0 > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and all $I \in \mathcal{X}_n$: $A(I) \notin [r, R_0]$. This implies:

$$\mu(I) \leq 2^{-nR_0} \text{ or } \mu(I) \geq 2^{-nr}. \tag{18}$$

Choose $R_0 = \max(2r, 2)$. Since A is unbounded in the neighbourhood of $|I| = 0$, there exists a dyadic $|I|$ of order $n \geq 2$ such that $R_0 < A(I) < +\infty$, *i.e.* $0 < \mu(I) \leq 2^{-nR_0}$.

One may divide I into two dyadic intervals I_1 and I_2 of order $n+1$. Since $I_1 \subset I$, $\mu(I_1) \leq \mu(I)$, and thus $\mu(I_1) \leq 2^{-nR_0}$. However, $R_0.n > r.(n+1)$ which entails $\mu(I_1) < 2^{-(n+1)r}$. From (18), one deduces: $\mu(I_1) \leq 2^{-(n+1)R_0}$. Likewise, $\mu(I_2) \leq 2^{-(n+1)R_0}$.

An obvious induction shows that, for any dyadic interval J of order $n+m$, $m \geq 0$, included in I: $\mu(J) \leq 2^{-(n+m)R_0}$. Thus, $\mu(I) \leq 2^{m-(n+m)R_0}$. Since $R_0 > 1$, one gets, by letting m tend m to infinity, $\mu(I) = 0$, which contradicts the assumption. $\square$

The case $q = 0$ appears to be more difficult to deal with. If one assumes in addition to the above that μ is absolutely continuous, then the following results hold:

Proposition 4.5. *Assume* $A(I) = \frac{\log \mu(I)}{\log |I|}$, *where* μ *is an absolutely continuous positive measure non identically zero. Then:*

$$\tau^d(0) = \inf_{\alpha \in \mathbb{R}} \{-f_g^d(\alpha)\} = -1.$$

As a consequence, for all $q \in \mathbb{R}$*:*

$$\tau^d(q) = \inf_{\alpha \in \mathbb{R}} \{q\alpha - f_g^d(\alpha)\}.$$

Proof. Let us first compute the value of $\tau^d(0)$:

By definition, $\tau^d(0) = -\dim(\text{support}(\mu))$. Since μ is absolutely continuous and non zero, $\tau^d(0) = -1$.

Let us now evaluate the limit of $\tau_R(0)$ when R tends to infinity.

The assumptions on μ entail that there exists an integrable positive function such that $\mu(I) = \int_I f(x)dx$. Since μ is not zero, there exist $\epsilon > 0$ and a bounded Borel set B of positive measure such that $f(x) \geq \epsilon.1_{x \in B}$. Set

$$\nu(I) = \epsilon \int_I 1_{x \in B} dx.$$

Since $\nu \leq \mu$, for any interval I with diameter smaller than 1, we deduce that

$$A(I) \leq \frac{\log \nu(I)}{\log |I|}.$$

Set

$$u_n = \sharp \left\{ I \in \mathcal{X}_n : \nu(I) \geq 2^{-n} \frac{\epsilon |B|}{2} \right\}.$$

We have $\epsilon|B| = \sum\{\nu(I) : I \in \mathcal{X}_n\}$. Since, for all $I \in \mathcal{X}_n$, $\nu(I) \leq 2^{-n}\epsilon$, we obtain: $\epsilon|B| \leq 2^{-n}\epsilon\, u_n + \frac{1}{2}\, 2^{-n}\epsilon|B|\, (2^n - u_n)$ and thus:

$$u_n \geq 2^n K$$

where $K = |B|/2$ is strictly positive.

As a consequence, $\sharp\left\{ I \in \mathcal{X}_n : A(I) \leq 1 + \frac{\log \frac{\epsilon|B|}{2}}{-n \log 2} \right\} \geq 2^n K$. Choose n large enough so that $\frac{\log \frac{\epsilon|B|}{2}}{-n \log 2} \leq 1$: $\sharp\left\{ I \in \mathcal{X}_n : A(I) \in [0,2] \right\} \geq 2^n K$ and thus:

$$S^0(n, [0,2]) \geq 2^n K$$

Taking logarithms and passing to the lim inf, one gets: $\tau_2^d(0) \leq \ -1$. The result follows since $R \to \tau_R^d(0)$ is not increasing with upper limit -1.

The second part of the proposition follows from proposition 4.2. $\square$

4.5 Continuous Multifractal Formalism

4.5.1 Continuous Large Deviation Spectrum

The very definition of the large deviation spectrum entails that it depends on a choice of a partition of the support of the signal (usually the dyadic intervals). This is a drawback, since, in general, the theoretical spectrum may indeed be different with different choices of partitions for the same signal. Even in the case where the theoretical spectrum does not depend on the partition, numerical estimation will typically yield different results with different choices of intervals.

The continuous large deviation spectrum was introduced in [VT04] as a means to define a partition-free spectrum. The idea is to consider all sets of a given length η instead of restricting to dyadic intervals. The continuous large deviation spectrum then measures the asymptotic behaviour, when η tends to 0, of the density of the union of intervals of size η having a coarse-grained exponent close to a given value.

Even though going from a discrete setting to a continuous one may appear as a complication, it actually eases the estimation of the large deviation spectrum, by yielding more robust results on numerical data. This is similar to the dichotomy discrete *vs* continuous wavelet transform: it is well-known that the continuous wavelet transform provides better results as far as estimation is concerned (at the expense of an additional computational burden). As in the continuous wavelet case, the numerical computation of the continuous spectrum is performed by considering all samples at each scale, instead of using an averaging or sub-sampling as in the usual case.

As above, we consider a general set function A instead of specific coarse-grained exponents.

As compared to the discrete case, we replace the sets $\mathcal{X}_n$ by the sets $\mathcal{R}_\eta$, defined, for all $\eta \in (0,1)$, by:

$$\mathcal{R}_\eta = \{I \in \mathcal{X} : |I| = \eta\}.$$

Recall that, for all set of intervals $\mathcal{T}$, we denote $\cup\mathcal{T} = \cup\{I : I \in \mathcal{T}\}$.

For any subset $\mathcal{T}$ of $\mathcal{R}_\eta$ and any $t \in \cup\mathcal{T}$, the largest interval containing t and included in $\cup\mathcal{T}$ has length not smaller than η. In addition, these intervals are disjoint. As a consequence, there are finitely many of them. This entails that $\cup\mathcal{T}$ is measurable. This allows to define the equivalent of $N(\eta, E)$:

$$N^c(\eta, E) = \frac{1}{\eta} \left|\cup\{I \in \mathcal{R}_\eta : A(I) \in E\}\right|.$$

Similarly to the discrete case, we seek to characterize the behavior of $N^c(\eta, E)$ when $\eta \to 0$. In that view, consider

$$F_g^c(E) = \limsup_{\eta \to 0} \frac{\log N^c(\eta, E)}{-\log \eta}.$$

The continuous large deviation spectrum associated to A is defined as:

$$f_g^c(\alpha) = \lim_{\epsilon \to 0} F_g^c([\alpha - \epsilon, \alpha + \epsilon]).$$

Roughly speaking, if the union of the intervals I of length η for which $A(I)$ belongs to $[\alpha_0 - \epsilon, \alpha_0 + \epsilon]$ has measure proportional to $\eta.\eta^{-r.n}$ for all sufficiently small ϵ, then $f_g^d(\alpha_0) = r$.

4.5.2 Continuous Legendre Multifractal Spectrum

To adapt the quantities S^q defined in the discrete case in this setting, one uses the notion of *packing* [VT04]: a family $\mathcal{R} \subset \mathcal{X}$ is a packing of $[0,1]$ if the intervals of $\mathcal{R}$ have pairwise disjoint interiors. One then define, for any $\mathcal{R} \subset \mathcal{X}$:

$$H^q(\mathcal{R}) = \sup\left\{ \sum\{|u|^{qA(u)} : u \in \mathcal{R}' \text{ and } A(u) < +\infty\} : \mathcal{R}' \text{is a packing of } [0,1] \text{ drawn from } \mathcal{R} \right\} \quad (19)$$

with the conventions: $|u|^{+\infty} = 0$, $|u|^{-\infty} = +\infty$, $H^q(\emptyset) = 0$.

Set

$$\tau^c(q) = \liminf_{\eta \to 0} \frac{\log H^q(\mathcal{R}_\eta)}{\log \eta}.$$

The continuous Legendre spectrum associated to A is the Legendre transform of τ^c:

$$L_c(\alpha) = \sup_{q \in \mathbb{R}} \{q\alpha - \tau^c(q)\}.$$

Remark that, when $\mathcal{R}$ is a packing, $H^q(\mathcal{R})$ may be written

$$H^q(\mathcal{R}) = \sum \left\{ |u|^{qA(u)} : u \in \mathcal{R} \text{ and } A(u) < +\infty \right\}.$$

This shows that H^q is indeed a generalization of S^q: $\mathcal{X}_n$ is a packing, thus

$$H^q(\mathcal{X}_n) = \sum \left\{ |u|^{qA(u)} : u \in \mathcal{X}_n \text{ and } A(u) < +\infty \right\} = S^q(n).$$

4.6 Application of Section 3

The results of Section 3 apply to the quantities defined in Section 4.4.1 in a continuous frame with the following correspondences:

$$L = (0, 1],$$
$$h(\eta) = 1/|\log \eta|,$$
$$U_\eta(E) = \log N^c(\eta, E).$$

Remark that, for any disjoint E_1, E_2, $N^c(\eta, E_1 \cup E_2) = N^c(E_1) \cup N^c(E_2)$. As a consequence, $E \to N^c(\eta, E)$ is non-increasing and sub-additive. This means that U_η is k-stable with $k = \log 2$ for all η.

One easily checks that, with the notations of Section 3:

$$F_U = F_g^c \text{ and } F_U^{loc} = f_g^c.$$

Set:

$$V_\eta(E) = \log H^q(\mathcal{R}_\eta \cap A^{-1}(E)).$$

where H^q is defined in Section 4.5.2. We note that, for all disjoint E_1, E_2, $H^q(\mathcal{R}_\eta \cap A^{-1}(E_1 \cup E_2)) = H^q(\mathcal{R}_\eta \cap A^{-1}(E_1)) + H^q(\mathcal{R}_\eta \cap A^{-1}(E_2))$. This entails that $E \to H^q(\mathcal{R}_\eta \cap A^{-1}(E))$ is non-decreasing and sub-additive.

Thus, V_η is k-stable with $k = \log 2$. Furthermore,

$$\tau^c(q) = -F_V(\mathbb{R}^+).$$

Set for convenience:

$$\tau_R^c(q) = -F_V([0, R]) = \liminf_{\eta \to 0} \frac{\log H^q(\mathcal{R}_\eta \cap A^{-1}([0, R]))}{\log \eta}.$$

We wish to apply lemma 3.1. In that view, we shall make use of the following result from [VT04], whose proof we recall for completeness:

For all $E \in \mathcal{P}(\mathbb{R})$:

$$\frac{1}{2} N^c(\eta, E) \inf_{\alpha \in E} \eta^{q\alpha} \leq H^q(\mathcal{R}_\eta \cap A^{-1}(E)) \leq N^c(\eta, E) \sup_{\alpha \in E} \eta^{q\alpha}. \tag{20}$$

Proof. Set $\mathcal{T} = \mathcal{R}_\eta \cap A^{-1}(E)$.

We start with the right-hand side inequality. Let $E \in \mathcal{P}(\mathbb{R})$. For any packing $\mathcal{R}$ drawn from $\mathcal{T}$, one may write:

$$\sum \{\eta^{qA(u)} : u \in \mathcal{R}\} \leq \sup_{u \in \mathcal{R}} \{\eta^{qA(u)}\} \sharp \mathcal{R}.$$

However, $\sharp\mathcal{R} \leq \eta^{-1} |\cup \mathcal{T}| \leq N^c(\eta, E)$, and, for all $u \in \mathcal{R}$, $A(u) \in E$. Thus

$$\sup_{u \in \mathcal{R}} \{\eta^{qA(u)}\} \leq \sup_{\alpha \in E} \eta^{q\alpha},$$

which was to be proved.

Now, $\cup\mathcal{T}$ is a union of intervals with same length, and one may write $\cup\mathcal{T}$ as: $\cup\mathcal{T} = \cup_{k=1}^n J_k$ where the J_k are non-empty, closed, and disjoint.

We show that, for all J_k, one may extract from $\mathcal{T}$ a packing $\mathcal{R}_k$ such that $\eta\sharp\mathcal{R}_k \geq |J_k|/2$ and $\cup\mathcal{R}_k \subset J_k$.

Indeed, let p be such that $|J_k| \in (2p\eta, 2(p+1)\eta]$. Choose $a_0 ... a_p$ in J_k such that $a_q > a_{q-1} + 2\eta$ for $1 \leq q \leq p$. We know that, for $0 \leq q \leq p$, there exists an interval $L_q \in \mathcal{T}$ such that $a_q \in L_q$. It is easily checked that the L_q are disjoint and included in J_k. Set $\mathcal{R}_k = \{L_q, q = 0..p\}$. Since $\sharp\mathcal{R}_k = (p+1)$, one has $\eta\sharp\mathcal{R}_k \geq |J_k|/2$. Furthermore, $\cup\mathcal{R}_k \subset J_k$.

Consider now the union $\mathcal{R}$ of $\mathcal{R}_k$ for $k = 1..n$: it is easy to see that $\mathcal{R}$ is a packing of $[0,1]$ drawn from $\mathcal{T}$, whose cardinal is not smaller than $|\mathcal{T}|/2\eta$. Since

$$\inf_{u \in \mathcal{R}} \{\eta^{qA(u)}\} \sharp \mathcal{R} \leq \sum \{\eta^{qA(u)} : u \in \mathcal{R}\},$$

one deduces:

$$\frac{1}{2} \frac{|\mathcal{T}|}{\eta} \inf_{u \in \mathcal{R}} \{\eta^{qA(u)}\} \leq \sum \{\eta^{qA(u)} : u \in \mathcal{R}\}.$$

□

Set $g(\alpha) = -q\,\alpha$. Taking logarithms, the inequalities (20) read:

$$-\ln(2) + U_\eta(E) + \frac{\inf_{\alpha \in E} g(t)}{h(\eta)} \leq V_\eta(E) \leq U_\eta(E) + \frac{\sup_{\alpha \in E} g(t)}{h(\eta)}.$$

Lemma 3.1 applies. Writing relation (12) in this frame, one finds:

Proposition 4.6. *For any function $A : \mathcal{X} \to \overline{\mathbb{R}}^+$ and all $q \in \mathbb{R}$,*

$$\inf_{\alpha \in \mathbb{R}} \{\alpha q - f_g^c(\alpha)\} = \lim_{R \to \infty} \tau_R^c(q). \tag{21}$$

Recall that, since F_V is non-decreasing, $\lim_{R\to\infty} \tau_R^c(q) \leq \tau^c(q)$. We look now for conditions that guarantee equality of these two terms. It is always verified when $q > 0$:

Proposition 4.7. *For any function $A : \mathcal{X} \to \overline{\mathbb{R}}^+$ and for all $q > 0$*

$$\inf_{\alpha \in \mathbb{R}} \{\alpha q - f_g^c(\alpha)\} = \tau^d(q). \tag{22}$$

Proof. For all $\eta > 0$ and for all packing $\mathcal{R}$ drawn from $\mathcal{R}_\eta$, $\sharp\mathcal{R} \leq \eta^{-1}$. This entails:

$$H^q(\mathcal{R}) \leq H^q(\mathcal{R}_\eta \cap A^{-1}([0, R])) + 2^n \, 2^{-qR}. \tag{23}$$

Taking logarithms, dividing by $\log \eta$ and passing to the lim sup when $\eta \to 0$:

$$\min\{\tau_R^c(q), qR - 1\} \leq \tau^c(q)$$

for all R. When R tends to $+\infty$ so does $qR - 1$ (recall that $q > 0$), and thus $\tau^c(q) = \lim_{R \to \infty} \tau_R^c(q)$. □

Additional assumptions on A are necessary for (22) to hold for all q. One may for instance require that A is bounded for sufficiently small $|I|$:

Proposition 4.8. *Assume there exists $\eta_0 > 0$ such that $\sup\{A(I) : |I| \leq \eta_0\} < +\infty$. Then*

$$\inf_{\alpha \in \mathbb{R}} \{\alpha q - f_g^c(\alpha)\} = \tau(q) \text{ for all } q.$$

This result obviously corresponds to the one in proposition 4.3. Again, one easily checks that it applies in the case of multinomial measures, *i.e.* the weak multifractal formalism for continuous spectra holds for such measures.

When the assumption of the previous proposition does not hold, we do know the value of $\tau^c(q)$ for $q < 0$:

Proposition 4.9. *If, for all $\eta > 0$ and $M \in \mathbb{R}$ there exists an interval I such that $|I| \leq \eta$ and $A(I) > M$, then for all $q < 0$:*

$$\tau^c(q) = -\infty.$$

Proof. For all $M > 0$, there exists a sequence I_n such that $A(I_n) > M$ and $|I_n| \to 0$. As a consequence, $H^q(\mathcal{R}_{|I_n|}) \geq 2^{-qnM}$, and thus $\frac{\log H^q(\mathcal{R}_{|I_n|})}{-\log |I_n|} \leq qM$. Taking lim inf, $\tau^d(q) \leq qM$. □

There exists cases where A is not bounded for small $|I|$ and where $\lim_{R \to \infty} \tau_R^c(q) \neq -\infty$ for $q < 0$.

Example Define the function A on all closed intervals of $[0, 1]$ of positive length by

$$A([a, b]) = \begin{cases} 1 & \text{for } a > 0 \\ 1/(b - a) & \text{for } a = 0. \end{cases}$$

Let $q \in \mathbb{R}$ and fix $R > 1$.

For all $\eta < 1/R$, for all interval I with length η: $A(I) \in [0, R]$ if and only if $0 \notin I$. In that case, $A(I) = 1$.

A packing of intervals with length η contains at most η^{-1} intervals. As a consequence,

$$H^q(\mathcal{R}_\eta \cap A^{-1}([0,R])) \leq \eta^{q-1}.$$

Besides, the packing of intervals $\{[k\eta,(k+1)\eta] : k = 1..\lfloor \eta^{-1} \rfloor - 1\}$ is a packing of intervals of $[0,1]$ with cardinal $\lfloor \eta^{-1} \rfloor - 1$ which does not contain 0. This implies that:

$$(\lfloor \eta^{-1} \rfloor - 1)\eta^q \leq H^q(\mathcal{R}_\eta \cap A^{-1}([0,R])).$$

Finally, we get:

$$\tau_R^c(q) = q - 1.$$

The sequence $A([0,1/n])$ tends to $+\infty$. Proposition 4.9 then entails that, for all $q < 0$:

$$\tau^c(q) = -\infty.$$

We thus have an example where (22) does not hold for all q.

Remark that the sequence $A([0,2^{-n}])$ tends to $+\infty$. From proposition 4.4, we get: $\tau^d(q) = -\infty$. However, $\tau_d^R(q) \geq \tau_c^R(q) \geq q - 1$: this provides thus a case where (16) does not hold either for all q.

More can be said under usual assumptions on A:

Proposition 4.10. *Let $A(I) = \frac{\log \mu(I)}{\log |I|}$ where μ is a positive measure. Then*

$$\inf_{\alpha \in \mathbb{R}} \{\alpha q - f_g^c(\alpha)\} = \tau(q) \text{ for all } q \neq 0.$$

Proof. When the assumptions of proposition 4.7 or of proposition 4.8 are in force, then the desired equality holds.

We still have to deal with the more delicate case where $q < 0$ and for all $\eta_0 > 0$:

$$\sup\{A(I) : |I| \leq \eta_0\} = +\infty.$$

We know from proposition 4.9 that, in this case, $\tau^c(q) = -\infty$. We would thus like to show that $\lim_R \tau_R^c(q) = -\infty$. This is in fact a consequence of theorem 4.1: just remark that $\tau_R^c(q) \leq \tau_R^d(q)$. □

5 2-microlocal Analysis

5.1 Continuous 2-microlocal Formalism

5.1.1 Notations

The continuous 2-microlocal formalism [LS04] is based on the characterization of 2-microlocal spaces from wavelet coefficients given in [Jaf91].

We assume given an admissible wavelet ψ with N vanishing moments and r first derivatives having fast decay. We also fix a real number x_0. As usual, we denote:

$$\psi_{(a,b)}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \tag{24}$$

the dilated and translated versions of ψ.

The 2-microlocal frontier of the function f at x_0 is characterized as follows: for all couple $(\sigma, s') \in \mathbb{R}^2$ such that $r + s + \inf(s', 1) > 0$ and $N > \max(s, s + s')$, $\sigma \leq \boldsymbol{\sigma}(s')$ is equivalent to:

$$\sigma \leq \liminf_{(a,b)\to 0} \frac{s' \log(a + |b|) + \log\left|\left\langle f, a^{-1/2}\psi_{(a,b+x_0)}\right\rangle\right|}{\log a}.$$

One deduces that, if $r + \boldsymbol{\sigma}(s') - s' + \inf(s', 1) > 0$ and $N > \max(\boldsymbol{\sigma}(s') - s', \boldsymbol{\sigma})$, then

$$\boldsymbol{\sigma}(s') = \liminf_{(a,b)\to 0} \frac{s' \log(a + |b|) + \log\left|\left\langle f, a^{-1/2}\psi_{(a,b+x_0)}\right\rangle\right|}{\log a}. \tag{25}$$

where $\boldsymbol{\sigma}$ is the 2-microlocal frontier of f at x_0.

Strictly speaking, one should define $\boldsymbol{\sigma}_\psi(s')$ as being equal to the right-hand side term of equation (25) and remark that $\boldsymbol{\sigma}_\psi$ and $\boldsymbol{\sigma}$ coincide on a certain subset of the plane. However, we shall in the sequel assume that ψ is sufficiently regular (*i.e.* that r and N are sufficiently large) so that, for all relevant s', (25) is verified.

The continuous 2-microlocal spectrum is defined as:

$$\chi(\rho, \epsilon) = \liminf_{(a,b)\to 0} \left\{ \frac{\log\left|\left\langle f, a^{-1/2}\psi_{(a,b+x_0)}\right\rangle\right|}{\log a} : (a, b) \in D^w(\rho, \epsilon) \right\}$$

where

$$D^w(\rho, \epsilon) = \{(a, b) \in (-1, 1)^2, \text{ such that } a^{\rho+\epsilon} \leq \max(a, |b|) \leq a^{\rho-\epsilon}\}.$$

For ϵ small enough, one may give an explicit description of the set $D^w(\rho, \epsilon)$ as follows:

- For $\rho \in (0, 1)$: $(a, b) \in D^w(\rho, \epsilon) \iff a^{\rho+\epsilon} \leq |b| \leq a^{\rho-\epsilon}$.
- For $\rho = 0$: $(a, b) \in D^w(\rho, \epsilon) \iff a^{\rho+\epsilon} \leq |b|$.
- For $\rho = 1$: $(a, b) \in D^w(\rho, \epsilon) \iff |b| \leq a^{\rho-\epsilon}$.

The *2-microlocal spectrum* of f at x_0 is defined by

$$\chi(\rho) = \lim_{\epsilon\to 0} \chi(\rho, \epsilon).$$

In order to facilitate the application of the results of Section 3, we shall define a localized version of the $D^w(\rho,\epsilon)$: taking $\epsilon = 0$ one gets

$$d(\rho) = \{(a,b) \in (0,1)^2 / |b|^\rho = \max(a,|b|)\}.$$

When $\rho \in (0,1)$,

$$d(\rho) = (-1,0] \times \{0\} \cup \{(a, a^{1/\rho})/a \in (0,1)\}$$

is the graph of a continuous function.

Define then, for $E \in \mathcal{P}(\mathbb{R})$, the subset D_E of $(0,1] \times \mathbb{R}$:

$$D_E = \bigcup_{\rho \in E} d(\rho). \tag{26}$$

It is easy to check that $D_{[\rho-\epsilon,\rho+\epsilon]} = D^w(\rho,\epsilon)$.

We make a final remark that $\chi(\rho,\epsilon)$ may be written as follows:

$$\chi(\rho,\epsilon) = \liminf_{\substack{(a,b)\to 0 \\ a>0 \ b\in\mathbb{R}}} \frac{\log\left(\left|\left\langle f, a^{-1/2}\psi_{(a,b+x_0)}\right\rangle\right| .1_{(a,b)\in D^w(\rho,\epsilon)}\right)}{\log a},$$

with the convention $\log 0 = -\infty$.

The next elementary inequalities will be useful in the sequel:

Proposition 5.1. $\forall s' \in \mathbb{R}, \forall \rho \in [0,1], \forall (a,b) \in D^w(\rho,\epsilon) \cap (0,1] \times [-1,1]$:

$$2^{-|s'|} a^{\epsilon|s'|+\rho s'} \leq (a+|b|)^{s'} \leq 2^{|s'|} a^{-\epsilon|s'|+\rho s'}.$$

Proof. For all $(a,b) \in D^w(\rho,\epsilon)$:

$$a^{\rho+\epsilon} \leq \max(a,|b|) \leq a+|b| \leq 2\max(a,|b|) \leq 2a^{\rho-\epsilon}.$$

Putting to the s' power the inequalities $a^{\rho+\epsilon} \leq a+|b| \leq 2a^{\rho-\epsilon}$ yields the result.

□

5.1.2 Application of Section 3

The results of Section 3 apply with:

$$L = (0,1] \times \mathbb{R},$$

$$t = (a,b),$$

$$h(a,b) = \frac{1}{|\log a|},$$

$$U((a,b),E) = \log(|\left\langle f, a^{-1/2}\psi_{(a,b+x_0)}\right\rangle|.1_{(a,b)\in D_E}),$$
$$V((a,b),E) = \log(|\left\langle f, a^{-1/2}\psi_{(a,b+x_0)}\right\rangle|.(a+|b|)^{s'}.1_{(a,b)\in D_E}).$$

The functions U and V are stable. $F_U([\rho-\epsilon,\rho+\epsilon])$ corresponds to $-\chi_\epsilon(\rho)$, and $F_U^{loc}(\rho)$ to $-\chi(\rho)$. Furthermore, $F_V([0,1]) = F_V(\mathbb{R}) = -\sigma(s')$.

Let $g(t) = -s't$. Proposition 5.1 may be written as:
$\forall (a,b) \in (0,1]\times[-1,1]$:

$$-s' + U(a,b,I) + \frac{m(g,I)}{h(t)} \leq V(a,b,I) \leq s' + U(a,b,I) + \frac{M(g,I)}{h(t)}.$$

This implies that U and V fulfill condition (9) with $g(\rho) = -\rho s'$. We may thus write equality (12) in this context, noting that $\chi(\rho)$ is equal to $+\infty$ when $\rho \notin [0,1]$ and that $-\sigma(s') = F_V([0,1]) = \lim_{R\to\infty} F_V([-R,R])$:

$$\max_{\rho\in[0,1]}\{-\chi(\rho) - \rho s'\} = -\sigma(s').$$

One finally reaches the following result:

Proposition 5.2. *For all $s' \in \mathbb{R}$:*

$$\sigma(s') = \min_{\rho\in[0,1]}\{\chi(\rho) + \rho s'\}.$$

This is the wavelet version of the 2-microlocal formalism, as exposed in [LS04]. It allows for an easy computation of the 2-microlocal frontier of many classical functions including for instance the Weierstrass function or fractal interpolation functions.

5.2 Discrete 2-microlocal Formalism

5.2.1 Classical Formalism

We shall make use of the following notation:

$$\psi_{j,k}(x) = \psi_{(2^{-j},k2^{-j})}(x) = 2^{j/2}\psi(2^jx - k). \tag{27}$$

The discrete 2-microlocal formalism is based on the discrete wavelet characterization of the 2-microlocal spaces given in [Jaf91]. As before, we assume that ψ is sufficiently regular so that, for all relevant values of s':

$$\boldsymbol{\sigma}(s') = \liminf_{(2^{-j},k2^{-j}-x_0)\to 0} \frac{\log_2(|\left\langle f, 2^{-j/2}\psi_{jk}\right\rangle|.(2^{-j}+|k2^{-j}-x_0|)^{s'}}{-j}. \tag{28}$$

In order to get an expression that looks more like (25), we set:

$$L_d(x_0) = \{2^{-j}, k2^{-j} - x_0, j > 0\,,\ k \in \mathbb{Z}\}.$$

Equation (28) may be written as:

$$\sigma(s') = \liminf_{(a,b)\to 0} \left\{ \frac{\log\left(\left|\left\langle f, a^{1/2}\psi_{(a,b+x_0)}\right\rangle\right| .(a+|b-x_0|)^{s'}\right)}{\log a} : (a,b) \in L_d(x_0) \right\}. \tag{29}$$

As above, we define:

$$\chi^d(\rho,\epsilon) = \liminf_{\substack{(a,b)\to 0\\ (a,b)\in L_d(x_0)}} \frac{\log\left(\left|\left\langle f, a^{1/2}\psi_{(a,b+x_0)}\right\rangle\right| .1_{(a,b)\in D_{[\rho-\epsilon,\rho+\epsilon]}}\right)}{\log a},$$

and

$$\chi^d(\rho) = \lim_{\epsilon\to 0} \chi^d(\rho,\epsilon).$$

As compared to the continuous 2-microlocal spectrum, the discrete one is less "robust", since its computation involves a fewer number of wavelet coefficients. However, it may be prescribed more easily, since the $\langle f, \psi_{jk}\rangle$ may be chosen as desired. The continuous 2-microlocal spectrum should then be more adapted for evaluating the frontier of a function at a given point, while the discrete one would be more fitted in situations where one wishes to obtain a given frontier (this is for instance the case in applications such as signal denoising, see [Ech07]).

5.2.2 Application of Section 3

As was the case for the continuous spectrum, the results of Section 3 apply. The same correspondences may be used, with the exception that we set here $L = L_d(x_0)$. One then easily gets the following result:

Proposition 5.3. *For all $s' \in \mathbb{R}$:*

$$\sigma^d(s') = \min_{\rho\in[0,1]} \{\chi^d(\rho) + \rho s'\}.$$

Again, this formalism yields an easy way to compute the 2-microlocal frontier of functions such as fractal interpolation functions.

6 Prescribing Multifractal and 2-microlocal Spectra

As another example of the profound similarities between the multifractal and 2-microlocal formalism, we consider in this section the problem of prescribing the spectra in the discrete frame. Both the 2-microlocal and the multifractal spectra may be written in the form studied in Section 3: $F_U(E) = \limsup_{t\to 0} h(t)U(t,E)$. As a consequence, the prescription of both spectra share some common features. However, as we will see, the constraints on the function U are not the same in the two cases.

6.1 General Frame

This section gives a result that be will useful for the prescription of both spectra.

The following distance over $\overline{\mathbb{R}}$ will be convenient in order to place ourselves in the most general frame: $d_{\overline{\mathbb{R}}}(x, y) = |\arctan x - \arctan y|$, with the convention $\arctan(\pm\infty) = \pm\pi/2$. It is easy to check that $d_{\overline{\mathbb{R}}}$ and the usual distance are topologically equivalent on $\mathbb{R}$. Another useful remark is that $d_{\overline{\mathbb{R}}}(x, y) \leq |x - y|$.

The heuristic idea is as follows: assume we wish to construct a function whose "spectrum" is g on an interval I. We give ourselves a sequence I_n of intervals that will "rotate" on I, and whose diameter will tend to 0. On each I_n, we approximate g by its maximum over I_n. This will be meaningful whenever g is upper semi-continuous. For a fixed interval $[\alpha - \epsilon, \alpha + \epsilon]$, assumption (30) below ensures that there exists a subsequence of $(I_n)_n$ made of intervals contained in $[\alpha - \epsilon, \alpha + \epsilon]$, and for which $x_n V_n(I_n)$ will be larger than $g(\alpha)$. Here $(x_n)_n$ is a positive sequence tending to 0. This entails $F^{loc}(\alpha) \geq g(\alpha)$.

For the reverse inequality, one needs to ensure that the values of V_n on intervals other than I_n will not affect F. Assumption (31) says that one may choose n_0 such that $I_{n_0} \subset [\alpha - \epsilon, \alpha + \epsilon]$ and that when $n \to \infty$, $x_n V_n(I_{n_0})$ will be not larger than $M(g, [\alpha - 2\epsilon, \alpha + 2\epsilon])$.

Lemma 6.1. *Let g be an upper semi-continuous function on the interval I. Let $(x_n)_n$ be a positive sequence tending to 0, and let V_n be k-stable for all $n \in \mathbb{N}$.*

Define F as: $F(E) = \limsup_{n\to\infty} x_n.V_n(E)$ for all $E \subset I$.

Let $(I_n)_n$ be a sequence of open intervals such that:

$$\begin{cases} \forall N \in \mathbb{N} : \ I \subset \cup_{n\geq N} I_n, \\ \lim_{n\to\infty} |I_n| = 0. \end{cases}$$

Assume that:

$$\lim_{n\to\infty} d_{\overline{\mathbb{R}}}(x_n V_n(I_n), M(g, I \cap I_n) = 0. \tag{30}$$

and that, for all n_0:

$$\limsup_{n\to\infty} x_n V_n(I_{n_0} \setminus I_n) \leq M(g, I_{n_0} \cap I). \tag{31}$$

Then $F^{loc} = g$.

Proof. The assumptions on $(I_n)_n$ entail that, for all $\alpha \in I$, there exists a subsequence $I_{\sigma(n)}$ such that, for all $n, \alpha \in I_{\sigma(n)}$.

As g is upper semi-continuous, $M(g, I \cap I_{\sigma(n)})$ tends to $g(\alpha)$ in $\overline{\mathbb{R}}$. Thus, by assumption, $x_{\sigma(n)} V_{\sigma(n)}(I_{\sigma(n)})$ tends to $g(\alpha)$ in $\overline{\mathbb{R}}$.

For all $\epsilon > 0$ and for all n such that $|I_{\sigma(n)}| < \epsilon$, the k-stability of $V_{\sigma(n)}$ entails that:

$$x_{\sigma(n)} V_{\sigma(n)}([\alpha - \epsilon, \alpha + \epsilon]) + k.x_{\sigma(n)} \geq x_{\sigma(n)} V_{\sigma(n)}(I_{\sigma(n)}).$$

Taking the lim sup when n tends to infinity, one gets $F([\alpha-\epsilon,\alpha+\epsilon]) \geq g(\alpha)$. This inequality is true for all $\epsilon > 0$, and thus $F(\alpha) \geq g(\alpha)$.

We show now that $F(\alpha) \leq g(\alpha)$.

Fix $\epsilon > 0$. There exists n_0 such that $I_{n_0} \subset [\alpha-\epsilon,\alpha+\epsilon]$. We divide the sequence $(I_n)_n$ into two subsequences I_{σ_1} and I_{σ_2} defined by:

$$\begin{aligned} \{I_{\sigma_1(n)} : n \in \mathbb{N}\} &= \{I_n : n \in \mathbb{N}, I_{n_0} \cap I_n = \emptyset\}, \\ \{I_{\sigma_2(n)} : n \in \mathbb{N}\} &= \{I_n : n \in \mathbb{N}, I_{n_0} \cap I_n \neq \emptyset\}. \end{aligned}$$

For all n, $I_{n_0} = I_{n_0} \setminus I_{\sigma_1(n)}$. Thus, by assumption:

$$\limsup_{n\to\infty} \left(x_{\sigma_1(n)} V_{\sigma_1(n)}(I_{n_0})\right) \leq M(g, I \cap I_{n_0}).$$

Besides, $V_{\sigma_2(n)}(I_{n_0}) \leq 2k + \max(V_{\sigma_2(n)}(I_{n_0} \setminus I_{\sigma_2(n)}), V_{\sigma_2(n)}(I_{\sigma_2(n)}))$ and thus

$$\limsup_{n\to\infty} x_{\sigma_2(n)} V_{\sigma_2(n)}(I_{n_0}) \leq \max\left\{M(g, I \cap I_{n_0}), \limsup_{n\to\infty}(x_{\sigma_2(n)} V_{\sigma_2(n)}(I_{\sigma_2(n)}))\right\}.$$

However, for n large enough, $I_{\sigma_2(n)} \subset [\alpha-2\epsilon,\alpha+2\epsilon]$. As a consequence,

$$\limsup_{n\to\infty} x_{\sigma_2(n)} V_{\sigma_2(n)}(I_{\sigma_2(n)}) \leq M(g, I \cap [\alpha-2\epsilon,\alpha+2\epsilon]).$$

The lim sup of both sequences I_{σ_1} and I_{σ_2} are not larger than $M(g, I \cap [\alpha-2\epsilon,\alpha+2\epsilon])$ and thus

$$F(I_{n_0}) \leq M(I \cap [\alpha-2\epsilon,\alpha+2\epsilon]).$$

Since F is not decreasing, $F(\alpha) \leq M(I \cap [\alpha-2\epsilon,\alpha+2\epsilon])$. Finally, let ϵ tend to 0 to obtain the result. □

Remark The shape of the function F in lemma 6.1 obviously fits in the frame of Section 3: set for instance $L = \{1/n, n \in \mathbb{N}\}$, $h(1/n) = x_n$ and $U(1/n, I) = V_n(I)$.

Remark For any interval I, it is easy to construct a sequence I_n such as the one described in theorem 6.1. If, for instance, $I = [0,+\infty]$, then, for k an integer between 0 and $+\infty$, one may divide the intervals $(-1/k, k)$ into sub-intervals of the form $((i-1)/k, (i+1)/k)$, where i is an integer varying from 0 to k^2-1:

$I_1 = (-1, 1)$
$I_2 = (-1/2, 1/2)$ $I_3 = (0, 1)$ $I_4 = (1/2, 3/2)$ $I_5 = (1, 2)$
$I_6 = (-1/3, 1/3)$ $I_7 = (0, 2/3]$ $I_8 = (1/3, 1)$ $\cdots$ $I_{14} = (7/3, 3)$
etc...

6.2 Multifractal Spectrum

The prescription of the Hausdorff spectrum has been considered in [LV98] et [Jaf89]. [VT04] has studied the prescription of various other related multifractal spectra. We focus here on the prescription of the discrete large deviation spectrum.

In a discrete frame, one is concerned with the values of the function A only for intervals of the form: $[(k-1)2^{-n}, k2^{-n}]$; $n \in \mathbb{N}$, $k \in 1..2^j$. To simplify notations, we shall write:

$$A_{nk} = A([(k-1)2^{-n}, k2^{-n}]).$$

By abuse of language, we shall speak of the "multifractal spectrum of the A_{nk}" in place of the multifractal spectrum of the measure or function whose associated intervals function A is such that $A([(k-1)2^{-j}, k2^{-j}]) = A_{nk}$ for all n, k.

Recall that $f_d^g = F^{loc}$, with $F(E) = \limsup_{n\to\infty} x_n V_n(E)$ where $x_n = 1/(n \log 2)$ and $V_n(E) = \log \#\{A_{nk}, A_{nk} \in E\}$.

6.2.1 Prescription on $\mathbb{R}$

Theorem 6.1. *Let $g : \mathbb{R} \to [0,1] \cup \{-\infty\}$ be an upper semi-continuous function. There exists a sequence of real numbers $(A_{jk})_{j\in\mathbb{N};k\in 1..2^j}$ such that g is the large deviation multifractal spectrum of the $(A_{jk})_{j,k}$.*

Proof. Let $(I_n)_n$ be a sequence of non-empty intervals such that $|I_n|$ tends to 0 and $\mathbb{R} = \cup_{n\geq N} I_n$ for all N. Let i_n be a sequence such that for all n , $i_n \in I_n$ (one may for instance choose the midpoint of I_n).

For all $n \in \mathbb{N}$, define A_{nk}, $k \in [1..2^n]$, by:

$$\text{If } M(g, I_n) \in [0,1] : \quad \begin{cases} A_{nk} = i_n & , \quad k = 1 \ldots \lfloor 2^{n.M(g,I_n)} \rfloor, \\ A_{nk} = 1 + \max(\cup_{p\leq n} I_p) & , \quad k = 1 + \lfloor 2^{n.M(g,I_n)} \rfloor \ldots 2^n. \end{cases}$$
$$\text{If } M(g, I_n) = -\infty : \quad A_{nk} = 1 + \max(\cup_{p\leq n} I_p) \quad , \quad k = 1 \cdots 2^n.$$

We shall apply lemma 6.1 to obtain the result.

For all n, we have: $\#\{A_{nk}, A_{nk} \in I_n\} = \lfloor 2^{n.M(g,I_n)} \rfloor$ with the convention $2^{-\infty} = 0$. The map $x \to \log\lfloor x \rfloor - \log x$ is bounded on $[1, +\infty)$. As a consequence, there exists c such that, for all $n \in \mathbb{N}$ such that $M(g, I_n) \neq -\infty$:

$$\left|V_n(I_n) - \log(2^{M(g,I_n)})\right| \leq c.$$

This entails that, for all n such that $M(g, I_n) \neq -\infty$:

$$|x_n V_n(I_n) - M(g, I_n)| \leq c x_n.$$

Furthermore, when $M(g, I_n) = -\infty$, then $V_n(I_n) = M(g, I_n)$.

Thus, $d_{\overline{\mathbb{R}}}(x_n V_n(I_n), M(g, I_n)))$ tends to 0.

Besides, it is easy to see that, for all n_0 and for all $n > n_0$: $\#\{A_{nk}, A_{nk} \in I_n \setminus I_{n_0}\} = 0$. As a consequence, $\limsup_{n\to\infty} x_n V_n(I_{n_0} \setminus I_n) = -\infty$. The assumptions of lemma 6.1 are in force, and the result follows.

□

Remark Conversely, Lemma 2.2 implies that the multifractal spectrum of any sequence $(A_{jk})_{j\in\mathbb{N};k\in 1..2^j}$ is an upper semi-continuous function from $\mathbb{R}$ to $[0,1]\cup\{-\infty\}$. Thus, we have:

Proposition 6.1. *The set of all large deviation multifractal spectra coincide with the one of upper semi-continuous functions from* $\mathbb{R}$ *to* $[0,1]\cup\{+\infty\}$

6.2.2 Prescription on $\overline{\mathbb{R}}$

In certain situations, it is convenient to extend the domain of the discrete multifractal spectrum to $\overline{\mathbb{R}}$. In that view, one sets:

$$f_g^d(+\infty) = \lim_{K\to+\infty} F_g^d([K,+\infty]),$$

and

$$f_g^d(-\infty) = \lim_{K\to-\infty} F_g^d([-\infty,K]).$$

Recall the following result from [VT04]:

Theorem 6.1. *If* $g:\overline{\mathbb{R}}\to[0,1]$ *is a large deviation multifractal spectrum, then the supremum of* g *is* 1.

We shall provide a result on the prescription of the large deviation multifractal spectrum in this frame. The proof simplifies greatly if we use the following "change of variable", that allows to deal smoothly with the particular cases $\alpha=\pm\infty$ in the definition of $f_g^d(\alpha)$: we set, for all $\theta\in[-\pi/2,\pi/2]$,

$$\widetilde{N}_\epsilon(\theta,j) = \#\{k \text{ such that } \arctan(A_{jk})\in[\theta-\epsilon,\theta+\epsilon]\}. \tag{32}$$

Define in an analogous way:

$$\widetilde{f}_g^d(\theta,\epsilon) = \limsup_n \frac{\log(\widetilde{N}_\epsilon(\theta,n))}{\log(2^n)}, \tag{33}$$

and:

$$\widetilde{f}_g^d(\theta) = \lim_{\epsilon\to0} \widetilde{f}_g^d(\theta,\epsilon). \tag{34}$$

It is then easy to check that, for all θ in $[-\pi/2,\pi/2]$:

$$\widetilde{f}_g^d(\theta) = f_g^d(\tan(\theta)).$$

with the convention $\tan(\pm\infty)=\pm\pi/2$.
We will show that $\widetilde{f}_g^d(\theta)$ has maximum 1 on $[-\pi/2,\pi/2]$.

Remark first that, for all θ, all ϵ and all j, $\widetilde{N}_\epsilon(\theta,j)\le 2^j$. As a consequence, 1 is an upper bound to $\widetilde{f}_g^d$ and thus to f_g^d.

Let us now show that $\widetilde{f}_g^d$ reaches the value 1.

We may divide $J_0 = [-\pi/2, \pi/2]$ into two intervals $[-\pi/2, 0]$ and $[0, \pi/2]$ such that at least one contains at least 2^{j-1} elements in the sequence $(\arctan A_{jk})_{j,k}$ for an infinite number of indices j. Denote this interval J_1.

Likewise, J_1 may be split into two intervals with diameter not larger than $\pi/4$ with at least one interval containing at least 2^{j-2} elements in the sequence $(\arctan A_{jk})_{j,k}$ for an infinite number of indices j. Denote J_2 this interval.

Iterating, one gets a sequence of non-empty nested intervals $(J_m)_{m\in\mathbb{N}}$ whose diameters tend to 0, which contain at least 2^{j-m} elements $\arctan A_{jk}$ for an infinite number of indices j.

Let $\{\theta_0\}$ denote their intersection, where $\theta_0 \in [-\pi/2, \pi/2]$. For all $\epsilon > 0$, there exists m such that $J_m \subset [\theta_0 - \epsilon, \theta_0 + \epsilon]$. As a consequence $[\theta_0 - \epsilon, \theta_0 + \epsilon]$ contains at least 2^{j-m} elements $\arctan A_{jk}$ for an infinite number of indices j. Thus, $\widetilde{f}_g^d(\theta_0, \epsilon) = 1$. This being true for all ϵ, we get $\widetilde{f}_g^d(\theta_0) = 1$. We conclude that $f_g^d(\tan\theta_0) = 1$, with the convention $\tan(\pm\infty) = \pm\pi/2$.

We will now prove the following result:

Theorem 6.2. *Let $g : \overline{\mathbb{R}} \to [0,1] \cup \{-\infty\}$ be an upper semi-continuous function defined on $\overline{\mathbb{R}}$ [1] with maximum 1. Then there exists a sequence of real numbers $(A_{jk})_{j\in\mathbb{N};k\in 1..2^j}$ such that g is the multifractal spectrum of the A_{jk}.*

We use the notations (32), (33) and (34) of the preceding part. Let us start by proving a preliminary result:

Lemma 6.2. *Let $\widetilde{g} : [-\pi/2; \pi/2] \to [0,1] \cup \{-\infty\}$ be an upper semi-continuous function on $[-\pi/2, \pi/2]$ with maximum 1.*

Then there exists a sequence of real numbers $(A_{jk})_{j\in\mathbb{N};k\in 1..2^j}$ such that $\widetilde{f}_g^d = \widetilde{g}$.

Proof. It is straightforward to construct a sequence of intervals $(I_n)_n$ such that $|I_n|$ tends to 0 and $[-\pi/2, \pi/2] = \cup_{n\geq N} I_n$ for all N. We also construct a sequence of real numbers i_n verifying $i_n \in I_n$ and $i_n \neq \pm\pi/2$.

Let θ_{max} be a point in $[-\pi/2, \pi/2]$ such that $\widetilde{g}(\theta_{max}) = 1$.

We build a sequence of points $(j_n)_n$ in $(-\pi/2, \pi/2)$ such that for all n and all $p \leq n$: $j_n \in I_p \Rightarrow \theta_{max} \in I_p$ ($j_n = \theta_{max}$ is a possible choice if $\theta_{max} \in (-\pi/2, \pi/2)$).

We now define the A_{nk} as:

$$\text{When } M(g, [-\pi/2, \pi/2] \cap I_n) \in [0,1] : \quad \begin{cases} A_{nk} = \tan i_n & , \quad k = 1 \cdots \lfloor 2^{n.M(g,I_n)} \rfloor, \\ A_{nk} = \tan j_n & , \quad k = \lfloor 2^{n.M(g,I_n)} \rfloor + 1 \cdots 2^n. \end{cases}$$

$$\text{When } M(g, [-\pi/2, \pi/2] \cap I_n) = -\infty : \quad A_{nk} = \tan j_n \quad , \quad k = 1 \cdots 2^n.$$

We have $\widetilde{f}_g^d = F^{loc}$ with $F(E) = \limsup_n x_n V_n(E)$ and $V_n(E) = \#\{k, \arctan(A_{nk}) \in [\theta - \epsilon, \theta + \epsilon]\}$.

[1] *i.e.* g is upper semi-continuous on $\mathbb{R}$ and it verifies in addition $g(+\infty) \geq \limsup_{x\to+\infty} g(x)$, $g(-\infty) \geq \limsup_{x\to-\infty} g(x)$.

For all n: $\#\{A_{nk}, \arctan A_{nk} \in I_n\} = \lfloor 2^{nM(g,I_n)} \rfloor$ with the convention $2^{-\infty} = 0$. As above, we get that: $d_{\overline{\mathbb{R}}}(x_n V_n(I_n), M(g, I_n))$ tends to 0 when n tends to infinity.

For all m such that $\theta_{max} \notin I_m$ and for all n such that $n > m$:

$$\#\{A_{nk}, A_{nk} \in I_n \setminus I_m\} = 0.$$

As a consequence, $\limsup_{n\to\infty} x_n V_n(I_m \setminus I_n) = -\infty$. Furthermore, if $\theta_{max} \in I_m$, then $M(g, I_n) = 1$. However, for all n,

$$\#\{A_{nk}, \arctan A_{nk} \in I_m \setminus I_n\} \le 2^n,$$

and thus $\limsup_{n\to\infty} x_n V_n(I_m \setminus I_n) \le M(g, I_n)$.

Lemma 6.1 yields that $\widetilde{f_g^d} = \widetilde{g}$. □

To obtain theorem 6.2, it now suffices to apply lemma 6.2 to $\widetilde{g}(\theta) = g(\tan(\alpha))$ with the convention $\tan(\pm\infty) = \pm\pi/2$.

Remark Conversely, lemma 2.2 yields that for any sequence of real numbers $(A_{jk})_{j\in\mathbb{N};k\in 1..2^j}$, the function $\widetilde{f_g^d}$ is upper semi-continuous. In other words, the multifractal spectrum of any sequence A_{jk} is an upper semi-continuous function on $\overline{\mathbb{R}}$. Furthermore, the multifractal spectrum of any sequence A_{jk} has to reach 1 on $\overline{\mathbb{R}}$ by theorem 6.1

To sum up, we have:

Proposition 6.2. *A function $g : \overline{\mathbb{R}} \to [0,1] \cup \{-\infty\}$ is a large deviation multifractal spectrum if and only if it is upper semi-continuous on $\overline{\mathbb{R}}$ and it reaches 1 on $\overline{\mathbb{R}}$.*

6.3 2-microlocal Spectrum

The prescription of the 2-microlocal frontier has been considered in [LS04, Mey98].

Let us recall that $-\chi = F_U^{loc}$, where $F_U(E) = \limsup_{t\to 0; t\in L} h(t)U(t, E)$, $L = \{2^{-j}, k2^{-j}\}$, $h(2^{-j}, k2^{-j}) = 1/j$ and, for $t \in L$: $U(t, E) = \log(|C(t)|.1_{t\in D_E})$.

Theorem 6.3. *Let $g : [0, 1] \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. Then there exists a distribution $f = \sum_{j,k} d_{jk}\psi_{jk}$ such that g is the 2-microlocal spectrum of f at x_0.*

Proof. We may assume without loss of generality that $x_0 = 0$. Let $(I_n)_n$ be a sequence of open intervals such that $|I_n|$ tends to 0 when n tends to infinity and $[0, 1] \subset \cup_{n\ge N} I_n$ for all N. Assume in addition that for all n: $I_n \cap [0, 1] \ne \emptyset$.
For all $n \in \mathbb{N}$ choose $t_n \in L_d \cap D_{I_n}$ such that:

- $\|t_n\|_\infty < \|t_{n-1}\|_\infty$
  (this condition ensures that each t_n is chosen only once).

- $\|t_n\|_\infty < 1/n$
  (this condition ensures that t_n tends to 0 when n tends to infinity).

To simplify notations, we let: $t_n = (2^{-j(n)}, k(n).2^{-j(n)})$. Note that $j(n)$ and $k(n)$ are integers since we assumed that $t_n \in L_d$.

Define finally d_{jk} by:

$$\begin{array}{ll} \forall n \in \mathbb{N} & : \ d_{j(n)k(n)} = 2^{j/2} 2^{-j(n)M(-g,I_n)}, \\ \text{If } (j,k) \notin \{(j(n),k(n)), n \in \mathbb{N}\} & : \ d_{jk} = 0. \end{array}$$

From the definition of the d_{jk}, it is easy to check that the 2-microlocal spectrum at 0 of $f = \sum_{j,k} d_{jk}\psi_{jk}$ reads $-\chi = F^{loc}$, where $F(E) = \limsup_{n\to\infty}(x_n V_n(E))$,

$$x_n = 1/(j(n)\log 2)$$

tends to 0 and

$$V_n(E) = \log(2^{-j/2} d_{j(n)k(n)}.1_{t_n \in D_E})$$

is semi-stable. For all $n \in \mathbb{N}$: $x_n V_n(I_n) = M(-g, I_n)$. Furthermore, for all (n_0, n), $t_n \in D_{I_n}$, and thus $V_n(I_{n_0} \setminus I_n) = -\infty$. The assumptions of Lemma 6.1 are verified. As a consequence, $\chi = g$. □

Remark The previous result may be extended to a function $g : [0,1] \to \mathbb{R} \cup \{\pm\infty\}$ by letting for instance:

- $\forall n \in \mathbb{N} \quad d_{j(n)k(n)} = 2^{-j(n)M(-g,I_n)}$ if $M(-g, I_n) > -\infty$ and $d_{j(n)k(n)} = 2^{j(n)n}$ if $M(-g, I_n) = -\infty$.

- If $(j,k) \notin \{(j(n),k(n)), n \in \mathbb{N}\}$ then $d_{jk} = 0$.

Proof. With the same notations as above:

$x_n V_n(I_n) = M(-g, I_n)$ if $M(-g, I_n) > -\infty$ and $x_n V_n(I_n) = n$ if $M(-g, I_n) > -\infty$.

The sequence $(d_{\overline{\mathbb{R}}}(x_n V_n(I_n), M(g, I \cap I_n))_n$ still tends to 0. Besides, we have again that $V_n(I_m \setminus I_n) = -\infty$. As a consequence, Lemma 6.1 still applies. □

Remark Conversely, Lemma 2.2 entails that the 2-microlocal spectrum of any distribution at any point is a lower semi-continuous function from $[0,1]$ to $\mathbb{R} \cup \{\pm\infty\}$.

We thus have the following proposition:

Proposition 6.3. *A function $g : [0,1] \to \mathbb{R} \cup \{+\infty\}$ is a 2-microlocal spectrum if and only if it is lower semi-continuous.*

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