

New rational extensions of solvable potentials with finite bound state spectrum

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Using the disconjugacy properties of the Schrödinger equation, it is possible to develop a new type of generalized SUSY QM partnership which allows to generate new solvable rational extensions for translationally shape invariant potentials having a finite bound state spectrum.

For this we prolong the dispersion relation relating the energy to the quantum number out of the physical domain until a disconjugacy sector. The prolonged excited states Riccati-Schrödinger (RS) functions are used to build Darboux-Bäcklund transforms which give regular isospectral extensions of the initial potential. We give the spectra of these extensions in terms of new orthogonal polynomials and study their shape invariance properties.

PACS numbers:

I. INTRODUCTION

The last three years have seen a substantial development of research works concerning the study of rational extensions of solvable quantum potentials, in particular because of their intimate relation with the recently discovered exceptional orthogonal polynomials (EOP) [4–27].

In [24–27] we have proposed a new scheme to generate the rational extensions of every primary translationally shape-invariant potentials (PTSIP) [1–3, 28]. They are obtained via Darboux-Bäcklund Transformations (DBT) based on negative eigenfunctions built from excited states of the initial hamiltonian. The regularity of corresponding Riccati-Schrödinger (RS) functions [28] is then directly verified by combining disconjugacy theorems and asymptotic analysis of the eigenfunctions. This approach which is systematic and generalizes the usual SUSY QM partnership can also be enlarged in a multistep version. In [24–27] these unphysical eigenfunctions are deduced from the excited states via discrete symmetries acting on the set of parameters of the initial potential. For the isotonic or trigonometric Darboux-Pöschl-Teller potentials, when the obtained extensions are strictly isospectral to the original potential, their eigenstates are (up to a gauge factor) the EOP discovered by Gomez-Ullate, Kamran and Milson [4].

For potentials presenting an infinite number of bound states associated to all integer values of the quantum number, the recourse to discrete symmetries is necessary to reach the disconjugacy sectors. However for PTSIP with a finite bound state spectrum $\{E_n, n \leq n_{\max}\}$ there exists another way. In this case, some disconjugacy sectors can be attained by prolonging the eigenstates ψ_n for values of the quantum number n going beyond $n_{\max}$. If the "dispersion relation", ie E_n as a function of n , goes then to negative values, the corresponding prolonged eigenstates ψ_n , although diverging at least at one extremity of the definition interval, may possibly be used to build regular extensions via DBT. Here again, disconjugacy theorems and asymptotic analysis allow to control the regularity of the corresponding RS function w_n . In the present paper we consider the four PTSIP for which the associated dispersion relation has the mentioned behaviour. We can share them into two groups: the Morse and hyperbolic Darboux-Pöschl-Teller potentials, which have a parabolic (quadratic) dispersion relation on the one hand and the Eckart and hyperbolic Rosen-Morse (HRM) potential for which the dispersion relation has the form of a second degree Laurent polynomial on the other hand. For the first group the prongation leads to only one disconjugacy sector and the extensions obtained are strictly isospectral to the initial potential. For the Morse potential we recover exactly the new extended potentials obtained very recently by Quesne [29]. Quesne shows in particular that if these extensions don't inherit of the translational shape invariance of the original potential (which one of the feature of the extensions associated to the EOP), they satisfy a kind of "enlarged shape invariance property". We prove that it is also the case for the similar extensions of the HDPT potential. For the Eckart potential, again we have only one disconjugacy sector with strictly isospectral extensions. This is no more the case for the HRM potential. In this last case, we have three distinct disconjugacy sectors, the two first corresponding to strictly isospectral extensions while the DBT built on eigenfunctions of the third sector are reverse SUSY QM partnership and then give only quasi isospectral extensions. For all the obtained strictly isospectral extensions, we give explicit expressions for the eigenstates in terms of new orthogonal polynomials.

II. DISCONJUGACY AND REGULAR EXTENSIONS OF ONE DIMENSIONAL POTENTIALS

If $\psi_\lambda(x; a)$ is an eigenstate of $\hat{H}(a) = -d^2/dx^2 + V(x; a)$, $a \in \mathbb{R}^m$, $x \in I \subset \mathbb{R}$, associated to the eigenvalue $E_\lambda(a)$ ($E_0(a) = 0$)

$$\psi_\lambda''(x; a) + (E_\lambda(a) - V(x; a)) \psi_\lambda(x; a) = 0, \quad (1)$$

then the Riccati-Schrödinger (RS) function $w_\lambda(x; a) = -\psi_\lambda'(x; a)/\psi_\lambda(x; a)$ satisfies the corresponding Riccati-Schrödinger (RS) equation [28]

$$-w_\lambda'(x; a) + w_\lambda^2(x; a) = V(x; a) - E_\lambda(a). \quad (2)$$

The set of Riccati-Schrödinger equations is preserved by the Darboux-Bäcklund Transformations (DBT), which are built from any solution $w_\nu(x; a)$ of the initial RS equation Eq(2) as [28, 32, 33]

$$w_\lambda(x; a) \xrightarrow{A(w_\nu)} w_\lambda^{(\nu)}(x; a) = -w_\nu(x; a) + \frac{E_\lambda(a) - E_\nu(a)}{w_\nu(x; a) - w_\lambda(x; a)}, \quad (3)$$

where $E_\lambda(a) > E_\nu(a)$. $w_\lambda^{(\nu)}$ is then a solution of the RS equation:

$$-w_\lambda^{(\nu)'}(x; a) + \left(w_\lambda^{(\nu)}(x; a)\right)^2 = V^{(\nu)}(x; a) - E_\lambda(a), \quad (4)$$

with the same energy $E_\lambda(a)$ as in Eq(2) but with a modified potential

$$V^{(\nu)}(x; a) = V(x; a) + 2w_\nu'(x; a), \quad (5)$$

which is called an extension of V .

The corresponding eigenstate of $\hat{H}^{(\nu)}(a) = -d^2/dx^2 + V^{(\nu)}(x; a)$ can be written

$$\psi_\lambda^{(\nu)}(x; a) = \exp\left(-\int dx w_\lambda^{(\nu)}(x; a)\right) \sim \frac{1}{\sqrt{E_\lambda(a) - E_\nu(a)}} \hat{A}(w_\nu) \psi_\lambda(x; a), \quad (6)$$

where $\hat{A}(a)$ is a first order operator given by

$$\hat{A}(w_\nu) = d/dx + w_\nu(x; a). \quad (7)$$

From V , the DBT generates a new potential $V^{(\nu)}$ (quasi)isospectral to the original one and its eigenfunctions are directly obtained from those of V via Eq(6). Nevertheless, in general, $w_\nu(x; a)$ and the transformed potential $V^{(\nu)}(x; a)$ are singular at the nodes of $\psi_\nu(x; a)$. For instance, if $\psi_n(x; a)$ ($\nu = n$) is a bound state of $\hat{H}(a)$, $V^{(n)}$ is regular only when $n = 0$, that is when $\psi_{n=0}$ is the ground state of $\hat{H}$, and we recover the usual SUSY partnership in quantum mechanics.

We can however envisage to use any other regular solution of Eq(2) as long as it has no zero on the considered real interval I , even if it does not correspond to a physical state. In particular, it is possible to use some eigenfunctions ψ_ν associated to negative eigenvalues $E_\nu(a) < 0$ [34]. This is due to the disconjugacy of the Schrödinger equation for these eigenvalues [27]. More precisely, a second order differential equation like Eq(1) is said to be disconjugated on I if every solution of this equation has at most one zero on I [35–37]. For a closed or open interval I , the disconjugacy of Eq(1) is equivalent to the existence of solutions of this equation which are everywhere non zero on I [35–37].

We have also the following result

Theorem [35, 36] If there exists a continuously differentiable solution on I of the Riccati inequation

$$-w'(x) + w^2(x) + G(x) \leq 0 \quad (8)$$

then the equation

$$\psi''(x) + G(x)\psi(x) = 0, \quad (9)$$

is disconjugated on I .

In our case, if $E_\lambda(a) \leq 0$, we have

$$-w'_0(x; a) + w_0^2(x; a) = V(x; a) \leq V(x; a) - E_\lambda(a), \quad (10)$$

$w_0(x; a)$ being continuously differentiable on I . The above theorem ensures the existence of nodeless solutions $\phi_\lambda(x; a)$ of Eq(1), that is, of corresponding regular RS functions $v_\lambda(x; a) = -\phi'_\lambda(x; a)/\phi_\lambda(x; a)$. To prove that a given solution $\phi_\lambda(x; a)$ belongs to this category, it is sufficient to determine the signs of the limit values $\phi_\lambda(0^+; a)$ and $\phi_\lambda(+\infty; a)$. If they are identical then ϕ_λ is nodeless and if they are opposite ϕ_λ presents then a unique zero on I . In the first case $V(x; a) + 2v'_\lambda(x; a)$ constitutes a perfectly regular (quasi)isospectral extension of $V(x; a)$.

Of particular interest is to obtain such solutions $\phi_\lambda(x; a)$ which are polynomials (up to a gauge factor) in order to build rational extensions of the initial potential V . In [25–27], we have shown that it is possible to generate such solutions by using specific discrete symmetries Γ_i which are covariance transformations for the considered family of potentials

$$\left\{ \begin{array}{l} a \xrightarrow{\Gamma_i} a_i \\ V(x; a) \xrightarrow{\Gamma_i} V(x; a_i) = V(x; a) + U(a). \end{array} \right. \quad (11)$$

Γ_i acts on the parameters of the potential and transforms the RS function of a physical excited eigenstate w_n into a unphysical RS function $v_{n,i}(x; a) = \Gamma_i(w_n(x; a)) = w_n(x; a_i)$ associated to the negative eigenvalue $\mathcal{E}_{n,i}(a) = \Gamma_i(E_n(a)) = U(a) - E_n(a_i) < 0$.

$$-v'_{n,i}(x; a) + v_{n,i}^2(x; a) = V(x; a) - \mathcal{E}_{n,i}(a). \quad (12)$$

To $v_{n,i}$ corresponds an unphysical eigenfunction of $\hat{H}(a)$

$$\phi_{n,i}(x; a) = \exp\left(-\int dx v_{n,i}(x; a)\right) \quad (13)$$

associated to the eigenvalue $\mathcal{E}_{n,i}(a)$.

Since $\mathcal{E}_{n,i}(a) < 0$, if $\phi_{n,i}(x; a)$ has the same sign at both extremities of I , then $v_{n,i}(x; a)$ can be used to build a regular extended potential (see Eq(5) and Eq(6))

$$V^{(n,i)}(x; a) = V(x; a) + 2v'_{n,i}(x; a) \quad (14)$$

(quasi)isospectral to $V(x; a)$. The eigenstates of $V^{(n,i)}$ are given by (see Eq(3))

$$\left\{ \begin{array}{l} w_k^{(n,i)}(x; a) = -v_{n,i}(x; a) + \frac{E_k(a) - \mathcal{E}_{n,i}(a)}{v_{n,i}(x; a) - w_k(x; a)} \\ \psi_k^{(n,i)}(x; a) = \exp\left(-\int dx w_k^{(n,i)}(x; a)\right) \sim \hat{A}(v_{n,i}) \psi_k(x; a) \end{array} \right. , \quad (15)$$

for the respective energies $E_k(a)$.

The nature of the isospectrality depends if $1/\phi_{n,i}(x; a)$ satisfies or not the appropriate boundary conditions. If it is the case, then $1/\phi_{n,i}(x; a)$ is a physical eigenstate of $\hat{H}^{(n,i)}(a) = -d^2/dx^2 + V^{(n,i)}(x; a)$ for the eigenvalue $\mathcal{E}_{n,i}(a)$ and we only have quasi-isospectrality between $V(x; a)$ and $V^{(n,i)}(x; a)$. If it is not the case, the isospectrality between $V^{(n,i)}(x; a)$ and $V(x; a)$ is strict.

This procedure can be viewed as a "generalized SUSY QM partnership" where the DBT can be based on excited states RS functions properly regularized by the symmetry Γ_j . In [25–27], it has been applied to exceptional PTSIP of the first and second categories [28]. In the particular case of the isotonic oscillator, the spectrum of the two first

series of extensions are given (up to a gauge factor) by the exceptional Laguerre polynomials initially discovered by Gomez-Ullate et al [6].

For some potentials which have finite bound states spectrum, it is possible to use an even more direct way to generate negative energies eigenstates. In this case, the eigenfunctions $\psi_n(x; a)$ satisfy the required Dirichlet boundary conditions to be acceptable physical eigenstates only for a finite number of values $\{0, \dots, n_{\max}\}$ of the quantum number n . Beyond this maximal value $n_{\max}$, ψ_n have a divergent behaviour at (at least) one extremity of the definition interval I . ψ_n has then to be rejected as eigenstate but it can still be used to build the corresponding DBT $A(w_n)$, the energy E_n (viewed as a function of the quantum number n) being extended to values of n greater than $n_{\max}$. If in this extended domain E_n reaches negative values we recover then a disconjugacy sector of the Schrödinger equation and ψ_n can be exempt of nodes. The DBT $A(w_n)$ gives then a regular extended potential (see Eq(5) and Eq(6))

$$V^{(n)}(x; a) = V(x; a) + 2w'_n(x; a), \quad (16)$$

(quasi)isospectral to $V(x; a)$ and its eigenstates are given by (see Eq(3))

$$\begin{cases} w_k^{(n)}(x; a) = -w_n(x; a) + \frac{E_k(a) - E_n(a)}{w_n(x; a) - w_k(x; a)} \\ \psi_k^{(n)}(x; a) = \exp\left(-\int dx w_k^{(n)}(x; a)\right) \sim \hat{A}(w_n) \psi_k(x; a). \end{cases} \quad (17)$$

III. PRIMARY TRANSLATIONALLY SHAPE INVARIANT POTENTIALS WITH FINITE BOUND STATES SPECTRUM

In [28], we have shown that all the PTSIP can be classified into two categories in which the potentials can be brought into a harmonic or isotonic form respectively, using a change of variable which satisfies a constant coefficient Riccati equation. Among them we have potentials admitting an infinite bound state spectrum [1, 2, 28]. This is naturally the case of the confining potentials as the harmonic and isotonic oscillators, the trigonometric DPT (or Scarf I) and the trigonometric Rosen-Morse potentials. For these ones the dispersion relation is a strictly increasing function on $n \in [0, +\infty[$ which is linear for the two first, parabolic for the third one and which is a second degree Laurent polynomial for the fourth one. This is also the case of the effective radial Kepler-Coulomb (ERKC) potential which admits an infinite bound state spectrum with negative energies and a continuous spectrum of scattering states with positive energies. The dispersion relation for the bound states (a second degree Laurent polynomial without a regular term) is also strictly increasing for $n \in [0, +\infty[$.

The other primary TSIP have a finite bound state spectrum. This is the case in the first category for the Morse, Eckart and hyperbolic Rosen-Morse potentials and in the second category for the hyperbolic Darboux-Pöschl-Teller (or Scarf II) potential [1, 2, 28].

A. Case of dispersion relation which is a second degree polynomial

1. Morse potential

For the Morse potential [1, 2, 28] ($x \in \mathbb{R}$)

$$V(y; a, b) = b^2 y^2 - 2\left(a + \frac{1}{2}\right) b y + a^2, \quad a, b > 0, \quad (18)$$

where $y = \exp(-x)$, we have a "dispersion relation" (energy E_n as a function of the quantum number $n \geq 0$) which is of parabolic type. Namely, we have ($a_n = a - n$)

$$E_n(a) = a^2 - a_n^2 = n(2a - n). \quad (19)$$

The bound states are obtained on the increasing part of E_n , that is for $n < a$ and we have exactly $[a]$ bound states ($[a]$ being the integer part of a) given by

$$\psi_n(x; a, b) = \psi_0(x; a_n, b) L_n^{2a_n}(2by), \quad (20)$$

where

$$\psi_0(x; a, b) = y^a e^{-by}. \quad (21)$$

The corresponding RS functions are given by

$$w_n(x; a, b) = w_0(x; a_n, b) - 2by \frac{L_{n-1}^{2a_n+1}(2by)}{L_n^{2a_n}(2by)}, \quad (22)$$

where

$$w_0(x; a, b) = a - by. \quad (23)$$

Beyond $n = a$, the ψ_n no longer correspond to physical eigenstates and the E_n function decreases. When n exceeds the value $2a$, E_n becomes negative and the corresponding Schrödinger equation enters in a disconjugacy regime.

Using

$$\begin{cases} L_n^\alpha(z) \xrightarrow{x \rightarrow 0^+} \frac{(\alpha+1)_n}{n!} \\ L_n^\alpha(z) \underset{x \rightarrow +\infty}{\sim} \frac{(-1)^n}{n!} z^n \end{cases}, \quad (24)$$

where $(X)_n = (X) \dots (X + n - 1)$ is the usual Pochhammer symbol [31], we find that the asymptotic behaviour of $\psi_n(x; a, b)$ at $+\infty$ and $-\infty$ is given by

$$\psi_n(x; a, b) \underset{x \rightarrow +\infty}{\sim} y^{n-a} \frac{(2a - 2n + 1) \dots (2a - n)}{n!} \xrightarrow{x \rightarrow +\infty} \pm\infty, \quad (25)$$

and

$$\psi_n(x; a, b) \underset{x \rightarrow -\infty}{\sim} \frac{(-1)^n}{n!} y^a e^{-by} \xrightarrow{x \rightarrow +\infty} 0^\pm, \quad (26)$$

where $\pm = (-1)^n$.

We see that ψ_n has the same sign at both ends of the definition interval, the disconjugacy of the Schrödinger equation that it satisfies implies that ψ_n has no zero on $\mathbb{R}$. We can then use the corresponding RS functions to build (quasi)isospectral extensions of the Morse potential respectively. They are given by

$$V^{(n)}(x; a, b) = V(y; a, b) - 2y \frac{dw_n(y; a_n, b)}{dy}, \quad (27)$$

where $w_n(y; a_n, b)$ is given in Eq(22), with $y = \exp(-\alpha x)$.

Since $1/\psi_n(x; a, b)$ is divergent at $-\infty$, it is not a physical eigenstate of $V^{(n)}$ which is then strictly isospectral to V and admits for bound state eigenfunctions ($k \in \{0, \dots, [a - 1]\}$, $z = 2by$)

$$\psi_k^{(n)}(x; a, b) = (w_n(x; a, b) - w_k(x; a, b)) \psi_k(x; a, b) = \psi_0(x; a_k, b) \frac{L_{n,k}^a(z)}{L_n^{2a_n}(z)}, \quad (28)$$

where, making use of the following recurrency properties of the Laguerre polynomials [30, 31]

$$\begin{cases} L_n^{(\alpha)}(z) + L_{n-1}^{(\alpha+1)}(z) = L_n^{(\alpha+1)}(z) \\ z L_{n-1}^{\alpha+1}(z) = (n + \alpha) L_{n-1}^\alpha(z) - n L_n^\alpha(z), \end{cases} \quad (29)$$

we can write

$$L_{n,k}^a(z) = (2a - k) L_n^{2a_n}(z) L_{k-1}^{2a_k}(z) - (2a - n) L_k^{2a_k}(z) L_{n-1}^{2a_n}(z), \quad (30)$$

which is a polynomial of degree $n + k - 1$.

2. HDPT potential

The dispersion relation of the hyperbolic Darboux-Pöschl-Teller (HDPT) potential [1, 2, 28] ($x > 0$)

$$V(x; \alpha, \beta) = \frac{(\alpha + 1/2)(\alpha - 1/2)}{\sinh^2 x} - \frac{(\beta + 1/2)(\beta - 1/2)}{\cosh^2 x} + V_0(\alpha, \beta), \quad \beta > \alpha + 1 > 1/2, \quad (31)$$

with $V_0(\alpha, \beta) = (\beta - \alpha - 1)^2$, is of the same parabolic type than in the preceding case

$$E_n(\alpha, \beta) = 4n(2a - n), \quad (32)$$

with $a = (\beta - \alpha - 1)/2$. Here again the bound states are obtained on the increasing part of E_n , that is, for $n < a$ and they are given by

$$\psi_n(x; \alpha, \beta) = \psi_0(x; \alpha, \beta) P_n^{(\alpha, -\beta)}(z), \quad (33)$$

where $z = \cosh 2x$ and

$$\psi_0(x; \alpha, \beta) = (\sinh x)^{\alpha+1/2} (\cosh x)^{-\beta+1/2}. \quad (34)$$

Using [30, 31]

$$\left(P_n^{(\alpha, \beta)}(x) \right)' = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{(\alpha+1, \beta+1)}(x), \quad (35)$$

we obtain for the corresponding RS functions

$$w_n(x; \alpha, \beta) = w_0(x; \alpha, \beta) - \sinh 2x (n + \alpha - \beta + 1) \frac{P_{n-1}^{(\alpha+1, 1-\beta)}(z)}{P_n^{(\alpha, -\beta)}(z)}, \quad (36)$$

where

$$w_0(x; \alpha, \beta) = -(\alpha + 1/2) \coth x + (\beta - 1/2) \tanh x. \quad (37)$$

As before, when n exceeds the value $2a$, the corresponding Schrödinger equation enters in a disconjugacy regime. Using

$$\begin{cases} P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!} \\ P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n! \Gamma(n+\alpha+\beta+1)} x^n + O(x^{n-1}), \end{cases} \quad (38)$$

from Eq(33) and Eq(34), we deduce for $n > \beta - \alpha - 1$

$$\begin{cases} \psi_n(x; \alpha, \beta) \underset{x \rightarrow 0^+}{\sim} x^{\alpha+1/2} \frac{(\alpha+1)_n}{n!} \rightarrow 0^+ \\ \psi_n(x; \alpha, \beta) \underset{x \rightarrow +\infty}{\sim} e^{(2n+\alpha-\beta+1)x} \frac{(2n+\alpha-\beta) \dots (n+\alpha-\beta+1)}{2^n n!} \rightarrow +\infty. \end{cases} \quad (39)$$

ψ_n having the same sign at both ends of the definition interval, ψ_n has no zero on $\mathbb{R}$. The DBT built on the corresponding RS functions generate isospectral extensions of the HDPT potential. They are given by

$$V^{(n)}(x; \alpha, \beta) = V(x; \alpha, \beta) + w'_n(x; \alpha, \beta), \quad (40)$$

where w'_n is given in Eq(36). Since $1/\psi_n(x; \alpha, \beta)$ is divergent at the origin, it is not a physical eigenstate of $V^{(n)}$ which is then strictly isospectral to V . It admits for bound state eigenfunctions

$$\psi_k^{(n)}(x; \alpha, \beta) = (w_n(x; \alpha, \beta) - w_k(x; \alpha, \beta)) \psi_k(x; \alpha, \beta) = \psi_0(x; \alpha + 1, \beta - 1) \frac{P_{n,k}^{(\alpha, \beta)}(z)}{P_n^{(\alpha, -\beta)}(z)}, \quad (41)$$

where $k < a$ and where

$$P_{n,k}^{(\alpha, \beta)}(z) = (k + \alpha - \beta + 1) P_n^{(\alpha, -\beta)}(z) P_{k-1}^{(\alpha+1, -\beta+1)}(z) - (n + \alpha - \beta + 1) P_{n-1}^{(\alpha+1, -\beta+1)}(z) P_k^{(\alpha, -\beta)}(z) \quad (42)$$

is a polynomial of degree $n + k - 1$.

B. Case of dispersion relation which is a second degree Laurent polynomial

1. Eckart potential

The Eckardt potential ($x \in]0, +\infty[$) can be written as [1, 2, 28]

$$V(x; a, b) = a(a+1)y^2 - 2by + V_0(a, b), \quad a^2 < b, \quad a, b > 0, \quad (43)$$

where $y = \coth x$ and

$$V(a, b) = \frac{b^2}{a^2} + a. \quad (44)$$

The dispersion relation corresponds to a second degree Laurent polynomial of the form ($a_n = a + n$)

$$\begin{aligned} E_n &= a^2 + \frac{b^2}{a^2} - a_n^2 - \frac{b^2}{a_n^2} \\ &= -\frac{n}{(n+a)^2} (n+2a) \left(n + \left(a - \frac{b}{a} \right) \right) \left(n + \left(a + \frac{b}{a} \right) \right). \end{aligned} \quad (45)$$

The bound states are obtained on the increasing part of E_n , that is for $0 \leq n < \sqrt{b} - a$ and they are given by

$$\psi_n(x; a, b) = \psi_0(x; \alpha_n, \beta_n) P_n^{(\alpha_n, \beta_n)}(y), \quad (46)$$

where

$$\psi_0(x; a, b) = (y-1)^{\frac{\alpha}{2}} (y+1)^{\frac{\beta}{2}} = e^{-bx/a} \sinh^a x, \quad (47)$$

with

$$\begin{cases} \alpha_n = -a_n + \frac{b}{a_n} \\ \beta_n = -a_n - \frac{b}{a_n} \end{cases}. \quad (48)$$

The corresponding RS functions are

$$w_n(x; a, b) = w_0(x; \alpha_n, \beta_n) + (y^2 - 1) \frac{n + \alpha_n + \beta_n + 1}{2} \frac{P_{n-1}^{(\alpha_n+1, \beta_n+1)}(y)}{P_n^{(\alpha_n, \beta_n)}(y)}, \quad (49)$$

where

$$w_0(x; \alpha, \beta) = \frac{\alpha}{2} (y+1) + \frac{\beta}{2} (y-1). \quad (50)$$

Beyond the value $n = a - \sqrt{b}$, the ψ_n do not correspond anymore to physical eigenstates and for $n > \frac{b}{a} - a$, E_n becoming negative, the corresponding Schrödinger equation enters in a disconjugacy regime.

Using Eq(38), from Eq(49) we deduce the following asymptotic behaviour for ψ_n ($\alpha_n + \beta_n = -2a - 2n$)

$$\begin{cases} \psi_n(x; a, b) \underset{x \rightarrow 0^+}{\sim} (-2a) \dots (-n - 2a + 1) x^{-(2n + \alpha_n + \beta_n)/2} \\ \psi_n(x; a, b) \underset{x \rightarrow +\infty}{\sim} (y-1)^{\frac{\alpha_n}{2}} \frac{(\alpha_n+1) \dots (\alpha_n+n)}{n!}. \end{cases} \quad (51)$$

When $n > \frac{b}{a} - a$, $\alpha_n + n < 0$ and $\alpha_n + \beta_n = -2a - 2n$. Consequently

$$\begin{cases} \psi_n(x; a, b) \xrightarrow{x \rightarrow 0^+} \pm\infty \\ \psi_n(x; a, b) \xrightarrow{x \rightarrow +\infty} 0^\pm, \end{cases} \quad (52)$$

with $\pm = (-1)^n$.

Since ψ_n has the same sign at both ends of the definition interval, the disconjugacy of the Schrödinger equation that it satisfies implies that ψ_n has no zero on $\mathbb{R}$. We can then use the corresponding RS functions to build isospectral extensions of the Eckart potential given by

$$V^{(n)}(x; a, b) = V(y; a, b) - 2(y^2 - 1) \frac{dw_n(y; a, b)}{dy}, \quad (53)$$

where w_n is given in Eq(49). $1/\psi_n(x; a, b)$ is divergent at $+\infty$ and is not a physical eigenstate of $V^{(n)}$ which is then strictly isospectral to V . The bound state eigenfunctions of $V^{(n)}$ are given by

$$\psi_k^{(n)}(x; a, b) = (w_n(x; a, b) - w_k(x; a, b)) \psi_k(x; a, b) = \psi_0(x; \alpha_k, \beta_k) \frac{P_{n,k}^{(\alpha, \beta)}(z)}{2P_n^{(\alpha_n, \beta_n)}(z)}, \quad (54)$$

where $k < \sqrt{b} - a$ and

$$\begin{aligned} P_{n,k}^{(\alpha, \beta)}(z) &= (\alpha_n(y+1) + \beta_n(y-1)) P_n^{(\alpha_n, \beta_n)}(z) P_k^{(\alpha_k, \beta_k)}(z) \\ &+ (y^2 - 1)(n + \alpha_n + \beta_n + 1) P_k^{(\alpha_k, \beta_k)}(z) P_{n-1}^{(\alpha_n+1, \beta_n+1)}(z) \\ &- [k \leftrightarrow n]. \end{aligned} \quad (55)$$

2. Hyperbolic Rosen-Morse (HRM) potential

Finally consider the HRM potential ($x \in \mathbb{R}$) which is given by [1, 2, 28]

$$V(x; a, b) = a(a+1)y^2 + 2by + V_0(a, b), \quad a^2 > b, \quad a, b > 0, \quad (56)$$

where $y = \tanh x$ and

$$V(a, b) = \frac{b^2}{a^2} - a. \quad (57)$$

As for the Eckart potential, the dispersion relation corresponds to a second degree Laurent polynomial

$$\begin{aligned} E_n &= a^2 + \frac{b^2}{a^2} - a_n^2 - \frac{b^2}{a_n^2} \\ &= -\frac{n}{(n-a)^2} (n-2a) \left(n - \left(a - \frac{b}{a} \right) \right) \left(n - \left(a + \frac{b}{a} \right) \right) \end{aligned} \quad (58)$$

($a_n = a - n$), but E_n is now singular at the positive value $n = a$.

As in the preceding cases, the bound states are obtained on the increasing part of E_n , that is, for $0 \leq n < a - \sqrt{b}$. They are given by

$$\psi_n(x; a, b) = \psi_0(x; \alpha_n, \beta_n) P_n^{(\alpha_n, \beta_n)}(y), \quad (59)$$

where

$$\psi_0(x; a, b) = (1-y)^{\frac{\alpha}{2}} (1+y)^{\frac{\beta}{2}} = \frac{e^{-bx/a}}{\cosh^a x}, \quad (60)$$

with

$$\begin{cases} \alpha_n = a_n + \frac{b}{a_n} \\ \beta_n = a_n - \frac{b}{a_n}. \end{cases} \quad (61)$$

The corresponding RS functions are

$$w_n(x; a, b) = w_0(x; \alpha_n, \beta_n) - (1-y^2) \frac{n + \alpha_n + \beta_n + 1}{2} \frac{P_{n-1}^{(\alpha_n+1, \beta_n+1)}(y)}{P_n^{(\alpha_n, \beta_n)}(y)}, \quad (62)$$

where

$$w_0(x; \alpha, \beta) = \frac{\alpha}{2} (1+y) - \frac{\beta}{2} (1-y). \quad (63)$$

Beyond the value $n = a - \sqrt{b}$, the ψ_n do not correspond anymore to physical eigenstates and E_n is negative when n belongs to the intervals $[a-b, a+b]$ and $[2a, +\infty]$.

Using [30, 31]

$$\begin{cases} P_n^{(\alpha, \beta)}(1) = \frac{(\alpha+1)_n}{n!} \\ P_n^{(\alpha, \beta)}(-1) = (-1)^n \frac{(\beta+1)_n}{n!}, \end{cases} \quad (64)$$

from Eq(62) we deduce the following asymptotic behaviour for ψ_n

$$\begin{cases} \psi_n(x; a, b) \underset{x \rightarrow -\infty}{\sim} (1+y)^{\frac{\beta_n}{2}} (-1)^n \frac{(\beta_n+1) \dots (\beta_n+n)}{n!} \\ \psi_n(x; a, b) \underset{x \rightarrow +\infty}{\sim} (1-y)^{\frac{\alpha_n}{2}} \frac{(\alpha_n+1) \dots (\alpha_n+n)}{n!}. \end{cases} \quad (65)$$

Consider first the case (i) where $a - \frac{b}{a} < n < a$. Then $a_n > 0$, $\alpha_n > 0$ and $\beta_n + n < 0$. Consequently

$$\begin{cases} \psi_n(x; a, b) \underset{x \rightarrow -\infty}{\rightarrow} +\infty \\ \psi_n(x; a, b) \underset{x \rightarrow +\infty}{\rightarrow} 0^+. \end{cases} \quad (66)$$

Consider now the case (ii) where $a + \frac{b}{a} > n > a$ ($a_n < 0$). In this case we have $\alpha_n + n < 0$ and $\beta_n > 0$. It results

$$\begin{cases} \psi_n(x; a, b) \underset{x \rightarrow -\infty}{\rightarrow} \pm\infty \\ \psi_n(x; a, b) \underset{x \rightarrow +\infty}{\rightarrow} 0^\pm, \end{cases} \quad (67)$$

where $\pm = (-1)^n$.

Finally consider the case (iii) $n > 2a$ ($a_n < 0$). We have $\alpha_n, \beta_n < 0$ and $\alpha_n + n, \beta_n + n > 0$. ψ_n is then divergent both at $+\infty$ and $-\infty$. Depending on the value of n (compared to $a + b/k$, $k \in \mathbb{N}^*$), we can then have the same sign or not for the limits of ψ_n .

When ψ_n has the same sign at both ends of the definition interval, the disconjugacy of the Schrödinger equation that it satisfies implies that ψ_n has no zero on $\mathbb{R}$. We can then use the corresponding RS functions to build isospectral extensions of the HRM potential given by

$$V^{(n)}(x; a, b) = V(y; a, b) + 2(1-y^2) \frac{dw_n(y; a, b)}{dy}, \quad (68)$$

where w_n is given in Eq(62).

In the cases (i) and (ii), $1/\psi_n(x; a, b)$ is divergent at $+\infty$ or $-\infty$ and is not a physical eigenstate of $V^{(n)}$ which is then strictly isospectral to V .

At the contrary in the case (iii), when $1/\psi_n(x; a, b)$ is regular, it also satisfies the required Dirichlet boundary conditions and constitutes the fundamental bound state of $V^{(n)}$. The DBT is then a backward SUSY partnership. We will not consider this case in the following.

In the cases (i) and (ii), the bound state eigenfunctions of $V^{(n)}$ are given by

$$\psi_k^{(n)}(x; a, b) = (w_n(x; a, b) - w_k(x; a, b)) \psi_k(x; a, b) = \psi_0(x; \alpha_k, \beta_k) \frac{P_{n,k}^{(\alpha, \beta)}(z)}{2P_n^{(\alpha_n, \beta_n)}(z)}, \quad (69)$$

where $k < a - \sqrt{b}$ and

$$\begin{aligned} P_{n,k}^{(\alpha, \beta)}(z) = & (1 - y^2) (k + \alpha_k + \beta_k + 1) P_n^{(\alpha_n, \beta_n)}(z) P_{k-1}^{(\alpha_k+1, \beta_k+1)}(z) \\ & - (\alpha_k(1 + y) - \beta_k(1 - y)) P_n^{(\alpha_n, \beta_n)}(z) P_k^{(\alpha_k, \beta_k)}(z) \\ & - [k \leftrightarrow n]. \end{aligned} \quad (70)$$

IV. SHAPE INVARIANCE

For the isotonic oscillator we have proven explicitly in [25, 27] that the shape invariance property of the initial potential is transmitted to all its strictly isospectral ($L1$ and $L2$ series) successive extensions. This is not the case for the extensions obtained from the potentials considered in [26] via the use of regularizing symmetries.

To look for such a property for the extensions Morse and HDPT potentials obtained above, we have to consider the superpartner of the potential $V^{(n)}(x; a)$ which is defined as

$$\tilde{V}^{(n)}(x; a) = V^{(n)}(x; a) + 2 \left(w_0^{(n)}(x; a) \right)', \quad (71)$$

$w_0^{(n)}(x; \omega, a)$ being the RS function associated to the ground level of $V^{(n)}$.

Since (see Eq(3))

$$w_0^{(n)}(x; a) = -w_n(x; a) - \frac{E_n(a)}{w_n(x; a) - w_0(x; a)}, \quad (72)$$

we have, using Eq(16)

$$\tilde{V}^{(n)}(x; a) = V(x; a) - 2 \left(\frac{E_n(a)}{w_n(x; a) - w_0(x; a)} \right)'. \quad (73)$$

We suppose that V is a TSIP satisfying

$$\tilde{V}(x; a) = V(x; a) + 2w_0'(x; a) = V(x; a - 1) + E_1(a). \quad (74)$$

Inserting Eq(74) into Eq(73), it results

$$\tilde{V}^{(n)}(x; a) = V(x; a - 1) + E_1(a) - 2W'(x; a), \quad (75)$$

where

$$W(x; a) = w_0(x; a) + \frac{E_n(a)}{w_n(x; a) - w_0(x; a)}. \quad (76)$$

A. Morse potential

Consider the case of the Morse potential. Using Eq(22) and Eq(23), Eq(76) gives ($z = 2by$)

$$W(x; a) = -a_n + by + z \frac{L_{n-2}^{2a_n+1}(z)}{L_{n-1}^{2a_n}(z)} \quad (77)$$

and since $a_n = (a-1)_{n-1}$, we deduce

$$W(x; a) = - \left(w_0(x; (a-1)_{n-1}) - z \frac{L_n^{2(a-1)_{n-1}-1}(z)}{L_{n-1}^{2(a-1)_{n-1}}(z)} \right) = -w_{n-1}(x; (a-1)_{n-1}). \quad (78)$$

Consequently

$$\begin{aligned} \tilde{V}^{(n)}(x; a) &= V(x; a-1) + 2w'_{n-1}(x; (a-1)_{n-1}) + E_1(a) \\ &= V^{(n-1)}(x; a_1) + E_1(a). \end{aligned} \quad (79)$$

This is not strictly speaking a shape invariance in the sense of Gendenshtein [3]. As noted by Quesne [29], we rather obtain a kind of enlarged shape invariance property where the SUSY QM partner of the n^{th} extended potential $V^{(n)}$ has not the functional form of $V^{(n)}$ (with translated parameters and an additional constant) but the one of the preceding extension $V^{(n-1)}$.

B. HDPT potential

For the HDPT potential, combining Eq(36) and Eq(37) in Eq(76), we find that W takes the form ($z = \cosh 2x$)

$$W(x; \alpha, \beta) = w_0(x; \alpha, \beta) + \frac{4nP_n^{(\alpha, -\beta)}(z)}{\sinh 2x P_{n-1}^{(\alpha+1, 1-\beta)}(z)}. \quad (80)$$

But $P_n^{(\alpha, -\beta)}(z)$ satisfies the differential equation

$$(1 - z^2) y''(z) + (\alpha + \beta - z(\alpha - \beta + 2)) y'(z) + n(n + \alpha - \beta + 1) y(z) = 0, \quad (81)$$

which, combined to Eq(35) gives

$$4nP_n^{(\alpha, -\beta)}(z) = (z^2 - 1)(n + \alpha - \beta + 2) P_{n-2}^{(\alpha+2, -\beta+2)}(z) + 2(\alpha + \beta - z(\alpha - \beta + 2)) P_{n-1}^{(\alpha+1, -\beta+1)}(z). \quad (82)$$

Substituting this result in Eq(80) and using Eq(37), we obtain

$$\begin{aligned} W(x; \alpha, \beta) &= \frac{\alpha + \beta + (\alpha - \beta + 3)z}{\sinh 2x} + \sinh 2x (n + \alpha - \beta + 2) \frac{P_{n-2}^{(\alpha+2, -\beta+2)}(z)}{P_{n-1}^{(\alpha+1, 1-\beta)}(z)} \\ &= -w_{n-1}(x; \alpha_1, \beta_1), \end{aligned} \quad (83)$$

that is,

$$\begin{aligned} \tilde{V}^{(n)}(x; \alpha, \beta) &= V(x; \alpha_1, \beta_1) + 2w'_{n-1}(x; \alpha_1, \beta_1) + E_1(a) \\ &= V^{(n-1)}(x; \alpha_1, \beta_1) + E_1(a). \end{aligned} \quad (84)$$

Again, $V^{(n)}$ satisfies the enlarged shape invariance property defined above.

V. CONCLUSION

For PTSIP with a finite number of bound states, we have enlarged the generalized SUSY QM partnership presented in [24–26, 28] by showing another way to obtain disconjugated unphysical eigenfunctions which may serve to generate regular rational extensions of these potentials. We have studied more explicitly those of these extended potentials which are strictly isospectral to the initial one. These results encompass in particular those obtained very recently by Quesne [29] for the Morse potential. The enlarged shape invariance property revealed by Quesne for the Morse potential extensions is proven to be shared by the HDPT potential.

VI. ACKNOWLEDGMENTS

I would like to thank C. Quesne for very stimulating and enlightening exchanges.

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