CHANG'S CONJECTURE MAY FAIL AT SUPERCOMPACT CARDINALS

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Abstract. We prove a revised version of Laver's indestructibility theorem which slightly improves over the classical result. An application yields the consistency of $(\kappa^+,\kappa) \not\twoheadrightarrow (\aleph_1,\aleph_0)$ when κ is supercompact. The actual proofs show that ω_1-regressive Kurepa-trees are consistent above a supercompact cardinal even though MM destroys them on all regular cardinals. This rather paradoxical fact contradicts the common intuition.

1. Introduction

A structure $\mathfrak{A}$ with a distinguished predicate R is said to be of *type* (λ,κ) if $|\mathfrak{A}| = \lambda$ and $|R^{\mathfrak{A}}| = \kappa$. The relation $(\lambda,\kappa) \to (\mu,\nu)$ then means that for every structure of type (λ,κ) there is an elementary equivalent structure of type (μ,ν). We have the classical theorem (see [1]):

1 Theorem (Vaught). *$(\lambda^+,\lambda) \to (\aleph_1,\aleph_0)$ holds for all cardinals λ.*

Consider a variation of this notion: the principle $(\lambda,\kappa) \twoheadrightarrow (\mu,\nu)$ means that every structure of type (λ,κ) has an elementary substructure of type (μ,ν). The relation $(\aleph_2,\aleph_1) \twoheadrightarrow (\aleph_1,\aleph_0)$ is usually called *Chang's conjecture.*

2 Theorem (Silver). *Chang's conjecture is independent of* ZFC.

Silver's result (see [6]) demonstrated in particular that Chang's conjecture is related to large cardinals and indiscernibles. Its exact consistency strength was later established by Donder [2] to be an ω_1-Erdős cardinal. This left open the possibility of whether a version of Chang's conjecture holds at or above a supercompact cardinal. The main contribution of this paper is the following.

3 Theorem. *$(\kappa^+,\kappa) \twoheadrightarrow (\aleph_1,\aleph_0)$ is independent of* ZFC *even if κ is a supercompact cardinal.*

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The proof uses a technique developed by Laver known as the *indestructibility theorem* for supercompact cardinals.[1] The key result from [9] reads as follows.

4 Theorem. *A supercompact cardinal κ can be made indestructible in the following sense: κ is supercompact in any generic extension $V^{\mathbb{P}}$ where $\mathbb{P}$ is a κ-directed-closed partial ordering.*

But we will argue later in this article that indestructibility under κ-directed-closed forcings is not enough for our purposes so we have to establish a result slightly more general than Theorem 4. It was shown in [7] that a supercompact cardinal is always destructible by λ-closed forcing for arbitrarily large λ, so we cannot hope to strengthen Laver's result by replacing the phrase 'κ-directed-closed' with 'λ-closed' even if λ is very large. The purpose of this note is to offer a strengthening of Theorem 4 that goes into a different direction.

5 Theorem. *A supercompact cardinal κ can be made indestructible in the following sense: κ is supercompact in any generic extension $V^{\mathbb{P}}$ where $\mathbb{P}$ is a partial ordering that is 'almost everywhere' κ-directed-closed.*

The exact meaning of 'almost everywhere' will be made precise in Definition 7. Once established, Theorem 5 will be applied to prove Theorem 3. In a generalization of this argument, we later go on to show that $(\lambda^+, \lambda) \twoheadrightarrow (\aleph_1, \aleph_0)$ can fail for all regular uncountable λ even in the presence of a supercompact cardinal.

2. Notation

The reader is assumed to have a strong background in set theory, especially regarding Easton extensions. As general references we recommend [5] and [8]. Let us make some remarks on the notations used in this paper.

We use an abbreviation in the context of elementary embeddings: $j : M \longrightarrow N$ means that j is a non-trivial elementary embedding from M into N such that M and N are transitive. The *critical point* of such an embedding, i.e. the first ordinal moved by j, is denoted by $\mathrm{cp}(j)$. We write jx for $j(x)$ in a context where too many parentheses might be confusing. An embedding $j : V \longrightarrow M$ is called *λ-supercompact* if $\kappa = \mathrm{cp}(j)$ is mapped above λ and M is closed under λ-sequences or equivalently if $j"\lambda \in M$. Remember that this is the same as saying that there is an ultrafilter on $[\lambda]^{<\kappa}$ that is *supercompact*, i.e. the set

[1] Recently, Apter and Hamkins have done a lot of work in this area and refined Laver's indestructibility theorem in various ways (see for example [4]).

$\{x \in [\lambda]^{<\kappa} : \alpha \in x\}$ is in the filter for every $\alpha < \lambda$ (fineness) and every regressive function on a filter set is constant on a filter set (normality). If κ is an infinite cardinal, then the poset $\mathbb{P}$ is called *κ-directed-closed* if any directed subset of size less than κ has a lower bound in $\mathbb{P}$ and called *κ-closed* if any $\mathbb{P}$-descending chain of length less than κ has a lower bound. $\mathbb{P}$ is *strategically κ-closed* if Player Nonempty has a winning strategy in the Banach-Mazur games of length less than κ. The following important theorem from [9] has become part of the set-theoretic folklore.

6 Theorem. *Let κ be supercompact. Then there is an $f : \kappa \longrightarrow H_\kappa$ such that for every x and every $\lambda \geq |\mathrm{TC}(x)|$ there is a λ-supercompact embedding $j : V \longrightarrow M$ such that $(jf)(\kappa) = x$.*

A function $f : \kappa \longrightarrow H_\kappa$ as in Theorem 6 is usually called *Laver function* or *Laver diamond.* It is a major tool in the proof of the Laver indestructibility theorem.

3. Revised Indestructibility

7 Definition. If $\mathbb{P}$ is a partial ordering then we always let $\theta = \theta_{\mathbb{P}}$ be the least regular cardinal such that $\mathbb{P} \in H_\theta$. Say that an $X \in [H_\theta]^{<\kappa}$ is $\mathbb{P}$-*complete* if every $(X, \mathbb{P})$-generic filter has a lower bound in $\mathbb{P}$. Define

$$\mathcal{H}(\mathbb{P}) = \{X \in [H_\theta]^{<\kappa} : X \text{ is } \mathbb{P}\text{-complete}\}.$$

Then a partial ordering $\mathbb{P}$ is called *almost κ-directed-closed* if $\mathbb{P}$ is strategically κ-closed and $\mathcal{H}(\mathbb{P})$ is in every supercompact ultrafilter on $[H_\theta]^{<\kappa}$.

Clearly, if a poset $\mathbb{P}$ is κ-directed-closed then it is almost κ-directed-closed. Thus, the following Theorem 9 is actually a bit stronger than the classical Laver indestructibility. We will present applications later in which this slight edge is crucial. Notice also that a closed unbounded subset of $[H_\theta]^{<\kappa}$ is in every supercompact ultrafilter on $[H_\theta]^{<\kappa}$. But there are more interesting examples:

8 Lemma. *Let $\nu < \kappa \leq \theta$ be regular cardinals. Then*

$$\mathrm{CF}(\geq \nu) = \{X \in [H_\theta]^{<\kappa} : \mathrm{cf}(\sup X \cap \kappa) \geq \nu\}$$

is in all supercompact ultrafilters on $[H_\theta]^{<\kappa}$. □

Lemma 8 will be exploited later in the applications. Now we prove the *revised indestructibility theorem.*

9 Theorem (Revised Laver indestructibility). *A supercompact cardinal κ can be made indestructible in the following sense: κ is supercompact in any generic extension $V^{\mathbb{P}}$ where $\mathbb{P}$ is almost κ-directed-closed.*

Proof. Of course, the proof is similar to the one given by Laver in [9]. To show that our slight variation works, we give the whole proof in detail. Let $f : \kappa \longrightarrow H_\kappa$ be as in Theorem 6. We construct an Easton support iteration $\mathbb{Q}_\kappa = (\mathbb{Q}_\alpha : \alpha < \kappa)$ of length κ. During this iteration, we inductively define a poset $\mathbb{Q}_\alpha$ and an ordinal λ_α. If γ is limit then $\mathbb{Q}_\gamma$ will be the Easton support limit of $(\mathbb{Q}_\alpha : \alpha < \gamma)$ and we define $\lambda_\gamma = \sup_{\alpha<\gamma} \lambda_\alpha$. In the successor step $\alpha + 1$ we pick a $\mathbb{Q}_\alpha$-name $\mathbb{P}$ for a partial ordering, where $\mathbb{P}$ is trivial except when

(i) $\lambda_\xi < \alpha$ for all $\xi < \alpha$,
(ii) $f(\alpha) = \langle \mathbb{P}, \lambda \rangle$, where $\mathbb{P}$ is a $\mathbb{Q}_\alpha$-name for a poset, and
(iii) $\Vdash_{\mathbb{Q}_\alpha}$ "$\mathbb{P}$ is almost α-directed-closed."

If (i)-(iii) are satisfied then we let $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha * \mathbb{P}$ and $\lambda_{\alpha+1} = \lambda$. We claim that this works. The rest of the proof is designed to show that κ is revised Laver indestructible in $V^{\mathbb{Q}_\kappa}$.

Now assume that $\mathbb{P}$ is a $\mathbb{Q}_\kappa$-name for a partial ordering that is almost κ-directed-closed, where θ is the least regular cardinal such that $\mathbb{P} \in H_\theta$. Remember that this means that $\mathbb{P}$ is strategically κ-closed and that $\mathcal{H}(\mathbb{P})$ is in every supercompact ultrafilter on $[H_\theta]^{<\kappa}$. Let $\gamma \geq \kappa$. We need to find a supercompact ultrafilter on $[\gamma]^{<\kappa}$ in $V^{\mathbb{Q}_\kappa * \mathbb{P}}$. To this end, let λ be a cardinal such that

- $\lambda > 2^\theta$ and
- $\Vdash_{\mathbb{Q}_\kappa * \mathbb{P}} \lambda > 2^{(\gamma^{<\kappa})}$.

Using the properties of the Laver function, pick in V a λ-supercompact embedding $j : V \longrightarrow M$ such that $(jf)(\kappa) = \langle \mathbb{P}, \lambda \rangle$. Notice that

$$j(\mathbb{Q}_\alpha : \alpha \leq \kappa) = (\mathbb{Q}_\alpha : \alpha \leq j\kappa).$$

Now conditions (i)-(iii) are satisfied in M when we replace κ for α and jf for f. Thus by elementarity,

$$\mathbb{Q}_{\kappa+1} = \mathbb{Q}_\kappa * \mathbb{P}$$

and note that $\mathbb{Q}_\delta$ is the trivial poset whenever $\kappa + 1 < \delta < \lambda$. So the final segment of the iteration $\mathbb{Q}_{j\kappa}$ after the $(\kappa + 1)$th step is by construction a strategically λ-closed forcing. But note that $j\mathbb{P}$ is also strategically λ-closed, so we can factor and get that

$$\mathbb{Q}_{j\kappa} * j\mathbb{P} = \mathbb{Q}_\kappa * \mathbb{P} * \mathbb{R}$$

is such that the factor $\mathbb{R}$ is strategically λ-closed. Now let $G \subseteq \mathbb{P}$ be generic over the model $V^{\mathbb{Q}_\kappa}$.

9.1 Claim. *j"G extends to a condition p_G in $j\mathbb{P}$.*

Proof. Notice that G is an $(H_\theta, \mathbb{P})$-generic filter, so by elementarity we have that j"G is a $(j\text{"}H_\theta, j\mathbb{P})$-generic filter. But

$$j\text{"}H_\theta \in j\mathcal{H}(\mathbb{P}) = \mathcal{H}(j\mathbb{P})$$

by the assumption that $\mathcal{H}(\mathbb{P})$ is in every supercompact ultrafilter on $[H_\theta]^{<\kappa}$. This implies that $j\text{"}H_\theta$ is $j\mathbb{P}$-complete and therefore j"G has a lower bound in $j\mathbb{P}$. □

We have found a master condition p_G for the final segment $\mathbb{R}$ of the iteration. This gives rise to the next claim:

9.2 Claim. *The embedding $j : V \longrightarrow M$ can be extended to*

$$j : V^{\mathbb{Q}_\kappa * \mathbb{P}} \longrightarrow M^{\mathbb{Q}_{j\kappa} * j\mathbb{P}}.$$

Proof. This is the classical extension lemma of Silver. We only need to see that if τ is a $\mathbb{Q}_\kappa * \mathbb{P}$-name then $j\tau$ becomes a $\mathbb{Q}_{j\kappa} * j\mathbb{P}$-name. □

The following is standard and we only sketch the argument: working in $V^{\mathbb{Q}_\kappa * \mathbb{P}}$ we construct a sequence $(r_\xi : \xi < \lambda)$ of $\mathbb{R}$-conditions below p_G such that for every $\mathcal{X} \subseteq [\gamma]^{<\kappa}$ in $V^{\mathbb{Q}_\kappa * \mathbb{P}}$ there is an r_ξ that decides if $j\text{"}\gamma$ is in $j\mathcal{X}$ or not. Similarly, we decide the statements that guarantee normality of the following filter:

$$\mathcal{U} = \{\mathcal{X} \subseteq [\gamma]^{<\kappa} : \exists \xi < \lambda \;\; r_\xi \Vdash j\text{"}\gamma \in j\mathcal{X}\}.$$

Then $\mathcal{U}$ is a supercompact ultrafilter on $[\gamma]^{<\kappa}$ in $V^{\mathbb{Q}_\kappa * \mathbb{P}}$. □

4. Regressive Kurepa-trees and transfer principles

Higher Kurepa-trees are natural counterexamples to model-theoretic transfer principles. For example, it is easy to check that the existence of an ω_2-Kurepa-tree negates the relation $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$. But in order to negate the principle $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_1, \aleph_0)$, we need a notion stronger than that of a regular ω_2-Kurepa-tree. The next definition is designed to serve this purpose.

10 Definition. For any tree T say that the level T_α is *non-stationary* if there is a function $f_\alpha : T_\alpha \longrightarrow T_{<\alpha}$ which is *regressive* in the sense that $f_\alpha(x) <_T x$ for all $x \in T_\alpha$ and if $x, y \in T_\alpha$ are distinct then $f_\alpha(x)$ or $f_\alpha(y)$ is strictly above the meet of x and y.

A λ-Kurepa-tree T is called *γ-regressive* if T_α is non-stationary for all limit ordinals $\alpha < \lambda$ with $\mathrm{cf}(\alpha) < \gamma$.

The notion of a regressive Kurepa-tree was first introduced in [7]. One can verify that an ω_1-regressive ω_2-Kurepa-tree is a counterexample to $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_1, \aleph_0)$. We actually prove the more general:

11 Lemma. *Let κ be regular and assume that there is an ω_1-regressive κ-Kurepa-tree T. Then $(\kappa^+, \kappa) \not\twoheadrightarrow (\aleph_1, \aleph_0)$.*

Proof. Let $\mathcal{B}$ be the set of cofinal branches of T and consider the structure $(\mathcal{B}, T)$ which is of type (κ^+, κ). Now assume towards a contradiction that $(\kappa^+, \kappa) \twoheadrightarrow (\aleph_1, \aleph_0)$ would hold, so we find a substructure

$$(\mathcal{A}, S) \prec (\mathcal{B}, T),$$

where $\mathcal{A}$ has size $\aleph_1$ and S has size $\aleph_0$. Define $\delta = \sup(\mathrm{ht}"S)$ and notice that $\mathrm{cf}(\delta) = \omega$. Hence T_δ is a non-stationary level of the tree T. A straightforward argument using the fact that there is a regressive 1-1 function defined on T_δ shows that $\mathcal{A}$ has size at most $|S| = \aleph_0$. This is a contradiction. $\square$

Let us give a quick summary of what is known about regressive Kurepa-trees. A result from [7] says that compact cardinals have considerable impact.

12 Theorem. *Assume that κ is a compact cardinal and $\lambda \geq \kappa$ is regular. Then there are no κ-regressive λ-Kurepa-trees.*

Our goal is to show that a supercompact cardinal κ is consistent with the existence of an ω_1-regressive κ-Kurepa-tree. This would be in contrast to Theorem 12. But another theorem from [7] indicates that we cannot succeed with the classical Laver indestructibility.

13 Theorem. *Under* MM*, there are no ω_1-regressive λ-Kurepa-trees for any uncountable regular λ.*

Indeed, if there were a κ-directed-closed partial ordering to add an ω_1-regressive κ-Kurepa-tree then such a forcing would preserve MM in particular. But that contradicts Theorem 13. So we had to develop 'revised Laver indestructibility' first in order to show the desired consistency. This motivates the construction in the following section.

5. An application

14 Lemma. *Let $\nu \leq \lambda$ where λ is regular. Then there is a λ-closed forcing $\mathcal{K}^\lambda_\nu$ that adds a ν-regressive λ-Kurepa-tree. Moreover, if κ is supercompact and $\nu < \kappa \leq \lambda$ then $\mathcal{K}^\lambda_\nu$ is almost κ-directed-closed.*

Proof. We describe the forcing $\mathcal{K}^\lambda_\nu$ and later show that it has the desired properties. One may assume the cardinal arithmetic $2^{<\lambda} = \lambda$, otherwise a preliminary Cohen-subset of λ could be added. Conditions of $\mathcal{K}^\lambda_\nu$ are pairs (T, h), where

(1) T is a tree of height $\alpha + 1$ for some $\alpha < \lambda$ and each level has size $< \lambda$.
(2) T is ν-regressive, i.e. if $\xi \leq \alpha$ is of cofinality less than ν then T_ξ is non-stationary over $T_{<\xi}$.
(3) $h : T_\alpha \longrightarrow \lambda^+$ is 1-1.

The condition (T, h) is stronger than (S, g) if

- $S = T \restriction \mathrm{ht}(S)$.
- $\mathrm{rng}(g) \subseteq \mathrm{rng}(h)$.
- $g^{-1}(\nu) \leq_T h^{-1}(\nu)$ for all $\nu \in \mathrm{rng}(g)$.

A generic filter G for $\mathcal{K}^\lambda_\nu$ will produce a ν-regressive λ-tree T_G in the first coordinate and the sets

$$b_\nu = \{x \in T_G : \text{there is } (T, h) \in G \text{ such that } h(x) = \nu\}$$

for $\nu < \lambda^+$ form a collection of λ^+-many mutually different λ-branches through the tree T_G. Notice also that the standard arguments for λ^+-cc go through here as we assumed $2^{<\lambda} = \lambda$.

It is shown in [7] that this forcing is λ-closed. So we are left with showing that if κ is supercompact and $\nu < \kappa \leq \lambda$ then $\mathcal{K}^\lambda_\nu$ is almost κ-directed-closed. By Lemma 8, it suffices to show that whenever the set $X \in [H_\theta]^{<\kappa}$ is such that $\mathrm{cf}(\sup X \cap \kappa) \geq \nu$ then X is $\mathbb{P}$-complete. So assume that X is like this and $K \subseteq \mathbb{P} \cap X$ be an $(X, \mathbb{P})$-generic filter. Define

$$T_K = \bigcup \{T : \text{there is } (T, h) \text{ in } K\}$$

and $\delta = \mathrm{ht}(T_K) = X \cap \kappa$. Then $\mathrm{cf}(\delta) \geq \nu$. Now extend T_K by defining the level T_δ such that every T_K-branch colored by the filter K has an extension in T_δ. Note that the only problem here could be (2) of the definition of $\mathcal{K}^\lambda_\nu$ but this condition does not apply to T_δ because $\mathrm{cf}(\delta) \geq \nu$. □

The supercompactness of κ is not really used in the construction of the forcing $\mathcal{K}^\lambda_\nu$ but we included it in the assumptions of Theorem 14 because the notion of 'almost κ-directed-closed', as we stated it, somehow depends on the supercompactness of κ. Using revised Laver indestructibility, we get an immediate consequence.

15 Theorem. *It is consistent with the supercompactness of κ that there is an ω_1-regressive κ-Kurepa-tree.*

Proof. By Theorem 9 and Lemma 14. □

Theorem 15 is curious in the light of Theorem 13: even though MM forbids the existence of ω_1-regressive Kurepa-trees on all uncountable regular cardinals, these trees are well consistent with a supercompact cardinal. Imagine the following scenario: assume that κ is supercompact and T an ω_1-regressive κ-Kurepa-tree. If we force MM over that model with the usual semiproper iteration [3], then T becomes an ω_2-Kurepa-tree in the extension which would have non-stationary levels at all ordinals that are ω-cofinal in the ground model. But note that the semiproper iteration to force MM will include Namba forcing at many stages of this iteration, so the resulting model of MM contains far more ω-cofinal ordinals than the ground model in which κ was still supercompact. This gives some indication as to why the ω_1-regressivity of T is suddenly impossible by Theorem 13. Regarding this phenomenon, see also [7].

Going back to the transfer principles, our main concern was the relation $(\kappa^+, \kappa) \twoheadrightarrow (\aleph_1, \aleph_0)$ when κ is supercompact. A positive consistency result is straightforward and well-known. Theorem 16 is probably folklore, but we sketch the proof for convenience.

16 Theorem. *Assume that κ is regular and $\theta > \kappa$ is measurable. Then*

$$V^{\mathrm{Coll}(\kappa,<\theta)} \models (\kappa^+, \kappa) \twoheadrightarrow (\aleph_1, \aleph_0).$$

Proof. Let $j : V \longrightarrow M$ be an embedding with critical point θ. Using standard arguments, this embedding can be extended to

$$j : V^{\mathrm{Coll}(\kappa,<\theta)} \longrightarrow M^{\mathrm{Coll}(\kappa,<j\theta)}.$$

It is now enough to show that, in $V^{\mathrm{Coll}(\kappa,<\theta)}$, every countable substructure N can be κ-*end-extended*, i.e. there is a proper elementary extension $\bar{N}$ such that $N \cap \kappa = \bar{N} \cap \kappa$. To see this, let f_i $(i < \omega)$ enumerate all functions from θ to κ that are in N. We may assume that N contains everything in sight, so we have for all $i < \omega$,

$$(jf_i)(\theta) \in N \cap \kappa.$$

By elementarity we can pick $\delta > \kappa$ such that $f_i(\delta) \in N \cap \kappa$ for all $i < \omega$. Now let $\bar{N}$ be the Skolem Hull of $N \cup \{\delta\}$. □

17 Corollary. *The supercompactness of κ is consistent with the model-theoretic transfer property $(\kappa^+, \kappa) \twoheadrightarrow (\aleph_1, \aleph_0)$.*

Proof. Let κ be a cardinal such that the supercompactness of κ is indestructible by κ-directed-closed forcing and such that $\theta > \kappa$ is measurable. Then κ remains supercompact in $V^{\mathrm{Coll}(\kappa,<\theta)}$. The corollary now follows from Theorem 16. □

The following is the reverse to Corollary 17 and yields a negative consistency result.

18 Corollary. *The supercompactness of κ is consistent with the model-theoretic transfer property $(\kappa^+, \kappa) \not\twoheadrightarrow (\aleph_1, \aleph_0)$.*

Proof. By Lemma 11 and Theorem 15. □

6. Global failure and GCH

We generalize the technique of the last section to show that, in the presence of a supercompact cardinal, the principle $(\lambda^+, \lambda) \twoheadrightarrow (\aleph_1, \aleph_0)$ can fail even for all regular uncountable λ.

19 Theorem. *It is consistent with the supercompactness of κ that*

(1) GCH *holds and*
(2) *there are ω_1-regressive λ-Kurepa-trees for all regular $\lambda \geq \aleph_1$.*

Proof. The reader is assumed to be familiar with the proof of Theorem 9. We only sketch the argument and content ourselves with pointing out the differences to the old construction. Start with a model in which κ is supercompact and GCH holds. Since we are only concerned with indestructibility for the forcings $\mathcal{K}^{\lambda}_{\omega_1}$, our 'Laver preparation' consists only of posets of the same form. So define the Easton support iteration $\mathbb{Q} = (\mathbb{Q}_\alpha : \alpha \in \mathrm{Ord})$ by letting $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha * \mathbb{P}$, where

- $\mathbb{P} = \mathcal{K}^{\alpha}_{\omega_1}$ if α is regular uncountable and
- $\mathbb{P}$ is trivial otherwise.

Using GCH and the standard Easton arguments, we know that all cardinals and cofinalities are preserved. The claim is that κ is still supercompact in $V^{\mathbb{Q}}$. Let $\gamma \geq \kappa$. Find a regular μ such that $\mathbb{Q}$ factors into $\mathbb{Q}_\mu * \mathbb{R}$, where the final segment $\mathbb{R}$ is $2^{(\gamma^{<\kappa})}$-closed and μ is much larger than γ. Now let $j : V \longrightarrow M$ be a $(2^\mu)^+$-supercompact embedding. Notice that $j(\mathbb{Q}_\mu) = \mathbb{Q}_{j\mu}$ by construction. We can factor

$$\mathbb{Q}_\mu = \mathbb{Q}_\kappa * \mathbb{Q}_{[\kappa+1,\mu]}.$$

If $H \subseteq \mathbb{Q}_\mu$ is generic, we let G be the restriction of H to the final segment $\mathbb{Q}_{[\kappa+1,\mu]}$. By the standard arguments, it is enough to show that j"G extends to a condition p_G in $j\mathbb{Q}_{[\kappa+1,\mu]} = \mathbb{Q}_{[j\kappa+1,j\mu]}$. This is similar to Claim 9.1 and using the fact that

$$\mathrm{CF}(\geq \omega_1) \subseteq \mathcal{H}(\mathbb{Q}_{[\kappa+1,\mu]}).$$

Note that $\mathbb{Q}_{[\mu+1,j\mu]}/G$ is the same as $\mathbb{Q}_{[\mu+1,j\mu]}$ below the condition p_G. Hence the closure properties of $\mathbb{Q}_{[\mu+1,j\mu]}$ help us decide the properties of the supercompact ultrafilter on $[\gamma]^{<\kappa}$ that lives in the model $V^{\mathbb{Q}_{j\mu}}$ (compare with Claim 9.2). We have shown that κ is γ-supercompact in $V^{\mathbb{Q}_\mu}$. Since the final segment $\mathbb{R}$ of the iteration is $2^{(\gamma^{<\kappa})}$-closed, it preserves γ-supercompactness and we are done. $\square$

20 Corollary. *It is consistent with the existence of a supercompact cardinal that $(\lambda^+, \lambda) \not\twoheadrightarrow (\aleph_1, \aleph_0)$ holds for all regular uncountable λ.*

Proof. By Lemma 11 and Theorem 19. $\square$

References

[1] C.C. Chang and Jerome Keisler. *Model theory.* North-Holland, 1990.

[2] Dieter Donder, Ronald Jensen, and Bernd Koppelberg. Some applications of the core model. In *Set theory and model theory*, volume 872 of *Lecture notes in Mathematics*, pages 55–97. Springer-Verlag, 1981.

[3] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's Maximum, saturated ideals, and nonregular ultrafilters I. *Annals of Mathematics*, 127:1–47, 1988.

[4] Joel Hamkins. The lottery preparation. *Annals of Pure and Applied Logic*, 101:103–146, 2000.

[5] Akihiro Kanamori. *The Higher Infinite.* Perspectives in Mathematical Logic. Springer-Verlag, 1997.

[6] Akihiro Kanamori and Menachem Magidor. The evolution of large cardinal axioms in set theory. In *Higher set theory*, volume 669 of *Lecture notes in Mathematics*, pages 99–275. Springer-Verlag, 1978.

[7] Bernhard König and Yasuo Yoshinobu. Kurepa-trees and Namba forcing. preprint, 2005.

[8] Kenneth Kunen. *Set theory. An introduction to independence proofs.* North-Holland, 1980.

[9] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel Journal of Mathematics*, 29:385–388, 1978.

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