OPTIMAL EIGENVALUES ESTIMATE FOR THE DIRAC OPERATOR ON DOMAINS WITH BOUNDARY

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Abstract. We give a lower bound for the eigenvalues of the Dirac operator on a compact domain of a Riemannian spin manifold under the MIT bag boundary condition. The limiting case is characterized by the existence of an imaginary Killing spinor.

1. Introduction

Let Ø be a compact domain in a n-dimensional Riemannian spin manifold (N^n, g) whose boundary is denoted by $\partial Ø$. In [HMR02], the authors studied four elliptic boundary conditions for the Dirac operator D of the domain Ø. More precisely, they prove a Friedrich-type inequality [Fri80] which relates the spectrum of the Dirac operator and the scalar curvature of the domain Ø. These boundary conditions are the following: the Atiyah-Patodi-Singer (APS) condition based on the spectral resolution of the boundary Dirac operator; a modified version of the APS condition, the mAPS condition; the boundary condition CHI associated with a chirality operator; and a Riemannian version of the MIT bag boundary condition. In fact, they show that, if the boundary $\partial Ø$ of Ø has non-negative mean curvature, then under the APS, CHI or mAPS boundary conditions, the spectrum of the classical Dirac operator of the domain Ø is a sequence of unbounded real numbers $\{\lambda_k : k \in \mathbb{Z}\}$ satisfying

$$\lambda_k^2 \geq \frac{n}{4(n-1)} \mathrm{R}_0, \tag{1}$$

where R_0 is the infimum of the scalar curvature of the domain Ø. Moreover, equality holds only for the CHI and the mAPS conditions and in these cases, Ø is respectively isometric to a half-sphere or it carries a non-trivial real Killing spinor and has minimal boundary. In the case of the MIT boundary condition, they show that the spectrum of the Dirac operator on Ø is an unbounded discrete set of complex numbers λ^{MIT} with positive imaginary part satisfying

$$|\lambda^{\mathrm{MIT}}|^2 > \frac{n}{4(n-1)} \mathrm{R}_0, \tag{2}$$

if the mean curvature of the boundary is non-negative. This result leads to the following question: can one improve this inequality in order to obtain some boundary geometric

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invariants on the right hand side of (2)? We show in this paper that such a result can be obtained. More precisely, we prove the following theorem:

Theorem 1. *Let $\emptyset$ be a compact domain of an n-dimensional Riemannian spin manifold (N^n, g) whose boundary $\partial\emptyset$ satisfies $\mathrm{H} > 0$. Under the* MIT *boundary condition $\mathbb{B}^-_{\mathrm{MIT}}$, the spectrum of the classical Dirac operator* D *on $\emptyset$ is an unbounded discrete set of complex numbers with positive imaginary part. Any eigenvalue λ^{MIT} satisfies*

$$|\lambda^{\mathrm{MIT}}|^2 \geq \frac{n}{4(n-1)}\,\mathrm{R}_0 + n\,\mathrm{Im}(\lambda^{\mathrm{MIT}})\,\mathrm{H}_0, \tag{3}$$

where H_0 is the infimum of the mean curvature of the boundary. Moreover, equality holds if and only if the associated eigenspinor is an imaginary Killing spinor on $\emptyset$ and if the boundary $\partial\emptyset$ is a totally umbilical hypersurface with constant mean curvature.

The proof of this theorem is based on a modification of the spinorial Levi-Civita connection which leads to a spinorial Reilly-type formula. This formula can be seen as a hyperbolic version of the Reilly inequality used in [HMR02].
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2. Geometric preliminaries

In this section, we give some standard facts about Riemannian spin manifolds with boundary. For more details, we refer to [BBW93] or [HMR02].
On a compact domain $\emptyset$ with smooth boundary $\partial\emptyset$ in a n-dimensional Riemannian spin manifold (N^n, g), denote by $\Sigma\emptyset$ the complex spinor bundle corresponding to the metric g and by ∇ its Levi-Civita connection acting on $\mathrm{T}\emptyset$ as well as its lift to $\Sigma\emptyset$. The map $\gamma : \mathbb{C}l\,(\emptyset) \longrightarrow \mathrm{End}(\Sigma\emptyset)$ is the Clifford multiplication where $\mathbb{C}l\,(\emptyset)$ is the Clifford bundle over $\emptyset$. The spinor bundle is endowed with a natural Hermitian scalar product, denoted by $\langle\, ,\, \rangle$, compatible with ∇ and γ. The Dirac operator is then the first order elliptic operator acting on sections of $\Sigma\emptyset$ locally given by

$$\begin{array}{rccl} \mathrm{D} : & \Gamma(\Sigma\emptyset) & \longrightarrow & \Gamma(\Sigma\emptyset) \\ & \psi & \longmapsto & \sum_{i=1}^n \gamma(e_i)\nabla_{e_i}\psi, \end{array}$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame of $\mathrm{T}\emptyset$.
Consider now the boundary $\partial\emptyset$ which is an oriented hypersurface of the domain $\emptyset$ with induced orientation and Riemannian structure. Since the normal bundle of $\partial\emptyset$ is trivial, the boundary itself is a spin manifold. This spin structure on the boundary allows to construct an intrinsic spinor bundle $\Sigma(\partial\emptyset)$ over $\partial\emptyset$ naturally endowed with a Hermitian metric, a Clifford multiplication $\gamma^{\partial\emptyset}$ and a spinorial Levi-Civita connection $\nabla^{\partial\emptyset}$. Moreover the restriction $\mathbf{S}(\partial\emptyset) := \Sigma\emptyset_{|\partial\emptyset}$ to the boundary of the spinor bundle $\Sigma\emptyset$ is a Dirac bundle, i.e. there exist on $\mathbf{S}(\partial\emptyset)$ a Hermitian metric denoted by $\langle\, ,\, \rangle$ compatible with the Levi-Civita connection $\nabla^{\mathbf{S}}$ and the Clifford multiplication $\gamma^{\mathbf{S}}$. The Clifford multiplication $\gamma^{\mathbf{S}} : \mathbb{C}l\,(\partial\emptyset) \longrightarrow \mathrm{End}(SS)$ is given by $\gamma^{\mathbf{S}}(X)\psi = \gamma(X)\gamma(\nu)\psi$ for all $X \in \Gamma(\mathrm{T}\emptyset)$ and $\psi \in \Gamma\,(SS)$. Similarly we can relate the Levi-Civita connection acting on $\Sigma\emptyset$ with that acting on $\mathbf{S}(\partial\emptyset)$ by the spinorial Gauss formula (see [Bär98]):

$$(\nabla_X\psi)_{|\partial\emptyset} = \nabla^{\mathbf{S}}_X\psi_{|\partial\emptyset} + \frac{1}{2}\gamma^{\mathbf{S}}(AX)\psi_{|\partial\emptyset},$$

for all $X \in \Gamma\big(\mathrm{T}(\partial\emptyset)\big)$, $\psi \in \Gamma(\Sigma\emptyset)$ and where $AX = -\nabla_X \nu$ is the shape operator of the boundary $\partial\emptyset$ with respect to the inner normal vector field ν. We can then define the boundary Dirac operator acting on $\mathbf{S}(\partial\emptyset)$ which is an elliptic first order differential operator locally given by

$$\mathrm{D}^{\mathbf{S}} = \sum_{j=1}^{n-1} \gamma^{\mathbf{S}}(e_j)\nabla^{\mathbf{S}}_{e_j}. \tag{4}$$

Recall that there is a standard identification

$$\mathbf{S}(\partial\emptyset) \equiv \begin{cases} \Sigma(\partial\emptyset) & \text{if n is odd} \\ \Sigma(\partial\emptyset) \oplus \Sigma(\partial\emptyset) & \text{if n is even} \end{cases}$$

Taking into account the relation between the Hermitian bundle $\mathbf{S}(\partial\emptyset)$ and $\Sigma(\partial\emptyset)$, one can see that

$$\nabla^{\mathbf{S}} \equiv \begin{cases} \nabla^{\partial\emptyset} & \text{if n is odd} \\ \nabla^{\partial\emptyset} \oplus \nabla^{\partial\emptyset} & \text{if n is even} \end{cases}$$

and

$$\gamma^{\mathbf{S}} \equiv \begin{cases} \gamma^{\partial\emptyset} & \text{if n is odd} \\ \gamma^{\partial\emptyset} \oplus -\gamma^{\partial\emptyset} & \text{if n is even} \end{cases}$$

3. The MIT boundary condition

First, note that on a closed compact Riemannian spin manifold, the classical Dirac operator has exactly one self-adjoint L^2 extention, so it has real discrete spectrum. In the setting of manifolds with boundary, a defect of self-adjointness appears. It is given by the Green formula

$$\int_{\emptyset} \langle \mathrm{D}\varphi, \psi\rangle dv(g) - \int_{\emptyset} \langle \varphi, \mathrm{D}\psi\rangle dv(g) = -\int_{\partial\emptyset} \langle \gamma(\nu)\varphi, \psi\rangle ds(g), \tag{5}$$

for all φ, $\psi \in \Gamma(\Sigma\emptyset)$. Furthermore, in this case, the Dirac operator has a closed range of finite codimension, but an infinite-dimensional kernel, which varies depending on the choice of the Sobolev space. We refer to [BBW93], [Lop53] or [HMR02] for a careful treatment of boundary conditions for elliptic operators.

The MIT bag boundary condition has first been introduced by physicists of the Massachusetts Institute of Technology in a Lorentzian setting (see [CJJ+74], [CJJT74] or [Joh75]). The Riemannian version of this condition has been studied in [HMR02] in order to get Friedrich estimates and in [HMZ02] because of its conformal covariance to give a conformal lower bound for the first eigenvalue of the intrinsic Dirac operator of hypersurfaces bounding a compact domain in a Riemannian spin manifold. Consider the pointwise endomorphism

$$i\gamma(\nu) : \Gamma(SS) \longrightarrow \Gamma(SS)$$

acting on the restriction to the boundary $\partial\emptyset$ of the spinor bundle over $\emptyset$ and where i is the fundamental imaginary number. This map is an involution, and so the bundle SS

splits into two eigensubbundles $V^{\pm}$ associated with the eigenvalues ± 1. We then have two associated orthogonal projections given by

$$\begin{array}{rccl} \mathbb{B}^{\pm}_{\mathrm{MIT}}: & Ł^2(SS) & \longrightarrow & Ł^2(V^{\pm}) \\ & \varphi & \longmapsto & \frac{1}{2}(\mathrm{Id} \pm i\gamma(\nu))\varphi. \end{array}$$

which define local elliptic boundary conditions for the Dirac operator D on the domain Ø. So under this boundary condition, the eigenvalue problem

$$\left\{ \begin{array}{ll} \mathrm{D}\varphi = \lambda^{\mathrm{MIT}}\varphi & \text{on } Ø \\ \mathbb{B}^{\pm}_{\mathrm{MIT}}\varphi = 0 & \text{along } \partial Ø \end{array} \right. \tag{6}$$

has a discrete spectrum with finite dimensional eigenspaces consisting of smooth spinor fields.

Remark 1. Under the MIT boundary condition $\mathbb{B}^{-}_{\mathrm{MIT}}$, the spectrum of the Dirac operator D is contained in the upper half complex plane $\{z \in \mathbb{C} \,/\, \mathrm{Im}(z) > 0\}$. Indeed, let λ^{MIT} be an eigenvalue of D under the MIT boundary condition and $\varphi \in \Gamma(\Sigma Ø)$ the associated spinor field, then taking $\psi = i\varphi$ in the Formula (5) leads to

$$2\,\mathrm{Im}(\lambda^{\mathrm{MIT}}) \int_{Ø} |\varphi|^2 dv(g) = \int_{\partial Ø} |\varphi|^2 ds(g) \tag{7}$$

Two possibilities can occur: we have either $\mathrm{Im}(\lambda^{\mathrm{MIT}}) > 0$ or $\mathrm{Im}(\lambda^{\mathrm{MIT}}) = 0$. If $\mathrm{Im}(\lambda^{\mathrm{MIT}}) = 0$, then the spinor field φ should vanish along the boundary $\partial Ø$ and by the unique continuation principle (see [BBW93]), it should be identically zero on the manifold Ø. This is impossible because the spinor φ is supposed to be an eigenspinor, so a non trivial field. The first case is the only possibility, i.e. $\mathrm{Im}(\lambda^{\mathrm{MIT}}) > 0$. For the boundary condition $\mathbb{B}^{+}_{\mathrm{MIT}}$, we can show that the imaginary part of all eigenvalues of the Dirac operator is negative.

4. The hyperbolic Reilly formula

In this section, we give a spinorial Reilly formula based on a modification of the spinorial Levi-Civita connection. Let $\alpha \in \mathbb{R}$, then we define the connection ∇^{α} acting on $\Sigma Ø$ by

$$\nabla^{\alpha}_X \varphi := \nabla_X \varphi + i\alpha\gamma(X)\varphi, \tag{8}$$

for all $\varphi \in \Gamma(\Sigma Ø)$ and $X \in \Gamma(\mathrm{T}Ø)$. We can now derive an integral version of the Schrödinger-Lichnerowicz formula using the modified connection ∇^{α}. Indeed, we have:

Proposition 2. *For all spinor fields* $\varphi \in \Gamma(\Sigma Ø)$*, we have:*

$$\langle (\nabla^{\alpha})^* \nabla^{\alpha} \varphi, \varphi \rangle_{Ł^2} = \langle \mathrm{D}^2 \varphi, \varphi \rangle_{Ł^2} - \langle \frac{\mathrm{R}}{4} \varphi, \varphi \rangle_{Ł^2} + n\alpha^2 ||\varphi||^2_{Ł^2} - \int_{\partial Ø} \langle \nabla^{\alpha}_{\nu} \varphi, \varphi \rangle ds(g), \tag{9}$$

where R *is the scalar curvature of the domain* Ø*.*

Proof: First note that the $Ł^2$-formal adjoint of the connection ∇^{α} is, by definition, given by

$$\langle (\nabla^{\alpha})^* \nabla^{\alpha} \varphi, \varphi \rangle_{Ł^2} = ||\nabla^{\alpha} \varphi||^2_{Ł^2} = \sum_{j=1}^{n} \int_{Ø} \langle \nabla^{\alpha}_{e_j} \varphi, \nabla^{\alpha}_{e_j} \varphi \rangle dv(g),$$

for all $\varphi \in \Gamma(\Sigma\text{Ø})$ and where $\{e_1, ..., e_n\}$ is a local orthonormal frame of TØ. An easy calculation using the compatibility properties of the Hermitian metric with the spinorial connection and the Clifford multiplication gives

$$\sum_{j=1}^{n}\langle\nabla^{\alpha}_{e_j}\varphi, \nabla^{\alpha}_{e_j}\varphi\rangle = \sum_{j=1}^{n}\Big(e_j\langle\nabla^{\alpha}_{e_j}\varphi, \varphi\rangle - \langle\nabla^{-\alpha}_{e_j}\nabla^{\alpha}_{e_j}\varphi, \varphi\rangle\Big),$$

and Stokes theorem leads to

$$\langle(\nabla^{\alpha})^{*}\nabla^{\alpha}\varphi, \varphi\rangle_{Ł^2} = \langle-\sum_{j=1}^{n}\nabla^{-\alpha}_{e_j}\nabla^{\alpha}_{e_j}\varphi, \varphi\rangle_{Ł^2} - \int_{\partial Ø}\langle\nabla^{\alpha}_{\nu}\varphi, \varphi\rangle ds(g).$$

We can now easily compute

$$\begin{aligned}\langle-\sum_{j=1}^{n}\nabla^{-\alpha}_{e_j}\nabla^{\alpha}_{e_j}\varphi, \varphi\rangle_{Ł^2} &= \langle-\sum_{j=1}^{n}\nabla_{e_j}\nabla_{e_j}\varphi, \varphi\rangle_{Ł^2} + n\alpha^2||\varphi||^2_{Ł^2}\\ &= \langle\nabla^{*}\nabla\varphi, \varphi\rangle_{Ł^2} + n\alpha^2||\varphi||^2_{Ł^2},\end{aligned}$$

and then the classical Schrödinger-Lichnerowicz formula (see [LM89]) leads to Identity (9). □

This formula is a first step to obtain Inequality (3). However, we have now to introduce the Dirac operator and the twistor operator associated with the connection ∇^{α}. The modified Dirac operator is locally defined by

$$\mathrm{D}^{\alpha}\varphi = \sum_{j=1}^{n}\gamma(e_j)\nabla^{\alpha}_{e_j}\varphi, \tag{10}$$

and the associated twistor operator by

$$\mathrm{P}^{\alpha}_{X}\varphi = \nabla^{\alpha}_{X}\varphi + \frac{1}{n}\gamma(X)\mathrm{D}^{\alpha}\varphi, \tag{11}$$

for all $X \in \Gamma(\text{TØ})$ and $\varphi \in \Gamma(\Sigma\text{Ø})$. Note that for $\alpha = 0$, the operators D^0 and P^0 are respectively the classical Dirac operator and the classical twistor operator which satisfy the relation (see [BHMM] or [Fri00] for example)

$$|\nabla\varphi|^2 = |\mathrm{P}\varphi|^2 + \frac{1}{n}|\mathrm{D}\varphi|^2$$

We can then check that the modified operators satisfy the same relation, i.e.

$$|\nabla^{\alpha}\varphi|^2 = |\mathrm{P}^{\alpha}\varphi|^2 + \frac{1}{n}|\mathrm{D}^{\alpha}\varphi|^2. \tag{12}$$

Indeed, if $\{e_1, ..., e_n\}$ is a local orthonormal frame of $\mathrm{T}\emptyset$, we have

$$\begin{aligned}|\mathrm{P}^\alpha\varphi|^2 &= \sum_{j=1}^{n}\langle\nabla^\alpha_{e_j}\varphi + \frac{1}{n}\gamma(e_j)\mathrm{D}^\alpha\varphi, \nabla^\alpha_{e_j}\varphi + \frac{1}{n}\gamma(e_j)\mathrm{D}^\alpha\varphi\rangle \\ &= |\nabla^\alpha\varphi|^2 - \frac{2}{n}|\mathrm{D}^\alpha\varphi|^2 + \frac{1}{n}|\mathrm{D}^\alpha\varphi|^2 \\ &= |\nabla^\alpha\varphi|^2 - \frac{1}{n}|\mathrm{D}^\alpha\varphi|^2,\end{aligned}$$

and so Identity (12) follows directly. We are now ready to establish the hyperbolic version of the spinorial Reilly formula given in [HMR02]. This formula can be seen as an analogous of the one used in [HMR03] to give a lower bound of the first eigenvalue of the intrinsic Dirac operator for hypersurfaces bounding a compact domain of a manifold with negative scalar curvature. More precisely, we prove:

Proposition 3. *For all $\varphi \in \Gamma(\Sigma\emptyset)$, we have:*

$$\begin{aligned}||\mathrm{P}^\alpha\varphi||^2_{\text{Ł}^2} &= \frac{n-1}{n}||\mathrm{D}^\alpha\varphi||_{\text{Ł}^2} - \langle\frac{\mathrm{R}}{4}\varphi, \varphi\rangle_{\text{Ł}^2} - n(n-1)\alpha^2||\varphi||^2_{\text{Ł}^2} \\ &+ \int_{\partial\emptyset}\langle\mathrm{D}^{\mathbf{S}}\varphi + \frac{n-1}{2}(2\alpha\, i\gamma(\nu)\varphi - \mathrm{H}\varphi), \varphi\rangle ds(g),\end{aligned} \tag{13}$$

where H *is the mean curvature of the boundary* $\partial\emptyset$ *of* $\emptyset$.

Proof: Observe first that the modified Dirac operator D^α is not formally self-adjoint. Indeed an easy calculation using (5) gives

$$\int_{\emptyset}\langle\mathrm{D}^\alpha\varphi, \psi\rangle dv(g) = \int_{\emptyset}\langle\varphi, \mathrm{D}^{-\alpha}\psi\rangle dv(g) - \int_{\partial\emptyset}\langle\gamma(\nu)\varphi, \psi\rangle ds(g), \tag{14}$$

for all φ, $\psi \in \Gamma(\Sigma\emptyset)$. However, we have:

$$\mathrm{D}^2\varphi = \mathrm{D}^{-\alpha}\mathrm{D}^\alpha\varphi - n^2\alpha^2\varphi,$$

and so substituting in Formula (9) gives

$$\langle(\nabla^\alpha)^*\nabla^\alpha\varphi, \varphi\rangle_{\text{Ł}^2} = \langle\mathrm{D}^{-\alpha}\mathrm{D}^\alpha\varphi, \varphi\rangle_{\text{Ł}^2} - \langle\frac{\mathrm{R}}{4}\varphi, \varphi\rangle_{\text{Ł}^2} - n(n-1)\alpha^2||\varphi||^2_{\text{Ł}^2} - \int_{\partial\emptyset}\langle\nabla^\alpha_\nu\varphi, \varphi\rangle ds(g).$$

The integration by parts formula (14) leads to

$$\begin{aligned}\langle(\nabla^\alpha)^*\nabla^\alpha\varphi, \varphi\rangle_{\text{Ł}^2} &= ||\mathrm{D}^\alpha\varphi||^2_{\text{Ł}^2} - \langle\frac{\mathrm{R}}{4}\varphi, \varphi\rangle_{\text{Ł}^2} - n(n-1)\alpha^2||\varphi||^2_{\text{Ł}^2} \\ &- \int_{\partial\emptyset}\langle\gamma(\nu)\mathrm{D}^\alpha\varphi + \nabla^\alpha_\nu\varphi, \varphi\rangle ds(g).\end{aligned}$$

With the help of Identity (12), we have

$$\begin{aligned}||\mathrm{P}^\alpha\varphi||^2_{\text{Ł}^2} &= \frac{n-1}{n}||\mathrm{D}^\alpha\varphi||_{\text{Ł}^2} - \langle\frac{\mathrm{R}}{4}\varphi, \varphi\rangle_{\text{Ł}^2} - n(n-1)\alpha^2||\varphi||^2_{\text{Ł}^2} \\ &- \int_{\partial\emptyset}\langle\gamma(\nu)\mathrm{D}^\alpha\varphi + \nabla^\alpha_\nu\varphi, \varphi\rangle ds(g).\end{aligned}$$

However the boundary term can be written

$$-\gamma(\nu)\mathrm{D}^{\alpha}\varphi-\nabla^{\alpha}_{\nu}\varphi=-\gamma(\nu)\mathrm{D}\varphi-\nabla_{\nu}\varphi+(n-1)\alpha\, i\gamma(\nu)\varphi,$$

and using the identity

$$-\gamma(\nu)\mathrm{D}\varphi-\nabla_{\nu}\varphi=\mathrm{D}^{\mathbf{S}}\varphi-\frac{n-1}{2}\mathrm{H}\varphi,$$

Formula (13) follows directly. □

We are now ready to prove Theorem 1.

5. The estimate

Proof of Theorem 1: Consider now a compact domain Ø of a Riemannian spin manifold such that the mean curvature H of the boundary satisfies $\mathrm{H}\geq 2\alpha$, for $\alpha>0$. By ellipticity of the MIT boundary condition $\mathbb{B}^{-}_{\mathrm{MIT}}$, consider a smooth spinor field $\varphi\in\Gamma(\Sigma Ø)$ solution of the eigenvalue boundary problem (6), i.e. φ satisfies

$$\begin{cases}\mathrm{D}\varphi=\lambda^{\mathrm{MIT}}\varphi & \text{on } Ø\\ \mathbb{B}^{-}_{\mathrm{MIT}}\varphi=0 & \text{along } \partial Ø\end{cases}\tag{15}$$

with $\mathrm{Im}(\lambda^{\mathrm{MIT}})>0$ by Remark 1. We now apply the hyperbolic Reilly formula (13) to the spinor field φ to get

$$\begin{aligned}||\mathrm{P}^{\alpha}\varphi||^2_{Ł^2} &= \left(\frac{n-1}{n}|\lambda^{\mathrm{MIT}}-n\alpha i|^2-n(n-1)\alpha^2\right)||\varphi||_{Ł^2}-\langle\frac{\mathrm{R}}{4}\varphi,\varphi\rangle_{Ł^2}\\ &\quad+\int_{\partial Ø}\langle\mathrm{D}^{\mathbf{S}}\varphi+\frac{n-1}{2}(2\alpha\, i\gamma(\nu)\varphi-\mathrm{H}\varphi),\varphi\rangle ds(g).\end{aligned}$$

Note that since $i\gamma(\nu)\varphi=\varphi$ along the boundary, we can compute

$$\langle\mathrm{D}^{\mathbf{S}}\varphi,\varphi\rangle=\langle\mathrm{D}^{\mathbf{S}}\varphi,i\gamma(\nu)\varphi\rangle=\langle i\gamma(\nu)\mathrm{D}^{\mathbf{S}}\varphi,\varphi\rangle=-\langle\mathrm{D}^{\mathbf{S}}\left(i\gamma(\nu)\varphi\right),\varphi\rangle=-\langle\mathrm{D}^{\mathbf{S}}\varphi,\varphi\rangle,$$

and so the preceding formula gives

$$\begin{aligned}&||\mathrm{P}^{\alpha}\varphi||^2_{Ł^2}+\frac{n-1}{2}\int_{\partial Ø}(\mathrm{H}-2\alpha)|\varphi|^2ds(g)=\\ &\qquad\frac{n-1}{n}\left(|\lambda^{\mathrm{MIT}}|^2-2n\,\alpha\,\mathrm{Im}(\lambda^{\mathrm{MIT}})\right)||\varphi||_{Ł^2}-\langle\frac{\mathrm{R}}{4}\varphi,\varphi\rangle_{Ł^2}\end{aligned}\tag{16}$$

The assumption on the mean curvature gives:

$$|\lambda^{\mathrm{MIT}}|^2-2n\,\alpha\,\mathrm{Im}(\lambda^{\mathrm{MIT}})\geq\frac{n}{4(n-1)}\,\mathrm{R}_0.$$

For $\alpha_0=\frac{1}{2}\,\mathrm{H}_0$, where $\mathrm{H}_0=\inf_{\partial Ø}(\mathrm{H})$, we get Inequality (16). Suppose now that equality is achieved, thus

$$||\mathrm{P}^{\alpha_0}\varphi||^2_{Ł^2}=0\quad\text{and}\quad\frac{n-1}{2}\int_{\partial Ø}(\mathrm{H}-2\alpha_0)|\varphi|^2ds(g)=0.$$

Moreover the spinor field φ is a solution of (15), so it satisfies the Killing equation

$$\nabla_X\varphi=-\frac{\lambda^{\mathrm{MIT}}}{n}\gamma(X)\varphi,\qquad\text{for all }\ X\in\Gamma(\mathrm{T}Ø).$$

Since such a spinor field has no zeroes (see [Fri00]), the mean curvature of the boundary is constant with $\mathrm{H} = 2\alpha_0$. Furthermore, it is a well-known result [BFGK90] that, in this case, the eigenvalue λ^{MIT} has to be either real or purely imaginary. Here we have $\mathrm{Im}(\lambda^{\mathrm{MIT}}) > 0$, then $\lambda^{\mathrm{MIT}} \in i\mathbb{R}^+_*$. The domain $\varnothing$ is in particular an Einstein manifold. We now show that the boundary has to be totally umbilical. Indeed, note that we have for all $X \in \Gamma(\mathrm{T}(\partial\varnothing))$:

$$\begin{aligned}\nabla_X(i\gamma(\nu)\varphi) &= i\gamma(\nabla_X\nu)\varphi + i\gamma(\nu)\nabla_X\varphi \\ &= i\gamma(\nabla_X\nu)\varphi + \alpha_0\gamma(\nu)\gamma(X)\varphi \\ &= i\gamma(\nabla_X\nu)\varphi - \alpha_0\gamma(X)\gamma(\nu)\varphi \\ &= i\gamma(\nabla_X\nu)\varphi + i\alpha_0\gamma(X)\varphi.\end{aligned}$$

However along the boundary we have $i\gamma(\nu)\varphi = \varphi$, so we obtain

$$\gamma(\nabla_X\nu)\varphi = -2\alpha_0\gamma(X)\varphi.$$

Since the spinor field φ has no zeros, we have $\mathrm{A}(X) = -\nabla_X\nu = 2\alpha X$ and the boundary is totally umbilical.We can again show that in the equality case, we have $\mathrm{Im}(\lambda^{\mathrm{MIT}}) = n\alpha_0$. In fact, just note that the boundary term can be rewritten as

$$\int_{\partial\varnothing}\langle\mathrm{D}^{\mathbf{S}}\varphi - \frac{n-1}{2}\mathrm{H}\varphi + (n-1)\alpha_0\varphi, \varphi\rangle ds(g) = -\int_{\partial\varnothing}\langle\nabla_\nu\varphi + \gamma(\nu)\mathrm{D}\varphi - (n-1)\alpha_0\varphi, \varphi\rangle ds(g).$$

This term is zero since we have equality in (16). Now using that the spinor field φ is an imaginary Killing spinor satisfying (6) gives

$$\nabla_\nu\varphi + \gamma(\nu)\mathrm{D}\varphi = \frac{n-1}{n}\mathrm{Im}(\lambda^{\mathrm{MIT}})\varphi.$$

Substituting in the preceding identity gives

$$(n-1)\int_{\partial\varnothing}(\alpha_0 - \frac{\mathrm{Im}(\lambda^{\mathrm{MIT}})}{n})|\varphi|^2 ds(g) = 0,$$

and since φ has no zeroes, $\mathrm{Im}(\lambda^{\mathrm{MIT}}) = n\alpha_0 = \frac{n\mathrm{H}_0}{2}$. □

Remark 2.

(1) The orthogonal projection $\mathbb{B}^+_{\mathrm{MIT}}$ defines a local elliptic boundary condition for the Dirac operator D of $\varnothing$. We can easily check that in this case, the imaginary part of an eigenvalue λ^{MIT} of D satisfies $\mathrm{Im}(\lambda^{\mathrm{MIT}}) < 0$. Inequality (3) is then given by

$$|\lambda^{\mathrm{MIT}}|^2 \geq \frac{n}{4(n-1)}\,\mathrm{R}_0 - n\,\mathrm{Im}(\lambda^{\mathrm{MIT}})\,\mathrm{H}_0.$$

(2) For $\mathrm{H}_0 = 0$, we obtain Inequality (2). In fact, if we suppose that equality is achieved, Theorem 1 implies $\mathrm{Im}(\lambda^{\mathrm{MIT}}) = \frac{n\mathrm{H}_0}{2} = 0$ which is impossible by Remark 1.

(3) Note that the Riemannian spin manifolds with an imaginary Killing spinor with Killing number $i\alpha$ have been classified by H. Baum in [Bau89a] and [Bau89b]. Such manifolds are called pseudo-hyperbolic and they are given by

$$(\mathbb{R}\times_{\exp}\mathrm{M}_0, g) = (\mathbb{R}\times\mathrm{M}_0, dt^2 \oplus e^{-4\alpha t}g_{\mathrm{M}_0}),$$

where $(\mathrm{M}_0, g_{\mathrm{M}_0})$ is a complete Riemannian spin manifold carrying a non-trivial parallel spinor. After suitable rescaling of the metric, we can assume that the Killing number is either $i/2$ or $-i/2$, i.e. we have

$$\nabla_X \phi = \pm \frac{i}{2}\gamma(X)\phi.$$

Moreover, constant mean curvature hypersurfaces in pseudo-hyperbolic manifolds are classified by the Hyperbolic Alexandrov Theorem proved in [Mon99] (see also [HMR03] for a proof using spinors). Indeed, such a hypersurface is either a round geodesic hypersphere (and, in this case, M_0 is flat and $\mathrm{H} > 1$) or a slice $\{s\} \times \mathrm{M}_0$ (and, in this case, M_0 is compact and $\mathrm{H} = 1$).

We can then prove the following corollary:

Corollary 4. *If the boundary of the compact domain $\emptyset$ is connected, there is no manifold satisfying the equality case in Inequality (3).*

Proof: If $\emptyset$ is a compact domain whith connected boundary achieving equality in (3), then there exists an imaginary Killing spinor on $\emptyset$ and the boundary $\partial\emptyset$ is a totally umbilical constant mean curvature hypersurface with $\mathrm{H} = 2\alpha$. However, using Remark (2).3, $\emptyset$ is a domain in a pseudo-hyperbolic space whose connected boundary is a slice $\{s\} \times \mathrm{M}_0$ and then $\emptyset$ is non-compact. $\square$

Remark 3. With a slight modification of the boundary condition, we give a domain $\emptyset$ whose boundary has two connected components carrying an imaginary Killing spinor field $\varphi \in \Gamma(\Sigma\emptyset)$ which satisfy

$$i\gamma(\nu_1)\varphi_{|\partial\emptyset_1} = \varphi_{|\partial\emptyset_1} \qquad \text{and} \qquad i\gamma(\nu_2)\varphi_{|\partial\emptyset_2} = -\varphi_{|\partial\emptyset_2}, \tag{17}$$

where ν_1 (resp. ν_2) is an inner unit vector field normal to $\partial\emptyset_1$ (resp. $\partial\emptyset_2$). First recall that one distinguishes two types of imaginary Killing spinors (see [Bau89a] and [Bau89b]). Indeed, if $\varphi \in \Gamma(\Sigma\emptyset)$ is an imaginary Killing spinor, denote by f its length function, then the function

$$q_\varphi(x) := f(x)^2 - \frac{1}{4\alpha^2}||\nabla f||^2$$

satisfies q_φ is constant and $q_\varphi \geq 0$. If $q_\varphi = 0$, φ is a Killing spinor of type I whereas if $q_\varphi > 0$, φ is a Killing spinor of type II. If (N^n, g) is a complete connected Riemannian spin manifold with an imaginary Killing spinor of type II associated with the Killing number $i\alpha$, then (N^n, g) is isometric to the hyperbolic space $\mathbb{H}^n_{-4\alpha^2}$. If (N^n, g) admits an imaginary Killing spinor of type I, then (N^n, g) is isometric to the warped product $(\mathbb{R} \times \mathrm{M}_0, dt^2 \oplus e^{-4\alpha t} g_{\mathrm{M}_0})$, where M_0 is a complete Riemannian spin manifold with a non-trivial parallel spinor field. Moreover, $q_\varphi = 0$ if and only if there exists a unit vector field ξ on N such that $\gamma(\xi)\varphi = i\varphi$. In fact, we can easily prove that the vector field ξ is the normal field of $\{t\} \times \mathrm{M}_0$ for all $t \in \mathbb{R}$. So consider the domain given by the warped product $\emptyset := ([a, b] \times \mathrm{M}_0, dt^2 \oplus e^{-4\alpha t} g_{\mathrm{M}_0})$, where M_0 is a compact spin manifold carrying a non-trivial parallel spinor field and with $-\infty < a < b < +\infty$. The domain $\emptyset$ carries an imaginary Killing spinor φ of type I, so there exists ξ normal to $\{t\} \times \mathrm{M}_0$ for all $t \in [a, b]$

such that $\gamma(\xi)\varphi = i\varphi$. The boundary of $\emptyset$ has two connected components which are slices $\{a\} \times \mathrm{M}_0$ and $\{b\} \times \mathrm{M}_0$ of $\emptyset$ and with mean curvature $\mathrm{H}_a = \mathrm{H}_b = 2\alpha$, where H_t is the mean curvature of a slice $\{t\} \times \mathrm{M}_0$. The spinor field φ clearly satisfies the boundary conditions (17).

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