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Jets, frames, and their Cartan geometry

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Abstract

Jet frames, that is a generalisation of ordinary frames on a manifold, are described in a language similar to that of gauge theory. This is achieved by constructing the Cartan geometry of a manifold with respect to the diffeomorphism symmetry. This point of vue allows to give new insights and interpretations in the theory of jet frames, in particular by making an interpolation between ordinary gauge theory concepts and pure jet ones.

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Introduction

A description of jet theory, and more precisely that of jet frames, described e.g. in [5] or [9], is proposed on the basis of Cartan type geometry : the geometry associated to a differentiable manifold M formally represented as the homogeneous space

$$M \simeq \mathrm{Diff}(M)/\mathrm{Diff}_x(M)$$

where $\mathrm{Diff}_x(M)$ are the diffeomorphisms that don't move a point $x \in M$, is constructed.

The interest of such a construction is that it realises a intermediate between the pure jet language [9] and the pure gauge theory language (principal fiber bundles). This gives an alternative description, in global terms, of the differential sequences given in [9], a gravity interpretation of the objects introduced, all being synthetised in some field theory of frames.

The first section, needed for both technical and notational purposes, is a short review and reformulation of the algebraic machinery exposed in [5], and alternatively in [3] and [1] in a closely related context.

The second section begins by recalling what are the jet frames of [5], or, as we shall see of [9]. We then describe an alternative viewpoint on the subject, based on a procedure of prolongation similar to that of [5] or [3], but here adapted to the infinite dimensional geometry of $\mathrm{Diff}(M)$. It allows to construct the so-called linear frames, of arbitrary order, the first order frames being the usual ones. See [7] for an example of the use of Cartan connection, i.e. the dual version of 2-frames and 3-frames there, in gravity.

The third section presents a field theory like treatment of the objects thus constructed. It is shown how to recover, in a simplified manner, the differential operators and sequences of [9], and a concrete description is given, in terms of symmetry, and deformations.

Contents

1 Algebraic preliminaries

Two functions $\phi, \phi' : \mathbb{R}^n \to \mathbb{R}^m$ are said equivalent to order k at $x \in \mathbb{R}^n$ if they have the same derivatives at x up to order k. The equivalence class is called a k-jet, and denoted $j^k_x(\phi)$.

1.1 Formal vector fields, Jet groups

• On $\mathbb{R}^n$ with coordinates x^a, $a = 1, \cdots, n$, the formal vector fields are the ($\partial_a = \frac{\partial}{\partial x^a}$ and sum on repeted index)

$$X = \sum_{k \geq -1} X_k \ \text{ with } \ X_k = \frac{1}{k!} X^a{}_{b_1 \cdots b_{k+1}} x^{b_1} \cdots x^{b_{k+1}} \partial_a$$

equiped with minus the ordinary Lie bracket of vector fields (the minus is taken by analogy with a group acting on one of its homogeneous space, see [5]). This defines a graded Lie algebra

$$\mathfrak{gl}_\infty = \bigoplus_{k \geq -1} \mathfrak{gl}_k \ \text{ with } \ [\mathfrak{gl}_k, \mathfrak{gl}_{k'}] \subset \mathfrak{gl}_{k+k'}$$

where $\mathfrak{gl}_k$ is the space of X_k's. The k's are "spins" with respect to the dilatation operator

$$[X_k, D] = k X_k, \ D = x^a \partial_a$$

• The jet group GL^k of order k is the space of $(k+1)$-jets of (orientation preserving) local diffeomorphisms g of $\mathbb{R}^n$ such that $g(0) = 0$. Denoting by $g^k = j^{k+1}_0(g)$ its elements, the group law is (formal successive derivations)

$$g^k g'^k = j^{k+1}_0(g \circ g')$$

By restrictions on the order of jets, we obtain projections $GL^k \to GL^{k-1}$ whose kernel GL_k is normal and abelian in GL^k, and we have

$$GL^k / GL_k \simeq GL^{k-1}, \ GL^k \simeq GL^{k-1} \ltimes GL_k$$

Recursively, the projections $GL^k \to GL^{k-1} \to \cdots \to GL^0 = GL_0$ induce the decomposition (factorisation of jets)

$$GL^k = GL^{k-1} \ltimes GL_k = (GL^{k-2} \ltimes GL_{k-1}) \ltimes GL_k = \cdots$$

and we shall denote this $GL^k = GL_0 \ltimes GL_1 \ltimes \cdots \ltimes GL_k$, in correspondance with the decomposition $g^k = g_0 g_1 \cdots g_k$.
Alternatively, letting H^k be the ∞-jets such that $j^{k+1}_0(g) = j^{k+1}_0(\mathrm{id})$, we obtain a normal

subgroup of GL^∞ which identifies $GL^k \simeq GL^\infty/H^k$. So, infinitesimally, we obtain the Lie algebra isomorphisms

$$\mathrm{Lie}H_k = \bigoplus_{l\geq k+1} \mathfrak{gl}_l,\ \mathrm{Lie}G_k = \bigoplus_{l\geq 0} \mathfrak{gl}_l / \bigoplus_{l\geq k+1} \mathfrak{gl}_l \simeq \bigoplus_{l\geq 0}^{k} \mathfrak{gl}_l$$

So, the product in GL^k is the truncation to $(k+1)$-jets of the product in GL^∞.

1.2 The jet action $\overline{\mathrm{Ad}}$

For $X \in \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$ written $X = \frac{d}{dt}_{|t=0} j_0^{k+1}(\phi_t)$ where $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ for each t on the path $t \to \phi_t$, $\phi_0 = \mathrm{id}$, and $g^{k+1} = j_0^{k+2}(g)$, $g(0) = 0$, define :

$$\overline{\mathrm{Ad}}(g^{k+1})X = \frac{d}{dt}_{|t=0} j_0^{k+1}(g \circ \phi_t \circ g^{-1}) \tag{1}$$

This is well defined since the result only depends on the $(k+2)$-jet of g. This is an action of GL^{k+1} on $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$. In particular $\overline{\mathrm{Ad}}(g_{k+1})$, $g_{k+1} \in GL_{k+1}$ is an isomorphism of degree k of $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$:

$$\overline{\mathrm{Ad}}(g_{k+1})\,(X_{-1} \oplus \cdots \oplus X_k) = X_{-1} \oplus \cdots \oplus X_{k-1} \oplus X_k + \alpha_k(X_{-1}) \tag{2}$$

where $\alpha_k \in \mathfrak{gl}_{k+1} \subset \mathfrak{gl}_k \otimes \mathfrak{gl}_{-1}^*$ thanks to $GL_{k+1} \simeq \mathfrak{gl}_{k+1}$, $k \geq 0$. We denote by $GL_{k,1}$ the group of degree k isomorphisms of $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$, then $GL_{k,1} \simeq \mathfrak{gl}_k \otimes \mathfrak{gl}_{-1}^*$, its action being given by the same formula (2). Finally, we obtain in this way an action of $GL^k \ltimes GL_{k,1}$ on $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$, which extends $\overline{\mathrm{Ad}}$, and still denoted $\overline{\mathrm{Ad}}$.

1.3 Spencer cohomology

Spencer cohomology [1] is the cohomology of the abelian Lie algebra of translations $\mathfrak{gl}_{-1}$ with values in $\mathfrak{gl}_\infty$, so Spencer cochains are $\mathfrak{gl}_\infty \otimes \Lambda^* \mathfrak{gl}_{-1}^*$. This space decomposes into a direct sum of the $\mathfrak{gl}_{k,l} = \mathfrak{gl}_k \otimes \Lambda^l \mathfrak{gl}_{-1}^*$. For a cochain α of form degree l, the coboundary operator is

$$\partial\alpha = \sum_{i=0}^{l} (-1)^i [X_i, \alpha(X_0, \cdots, \hat{X}_i, \cdots, X_l)],\ \ X_i \in \mathfrak{gl}_{-1},\ \ \partial^2 = 0 \tag{3}$$

where $\hat{}$ here denotes omission. In particular $\mathfrak{gl}_{k+1}$ appears as the kernel of $\mathfrak{gl}_{k,-1} \xrightarrow{\partial} \mathfrak{gl}_{k-1,2}$. More generally, Spencer ∂-cohomology is trivial [9], and so the particular sequences (for each k)

$$0 \longrightarrow \mathfrak{gl}_{k+1} \longrightarrow \mathfrak{gl}_{k,1} \xrightarrow{\partial} \mathfrak{gl}_{k-1,2} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathfrak{gl}_{k-n+1,n} \longrightarrow 0 \tag{4}$$

are exacts.
The $\overline{\mathrm{Ad}}$ action of GL^{k+1} on $\mathfrak{gl}_{-1} \oplus \cdots \mathfrak{gl}_k$ induces an action on Spencer cochains, that we still denote $\overline{\mathrm{Ad}}$, and given by, for $\alpha = \alpha_{-1} \oplus \cdots \oplus \alpha_k \in \mathfrak{gl}_{-1,l} \oplus \cdots \oplus \mathfrak{gl}_{k,l}$:

$$\overline{\mathrm{Ad}}(g)\alpha = \overline{\mathrm{Ad}}(g) \circ \alpha \circ \overline{\mathrm{Ad}}({g_0}^{-1}),\ g \in GL^{k+1},\ g = g_0.g_1.\cdots.g_{k+1}$$

1.4 Notations

For G a Lie group, a G-principal bundle P above the base space M will be denoted by

$$G \longrightarrow P \longrightarrow M$$

We shall think of this as a non linear version of a short exact sequence. For $g \in G$, the right action on $p \in P$ is denoted $R_g(p) = p.g$, and the vertical vector field on P induced by $X \in \mathrm{Lie}G$ is denoted $\hat{X}$.
The associated bundle E defined by a left action ρ of G on the space V will be denoted

$$E = P \times_\rho V$$

and its space of sections $\Gamma(E)$. The space of l-forms on M with values in the bundle E is denoted $\Omega^l(M, E)$, and the space of tensorial forms on P with values in V is denoted $\Omega^l_G(P, V)$. These two spaces are isomorphic.

2 Geometry of frames

Fix now an n-dimensional differentiable (and orientable) manifold M.

2.1 Jet frames

A $(k+1)$-jet frame above $x \in M$ is the $(k+1)$-jet at 0 of a (orientation preserving) local diffeomorphism $\phi : \mathbb{R}^n \to M$ such that $\phi(0) = x$. We shall denote this $e^k = j_0^{k+1}(\phi)$, and M^k the space of e^k's. The projection

$$\pi_{k,-1} : M^k \to M,\ e^k \mapsto x$$

where $e^k = j_0^{k+1}(\phi)$, $x = j_0^0(\phi) = \phi(0)$, and right action

$$M^k \times GL^k \to M^k,\ (e^k, g^k) \mapsto R_{g^k}(e^k) = e^k.g^k = j_0^{k+1}(\phi \circ g)$$

where $e^k = j_0^{k+1}(\phi)$, $g^k = j^{k+1}(g)$ with $g(0) = 0$, turns M^k into GL^k-principal bundle above M :

$$GL^k \longrightarrow M^k \longrightarrow M \tag{5}$$

More generally, for $k' < k$, the projection

$$\pi_{k,k'} : M^k \to M^{k'},\ e^k \mapsto e^{k'}$$

where $e^{k'} = j^{k'+1}(\phi)$, and right action

$$M^k \times GL_{k'+1} \ltimes \cdots \ltimes GL_k \to M^k,\ (e^k, g^{k'k}) \mapsto R_{g^{k'k}}(e^k) = e^k.g^{kk'} = j_0^{k+1}(\phi \circ g)$$

where $g^{k'k} = j_0^{k+1}(g)$ with $j_0^{k'+1}(g) = j_0^{k'+1}(\mathrm{id})$, defines on M^k the structure of a $GL_{k'+1} \ltimes \cdots \ltimes GL_k$-principal bundle above $M^{k'}$:

$$GL_{k'+1} \ltimes \cdots \ltimes GL_k \longrightarrow M^k \longrightarrow M^{k'} \tag{6}$$

We obtain in this way a tower of principal bundles :

$$M^k \longrightarrow M^{k-1} \longrightarrow \cdots \longrightarrow M^0 \longrightarrow M \tag{7}$$

Alternatively, since $GL_{k'+1} \ltimes \cdots \ltimes GL_k$ is a normal subgroup of GL^k, we have an induced principal structure on the quotient $M^k/GL_{k'+1} \ltimes \cdots \ltimes GL_k$ and this is isomorphic with $M^{k'}$. See e.g. [5] for a coordinate description of these bundles.

2.2 Interpretation : Induced linear frames

Let $k \geq -1$. Denoting by $\mathbb{R}^{n,k}$ the $(k+1)$-jet frames bundle of $\mathbb{R}^n$, and $O = j_0^{k+1}(\mathrm{id})$, we obtain the natural isomorphy :

$$T_O\mathbb{R}^{n,k} \simeq \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$$

because each $X = X_{-1} \oplus \cdots \oplus X_k \in \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$ can be written $X = \frac{d}{dt}_{|t=0} j_0^{k+1}(\phi_t)$. A $(k+2)$-jet frame $e^{k+1} = j_0^{k+2}(\phi)$ induces a locally defined isomorphism

$$\overline{\phi}_{k+1} : \mathbb{R}^{n,k} \to M^k,\ j_0^{k+1}(f) \mapsto j_0^{k+1}(\phi \circ f)$$

whose derivative $\overline{\phi}_{k+1*}$ at O only depends on $j_0^{k+2}(\phi) = e^{k+1}$. So, to each e^{k+1}, we can associate the isomorphism

$$e_{k+1} = \overline{\phi}_{k+1*|O} : \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k \to T_{e^k}M^k$$

We call the e_{k+1}'s linear frames (of order $k+2$). The definition of projections $\pi_{k,k-1}$, and (infinitesimal) right action of $M^k \to M$, show successively that e_{k+1} satisfies :

$$(i) \quad \pi_{k,k-1*}e_{k+1}\left(X_{-1} \oplus \cdots \oplus X_k\right) = e_k\left(X_{-1} \oplus \cdots \oplus X_{k-1}\right)$$
$$(ii) \quad e_{k+1}\left(X_0 \oplus \cdots \oplus X_k\right) = \hat{X}_0 \oplus \cdots \oplus \hat{X}_k$$

The properties (i) and (ii) above means respectively the right and left squares in the following diagram commute :

$$\begin{array}{ccccc}
\mathfrak{gl}_0 \oplus \cdots \mathfrak{gl}_k & \longrightarrow & \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k & \longrightarrow & \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_{k-1} \\
\downarrow \hat{} & & \downarrow e_{k+1} & & \downarrow e_k \\
T_0 M^k & \longrightarrow & T_{e^k} M^k & \longrightarrow & T_{e^{k-1}} M^{k-1}
\end{array}$$

where $T_0 M^k$ is the vertical tangent space of $M^k \to M$. Under the action of $g^{k+1} \in GL^{k+1}$, $g^{k+1} = j_0^{k+2}(g)$, ϕ_{k+1} becomes ϕ'_{k+1} with :

$$\begin{aligned}
\phi'_{k+1}(j_0^{k+1}(f)) &= j_0^{k+1}(\phi \circ g \circ f) = j_0^{k+1}(\phi \circ g \circ f \circ g^{-1} \circ g) \\
&= \phi_{k+1}(j_0^{k+1}(g \circ f \circ g^{-1})).g^k = (R_{g^k} \circ \phi_{k+1})(j_0^{k+1}(g \circ f \circ g^{-1}))
\end{aligned}$$

so, by derivation at O, we obtain the transformation of e_{k+1} :

$$e'_{k+1} = R_{g^k *} e_{k+1} \circ \overline{\mathrm{Ad}}(g^{k+1}) \tag{8}$$

2.3 Frame forms and their Structure equations

2.3.1 Frame form

On M^{k+1}, let u be a tangent vector at $e^{k+1} = j_0^{k+2}(\phi)$,

$$u = \frac{d}{dt}_{|t=0} j_0^{k+2}(\phi_t) \in T_{e^{k+1}} M^{k+1}$$

where $t \to \phi_t$ a path such that $\phi_0 = \phi$. From the jet point of vue, we define the frame form θ^k as

$$\theta^k(u) = \frac{d}{dt}_{|t=0} j_0^{k+1}(\phi^{-1} \circ \phi_t) \tag{9}$$

From the linear frame point of vue, the frame form is defined as

$$\theta^k(u) = {e_{k+1}}^{-1} \pi_{k+1,k*} u \tag{10}$$

where e_{k+1} is the linear frame induced by e^{k+1}. Formulas (9) and (10) agree since

$$\begin{aligned}
{e_{k+1}}^{-1} \pi_{k+1,k*} u &= {\overline{\phi}_{k+1}}^{-1}{}_* \frac{d}{dt}_{|t=0} j_0^{k+1}(\phi_t) = \frac{d}{dt}_{|t=0} \left({\overline{\phi}_{k+1}}^{-1} (j_0^{k+1}(\phi_t)) \right) \\
&= \frac{d}{dt}_{|t=0} j_0^{k+1}(\phi^{-1} \circ \phi_t)
\end{aligned}$$

The properties of the frame form θ^k on M^{k+1} are summarised in, see [5] :

The frame form $\theta = \theta^k$ on M^{k+1} is a $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$ valued one-form on M^{k+1} such that :

$$
\begin{aligned}
&(i) \quad {R_g}^* \theta = \overline{\mathrm{Ad}}(g^{-1})\theta, \ g \in GL^{k+1} \\
&(ii) \quad \theta(\hat{X}_0 \oplus \cdots \oplus \hat{X}_{k+1}) = X_0 \oplus \cdots X_k, \ \ker\theta = \ker \pi_{k+1,k*} = T_{k+1}M^{k+1} \\
&(iii) \quad \pi^*_{k+1,k}\theta^{k-1} = \theta^k \mod \mathfrak{gl}_k
\end{aligned}
$$

Properties (i) and (iii) follow directly from (9) and the definition of the right action and projection, and (ii) is a direct consequence of (10) and the fact that e_{k+1} is an isomorphism. We will sometimes omit the superscript k on θ^k when it is possible to do so. The frame form decomposes as $\theta^k = \theta_{-1} \oplus \theta_0 \oplus \cdots \oplus \theta_k$ with θ_l the $\mathfrak{gl}_l$ part.

In the limit $k \to +\infty$, we can think of the frame form as the Maurer-Cartan form on the group $\mathrm{Diff}(M)$, the translation part θ_{-1} corresponding to (the tangent space of) M in the formal quotient (see introduction) :

$$M \simeq \mathrm{Diff}(M)/\mathrm{Diff}_x(M) \tag{11}$$

and the $\theta_0 \oplus \theta_1 \oplus \cdots$ part corresponding to the Maurer-Cartan form on the 'structure group' $\mathrm{Diff}_x(M)$ of the formally defined principal bundle

$$\mathrm{Diff}_x(M) \longrightarrow \mathrm{Diff}(M) \longrightarrow \mathrm{Diff}(M)/\mathrm{Diff}_x(M) \simeq M$$

2.3.2 Structure equations, Bianchi identities

On M_{k+1}, the $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_{k-1}$-valued 2-form

$$\Theta^{k-1} = d\theta^k + \frac{1}{2}[\theta^k, \theta^k] \mod \mathfrak{h}_{k-1}$$

is tensorial and invariant under G_{k+1}, so it descends to a 2-form on M_k. It satisfies the structure equations, analogous to the Maurer-Cartan equations on a group manifold (recall the formal identification between the frame form and the Maurer-Cartan form of $\mathrm{Diff}(M)$) :

$$\Theta^{k-1} = d\theta^k + \frac{1}{2}[\theta^k, \theta^k] \mod \mathfrak{h}_{k-1} = 0 \tag{12}$$

This is proved in local coordinate form in [5], for $k = 0, 1$. One can also prove this directly in the same way one proves the Maurer-Cartan equations for a group.
By exterior differentiation of the term $d\theta^k + \frac{1}{2}[\theta^k, \theta^k]$, and use of the structure equations (12), one deduces the Bianchi type identities :

$$\left[\theta^k, d\theta^k + \frac{1}{2}[\theta^k, \theta^k]\right] = 0 \mod \mathfrak{h}_{k-1} \tag{13}$$

Note that, in contrast with gauge theory, the Bianchi identities are not the sole consequence of the structure equations.

2.4 Linear frames : reconstruction of the jet frames

We shall denote for later convenience $M = M_{-1}$.

2.4.1 1-frames

A 1-frame above $x \in M$ is an isomorphism

$$e_0 : \mathfrak{g}_{-1} \to T_x M$$

For e_0 and e'_0 above the same x, ${e_0}^{-1} \circ e'_0$ is an isomorphism of $\mathfrak{gl}_{-1}$ so can be written ${e_0}^{-1} \circ e'_0 = \overline{\mathrm{Ad}}(g_0)$ for $g_0 \in GL_0$. So, the space M_0 of 1-frames is a GL_0-principal bundle above M with projection $\pi_{0,-1} : e_0 \mapsto x$ and right action $e_0 \mapsto e_0.g_0 = R_{g_0}(e_0) = e_0 \circ \overline{\mathrm{Ad}}(g_0)$, which is isomorphic to M^0. The frame form $\theta = \theta_{-1}$ on M_0 is then defined as

$$\theta = e_0^{-1} \circ \pi_{0,-1*}$$

It satisfies the same properties as the frame form on M^0. So, we have a principal bundle structure

$$GL_0 \longrightarrow M_0 \longrightarrow M$$

such that, at the tangent space level, the following commutative and exact diagram occurs :

$$\begin{array}{ccccc} \mathfrak{gl}_0 & \longrightarrow & \mathfrak{gl}_0 \oplus \mathfrak{gl}_{-1} & \longrightarrow & \mathfrak{gl}_{-1} \\ \downarrow & & & \nearrow^{\theta_{-1}} & \downarrow {\scriptstyle e_0} \\ T_0 M_0 & \longrightarrow & T_{e_0} M_0 & \xrightarrow{\pi_{0,-1*}} & T_x M \end{array}$$

Note the well known fact [10, 3] that this is this last point which makes the difference between gravity and ordinary gauge theory.

2.4.2 k-frames, $k > 1$

Induction hypothesis

Assume now we have constructed spaces M_l of e_l's, for $0 \leq l \leq k$, which are isomorphic to the M^l, and so have the same structure and same properties as displayed previously. We denote $\pi_{k,l-1} : M_k \to M_{l-1}$ the projections, and $T_l M_k = \ker \pi_{k,l-1*}$. We shall construct the space M_{k+1} isomorphic to M^{k+1} by a prolongation procedure similar to those of [5], [3].

First prolongation of M_k

We define a $(k+2)$-frame above $e_k \in M_k$ as an isomorphism

$$e_{k+1} : \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k \to T_{e_k} M_k$$

such that the following diagram commute :

$$\begin{array}{ccccc}
\mathfrak{gl}_0 \oplus \cdots \mathfrak{gl}_k & \longrightarrow & \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k & \longrightarrow & \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_{k-1} \\
\downarrow \hat{} & & \downarrow e_{k+1} & & \downarrow e_k \\
T_0 M_k & \longrightarrow & T_{e_k} M_k & \longrightarrow & T_{e_{k-1}} M_{k-1}
\end{array}$$

Let $M_{k,1}$ be the space of the e_{k+1}'s.

Principal bundle structure

• For e_{k+1} and e'_{k+1} above the same e_k, the definition then implies that the isomorphism ${e_{k+1}}^{-1} \circ e'_{k+1}$ of $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$ is of degree k i.e.

$${e_{k+1}}^{-1} \circ e'_{k+1} = \overline{\mathrm{Ad}}(g_{k,1}),\ g_{k,1} \in GL_{k,1}$$

Alternatively, this means we have constructed, above $e_k \in M_k$, the commutative square :

$$\begin{array}{ccc}
\mathfrak{gl}_{-1} \oplus \cdots \mathfrak{gl}_k & \xrightarrow{\overline{\mathrm{Ad}}(g_{k,1})} & \mathfrak{gl}_{-1} \oplus \cdots \mathfrak{gl}_k \\
\downarrow e'_{k+1} & & \downarrow e_{k+1} \\
T_{e_k} M_k & \longrightarrow & T_{e_k} M_k
\end{array}$$

All this proves that the projection $\pi_{k+1,k} : e_{k+1} \mapsto e_k$, and right action $e_{k+1} \mapsto e_{k+1} \circ \overline{\mathrm{Ad}}(g_{k,1})$ identifie the principal bundle :

$$GL_{k,1} \longrightarrow M_{k,1} \longrightarrow M_k \tag{14}$$

• Next, consider e_{k+1}, e'_{k+1} above the same $x \in M$ for the projection $\pi_{k+1,-1} = \pi_{k,-1} \circ \pi_{k+1,k}$. Then e_{k+1}, e'_{k+1} are above e_k, e'_k with $e'_k = e_k.g^k$, $g^k \in GL^k$. For any g^{k+1} above g^k, with respect to the projection $GL^{k+1} \to GL^k$, we define $e''_{k+1} = R_{g^k *} e_{k+1} \circ \overline{\mathrm{Ad}}(g^{k+1})$ (see equation (8)). Then e''_{k+1} is a $(k+2)$-linear frame above e'_k.
So, by the preceding point, we have $g_{k,1} \in GL_{k,1}$ such that $e''_{k+1} \circ \overline{\mathrm{Ad}}(g_{k,1}) = e'_{k+1}$, and we obtain

$$e'_{k+1} = R_{g^k *} e_{k+1} \circ \overline{\mathrm{Ad}}(g^{k,1}) \tag{15}$$

with $g^{k,1} = g^{k+1}.g_{k,1}$ In one word, we have just constructed the commutative squares :

$$
\begin{array}{ccccc}
\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k & \xrightarrow{\overline{\mathrm{Ad}} g_{k,1}} & \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k & \xrightarrow{\overline{\mathrm{Ad}} g^{k+1}} & \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k \\
\downarrow{\scriptstyle e'_{k+1}} & & \downarrow{\scriptstyle e''_{k+1}} & & \downarrow{\scriptstyle e_{k+1}} \\
T_{e'_k} M_k & \xrightarrow{\quad \mathrm{id} \quad} & T_{e'_k} M_k & \xrightarrow{\quad {R_{g^k}}_*^{-1} \quad} & T_{e_k} M_k
\end{array}
$$

Thus, the projection $\pi_{k+1,-1}$ and the right action identifie $M_{k,1}$ as a $GL^{k,1}$-principal bundle above M :

$$GL^k \ltimes GL_{k,1} = GL^{k,1} \longrightarrow M_{k,1} \longrightarrow M \tag{16}$$

• The principal fibrations (14) and (16), are summarised in

$$
\begin{array}{ccccc}
GL_{k,1} & \longrightarrow & GL^{k,1} = GL^k \ltimes GL_{k,1} & \longrightarrow & GL^k \\
\| & & \downarrow & & \downarrow \\
GL_{k,1} & \longrightarrow & M_{k,1} & \xrightarrow{\pi_{k+1,k}} & M_k \\
 & & \downarrow{\scriptstyle \pi_{k+1,-1}} & & \downarrow{\scriptstyle \pi_{k,-1}} \\
 & & M & = & M
\end{array}
$$

Frame form

On $M_{k,1}$, we define the frame form θ^k as :

$$\theta^k = {e_{k+1}}^{-1} \pi_{k+1,k*}$$

Then, the definition of right action (15) and definition of $(k+2)$-frames are dually encoded in the properties :

$$
\begin{array}{ll}
(i) & {R_g}^* \theta^k = \overline{\mathrm{Ad}}(g^{-1}) \theta^k, \ g \in GL^{k,1} \\
(ii) & \theta^k(\hat{X}_0 \oplus \cdots \oplus \hat{X}_{k+1}) = X_0 \oplus \cdots X_k, \ \ \ker \theta^k = \ker \pi_{k+1,k*} = T_{k+1} M^{k+1} \\
(iii) & \pi^*_{k+1,k} \theta^{k-1} = \theta^k \quad \mathrm{mod}\ \mathfrak{gl}_k
\end{array}
$$

From this, we define the curvature form as :

$$\Theta^{k-1} = d\theta^k + \frac{1}{2}[\theta^k, \theta^k] \quad \mathrm{mod}\ \mathfrak{h}_{k-1} \tag{17}$$

Horizontality of the frame form (ii) then proves Θ^{k-1} is basic, $i_{\hat{X}} \Theta^{k-1} = 0$, $X = X_0 \oplus \cdots \oplus X_{k,1}$. Equivariance (i) proves that Θ^{k-1} is equivariant under GL^k :

$$R^*_{g^k} \Theta^{k-1} = \overline{\mathrm{Ad}}((g^k)^{-1}) \Theta^{k-1} \tag{18}$$

and transforms affinely under $GL_{k,1}$:

$$R^*_{g_{k,1}}\Theta^{k-1} = \Theta^{k-1} - \partial\alpha_k \circ \theta_{-1}$$

Finally, the recursive property (iii) and induction hypothesis prove the recursive identity :

$$\pi^*_{k+1,k}\Theta^{k-2} = \Theta^{k-1} \mod \mathfrak{gl}_{k-1} = 0 \tag{19}$$

All the properties of Θ^{k-1} are then equivalently encoded in the torsion map

$$t : M_{k,1} \to \mathfrak{gl}_{k-1,2} = \mathfrak{gl}_{k-1} \otimes \Lambda^2\mathfrak{gl}^*_{-1}$$

which maps e_{k+1} to $t_{e_{k+1}}$ with :

$$t_{e_{k+1}}(X_{-1}, Y_{-1}) = \Theta^{k-1}(\overline{e_{k+1}(X_{-1})}, \overline{e_{k+1}(Y_{-1})}) = d(\theta^{k-1})_{k-1}(e_{k+1}(X_{-1}).e_{k+1}(Y_{-1}))$$

where $\overline{e_{k+1}(X_{-1})}$ is a lift of $e_{k+1}(X_{-1}) \in TM_k$ to $TM_{k,1}$, and where $(\theta^{k-1})_{k-1}$ is the component of degree $(k-1)$ of the frame form θ^{k-1} on M_k. We then have the covariance properties :

$$t_{e_{k+1}.g^k} = \overline{\mathrm{Ad}}(g^{k^{-1}}) \circ t_{e_{k+1}} \circ \overline{\mathrm{Ad}}(g_0) \ , \ t_{e_{k+1}.g_{k,1}} = t_{e_{k+1}} - \partial\alpha_k \tag{20}$$

We summarise this by saying the following diagram is commutative and covariant under the GL^k action :

$$\begin{array}{ccc} GL_{k,1} \simeq \mathfrak{gl}_{k,1} & \longrightarrow & M_{k,1} \\ \big\downarrow \partial & & \big\downarrow t \\ \mathfrak{gl}_{k-1,2} & = & \mathfrak{gl}_{k-1,2} \end{array}$$

Reduction to M_k

Now, by evaluating the Bianchi identities of M_k (satisfied by the induction hypothesis)

$$\left[\theta^{k-1}, d\theta^{k-1} + \frac{1}{2}[\theta^{k-1}, \theta^{k-1}]\right] = 0 \mod \mathfrak{h}_{k-2}$$

on vectors $e_{k+1}(X_{-1}), e_{k+1}(Y_{-1}), e_{k+1}(Z_{-1})$, we obtain

$$\partial t_{e_{k+1}} = 0$$

so that the torsion at each e_{k+1} is a ∂-cocyle. This last property and the exactness of the ∂-sequence

$$\mathfrak{gl}_{k+1} \longrightarrow \mathfrak{gl}_{k,1} \xrightarrow{\partial} \mathfrak{gl}_{k-1,2} \xrightarrow{\partial} \mathfrak{gl}_{k-2,3} \tag{21}$$

at $\mathfrak{gl}_{k-1,2}$ then proves that we have $t_{e_1} = \partial\alpha_k$, for a $\alpha_k \in \mathfrak{gl}_{k,1} \simeq GL_{k,1}$. Thanks to equivariance (20), all this proves the existence of $(k+2)$-frames with null torsion, i.e. the map t has a kernel. We then simply define

$$M_{k+1} = t^{-1}(0)$$

that is M_k are the e_{k+1} such that $t_{e_{k+1}} = 0$. Then both the equivariance (20) and the exactness of (21) at $\mathfrak{gl}_{k,1}$ then prove that $M_{k+1} \to M_k$ is a subbundle of $M_{k,1} \to M_k$ with structure group $GL_{k+1} \simeq \mathfrak{gl}_{k+1}$. All these facts are summarised in the exact commutative diagram, which completes the diagram following equation (20) :

$$\begin{array}{ccccc}
\mathfrak{gl}_k \simeq GL_k & \longrightarrow & M_{k+1} & \longrightarrow & M_k \\
\downarrow & & \downarrow & & \| \\
\mathfrak{gl}_{k,1} \simeq GL_{k,1} & \longrightarrow & M_{k,1} & \longrightarrow & M_k \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle t} & & \\
\mathfrak{gl}_{k-1,2} & = & \mathfrak{gl}_{k-1,2} & & \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
\mathfrak{gl}_{k-2,3} & = & \mathfrak{gl}_{k-2,3} & &
\end{array}$$

The first column describes an exact Spencer ∂-sequence, the second the construction of M_{k+1}, and the first two lines the principal fibrations so obtained.

Structure of M_{k+1}

We have thus obtained an iterative fibration

$$\begin{array}{ccccc}
GL_{k+1} & \longrightarrow & GL^{k+1} & \longrightarrow & GL^k \\
\| & & \downarrow & & \downarrow \\
GL_{k+1} & \longrightarrow & M_{k+1} & \longrightarrow & M_k \\
 & & \downarrow & & \downarrow \\
 & & M & = & M
\end{array}$$

M_{k+1} is equiped with the frame form θ^k inherited from $M_{k,1}$, and now we have, as the torsion of e_{k+1} vanish :

$$\Theta^{k-1} = 0$$

i.e. the structural equations. So M_{k+1} has the same structure as M_k at the next order. Using the induction hypothesis $M_k \simeq M^k$, the map

$$e^{k+1} \mapsto e_{k+1}$$

defined in section 2.2, is then, by construction, an isomorphism of principal bundles, so $M_{k+1} \simeq M^{k+1}$.

3 Field theory of frames

3.1 Preliminaries : Local fields

3.1.1 Local Spencer cochains

• To order $k+2$, one obtains a local version of $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$ by defining the associated bundle

$$S_k = M_{k+1} \times_{\overline{\mathrm{Ad}}} \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k \tag{22}$$

which can be seen as a higher order tangent bundle above M. Local Spencer cochains are S_k-valued forms on M, i.e. elements of $\Omega^*(M, S_k)$. These are the basic fields of the theory.

• Owing to the structure of M_{k+1}, we can give alternative and useful descriptions of this. First, recall we have

$$\Omega^l(M, S_k) \simeq \Omega^l_{GL^{k+1}}(M_{k+1}, \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k)$$

Second, this isomorphy allows to associate to each $\alpha \in \Omega^l(M, S_k)$ the function $\tilde{\alpha}$ on M_{k+1} defined at each e_{k+1} by

$$\tilde{\alpha}_{|e_{k+1}}(X_1, \cdots X_l) = \alpha_{|e_{k+1}}(e_{k+2}(X_1), \cdots, e_{k+2}(X_l)) \Leftrightarrow \alpha = \tilde{\alpha} \circ \theta_{-1}$$

for any e_{k+2} above e_{k+1}, $X_i \in \mathfrak{gl}_{-1}$. We shall extend each $\tilde{\alpha}_{|e_{k+1}}$ to a null form on $\mathfrak{gl}_0 \oplus \cdots \oplus \mathfrak{gl}_k$, so that we will also write $\alpha = \tilde{\alpha} \circ \theta$. As $\tilde{\alpha}$ is then equivariant, this naturally defines an isomorphy between $\Omega^l(M, S_k)$ and the space of section of the bundle

$$M_{k+1} \times_{\overline{\mathrm{Ad}}} (\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k) \otimes \Lambda^l \mathfrak{gl}^*_{-1}$$

This last fact implies that we can define, point by point, an algebraic ∂-operator on $\Omega^l(M, S_k)$. Third, to α one can also associate the vector valued form $\bar{\alpha}$ defined as

$$\bar{\alpha}_{|e_{k+1}}(u_1, \cdots, u_l) = e_{k+1}(\alpha_{|e_{k+1}}(u_1, \cdots, u_l))$$

for $u_i \in TM_{k+1}$. This means $\Omega^l(M, S_k)$ is also isomorphic with the tensorial forms on M_{k+1} with values in tangent vector on M_k. Then, as on any space of Lie algebra valued forms, we can define the standard structure of differential graded Lie algebra, thus obtaining the algebraic as well as differential brackets of [9].

3.1.2 Linear Spencer sequences

For $\alpha \in \Omega^l(M, S_k)$, viewed as a tensorial form on M_{k+1}, we define :

$$d_\theta \alpha = d\alpha + [\theta, \alpha] \mod \mathfrak{h}_{k-1}$$

Then $d_\theta \alpha$ is still tensorial, and this defines a map

$$d_\theta : \Omega^l(M, S_k) \to \Omega^{l+1}(M, S_{k-1})$$

The structure equation $\Theta = 0$ on M_{k+1}, then proves that d_θ is nilpotent

$$d_\theta^2 \alpha = [\Theta, \alpha] \mod \mathfrak{h}_{k-2} = 0$$

thus giving the linear sequence

$$\Omega^0(M, S_k) \xrightarrow{d_\theta} \Omega^1(M, S_{k-1}) \xrightarrow{d_\theta} \cdots \longrightarrow \Omega^n(M, S_{k-n}) \xrightarrow{d_\theta} 0 \tag{23}$$

The proof of this is a straightforward application of the definitions. In the following, we shall complete this sequence to the linear Spencer sequence.

3.2 Symmetries

3.2.1 Diffeomorphisms

We denote by $\mathrm{Aut}(M)$ the group of (oriention preserving) diffeomorphisms of M. Let $f = f_{-1} \in \mathrm{Aut}(M)$.
From the jet viewpoint, f acts on M^k by

$$e^k = j_0^{k+1}(\phi) \mapsto f_k(e_k) = j_0^{k+1}(f \circ \phi)$$

Let us analyse this from the linear frame viewpoint. The action on M_0 is given by

$$e_0 \to f_0(e_0) = f_{-1*} e_0$$

Then f_0 satisfies $R_{g_0} \circ f_0 = f_0 \circ R_{g_0}$, $g_0 \in GL_0$, and $\pi_{0,-1} \circ f_0 = f_{-1} \circ \pi_{0,-1}$, so is a principal bundle automorphism. Moreover, we have

$$\begin{aligned} f_0^* \theta^{-1}{}_{|e_0} &= \theta^{-1}{}_{|f_0(e_0)} \circ f_{0*} = f_0(e_0)^{-1} \pi_{0,-1*} f_{0*} \\ &= f_0(e_0)^{-1} f_{-1*} \pi_{0,-1*} = e_0{}^{-1} \pi_{0,-1*} \\ &= \theta^{-1}{}_{|e_0} \end{aligned}$$

This shows that the action on M_1 defined by

$$e_1 \to f_1(e_1) = f_{0*} e_1$$

is well defined (i.e. e_1 is a 2-frame of null torsion). Recursively, we define f_{k+1} from f_k by :

$$f_{k+1}(e_{k+1}) = f_{k*}e_{k+1}$$

Exactly the same calculation as before proves this is well defined. Then the prolongated diffeomorphisms satisfies :

$$R_{g^k} \circ f_k = f_k \circ R_{g^k},\ \pi_{k,k-1} \circ f_k = f_{k-1} \circ \pi_{k,k-1}$$

and keep invariant the frame form (same calculation as for θ^{-1})

$$f_k{}^*\theta^{k-1} = \theta^{k-1} \tag{24}$$

We shall denote $j_k(f) = f_k$ the prolongated diffeomorphism.

3.2.2 Extended diffeomorphisms

• Now, denote by $\mathrm{Aut}(M_k)$ the automorphism group of $M_k \to M$ as a principal fiber bundle, that is :

$$f_k \in \mathrm{Aut}(M_k)\ :\ f_k \circ R_{g^k} = R_{g^k} \circ f_k$$

Then $\mathrm{Aut}(M_k)$ is a a subgroup of the group of diffeomorphisms of M_k which preserves the fibers of $M_k \to M$. The gauge group $\mathcal{GL}^k$ of $M_k \to M$ are the vertical automorphisms in $\mathrm{Aut}(M_k)$ i.e.

$$f_k \in \mathcal{GL}^k : f_k \circ R_{g^k} = R_{g^k} \circ f_k\ \ ,\ \ f_k(\pi_{k,-1}^{-1}(x)) = \pi_{k,-1}^{-1}(x)$$

for all $x \in M$. Similarly, we define the gauge group $\mathcal{GL}_k$ of $M_k \to M_{k-1}$, and we observe that the gauge group of $M_k \to M_{k'}$, $k' \le k$, is $\mathcal{GL}_{k'+1} \ltimes \cdots \ltimes \mathcal{GL}_k$, so that in particular

$$\mathcal{GL}^{k+1} \simeq \mathcal{GL}_0 \ltimes \cdots \ltimes \mathcal{GL}_k$$

As usual, gauge transformations, in $\mathcal{GL}^k$ say, are isomorphic with section of the adjoint bundle $M_k \times_{\mathrm{Ad}} GL^k$, thanks to the isomorphy $g^k \mapsto \tilde{g}^k$ defined by

$$g^k(e_k) = e_k.\tilde{g}^k(e_k) = R_{\tilde{g}^k(e_k)}(e_k)$$

• We define projections $f_{k+1} \mapsto f_k$ by

$$f_k(e_k) = \pi_{k+1,k}(f_{k+1}(e_{k+1}))$$

for any e_{k+1} above e_k. This is well defined thanks to the equivariance of f_{k+1}, and we have $\pi_{k+1,k} \circ f_{k+1} = f_k \circ \pi_{k+1,k}$. In other words, we have commutation in

$$\begin{array}{ccccc} GL_{k+1} & \longrightarrow & M_{k+1} & \xrightarrow{\pi_{k+1,k}} & M_k \\ \downarrow & & \downarrow f_{k+1} & & \downarrow f_k \\ GL_{k+1} & \longrightarrow & M_{k+1} & \xrightarrow{\pi_{k+1,k}} & M_k \end{array}$$

and we obtain the tower of commutative squares

$$\begin{array}{ccccccccc} M_{k+1} & \longrightarrow & M_k & \longrightarrow \cdots \longrightarrow & M_0 & \longrightarrow & M \\ \downarrow f_{k+1} & & \downarrow f_k & & \downarrow f_0 & & \downarrow f_{-1} \\ M_{k+1} & \longrightarrow & M_k & \longrightarrow \cdots \longrightarrow & M_0 & \longrightarrow & M \end{array}$$

Note that these projections are group morphisms from $\mathrm{Aut}(M_{k+1})$ to $\mathrm{Aut}(M_k)$. For f_{k+1}, f'_{k+1} projecting on the same f_k, the automorphism $f''_{k+1} = {f_{k+1}}^{-1} \circ f'_{k+1}$ then preserves the fibers of $M_{k+1} \to M_k$, and is thus a gauge transformation :

$$f'_{k+1} = f_{k+1} \circ f''_{k+1} \quad , \quad f''_{k+1} \in \mathcal{GL}_{k+1}$$

So we obtain a principal bundle

$$\mathcal{GL}_{k+1} \longrightarrow \mathrm{Aut}(M_{k+1}) \longrightarrow \mathrm{Aut}(M_k)$$

with gauge transformations projecting on the identity of $\mathrm{Aut}(M_k)$. More generally we obtain in this way principal bundles :

$$\mathcal{GL}_{k'+1} \ltimes \cdots \ltimes \mathcal{GL}_k \longrightarrow \mathrm{Aut}(M_k) \longrightarrow \mathrm{Aut}(M_{k'})$$

and in particular

$$\mathcal{GL}^{k+1} \longrightarrow \mathrm{Aut}(M_{k+1}) \longrightarrow \mathrm{Aut}(M) \tag{25}$$

This last bundle admits the global section given by $f_{-1} \to f_{k+1} = j_{k+1}(f_{-1})$. The section j_{k+1} enables us to construct, for $f_{k+1} \in \mathrm{Aut}(M_{k+1})$ projecting on $f_{-1} \in \mathrm{Aut}(M)$, the gauge transformation $g^{k+1} \in \mathcal{GL}^{k+1}$ defined by :

$$f_{k+1} = j_{k+1}(f_{-1}) \circ g^{k+1} \tag{26}$$

The equation (26) gives a global trivialization of (25), that is of the semi-direct product $\mathrm{Aut}(M_{k+1}) \simeq \mathrm{Aut}(M) \ltimes \mathcal{GL}^{k+1}$.

3.2.3 Synthesis

For $f_{k+1} \in \mathrm{Aut}(M_{k+1})$, we define the first Spencer operator

$$D_\theta f_{k+1} = {f_{k+1}}^* \theta^k - \theta^k$$

Then $D_\theta f_{k+1}$ is a tensorial $(\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k)$-valued 1-form, that is $D_\theta f_{k+1} \in \Omega^1(M, S_k)$. Indeed, for $X = X_0 \oplus \cdots \oplus X_{k+1}$, the equivariance of f_{k+1} implies :

$$i_{\hat{X}} D_\theta f_{k+1} = \theta^k(f_{k+1*}\hat{X}) - \theta^k(\hat{X}) = \theta^k(\hat{X}) - \theta^k(\hat{X}) = 0$$

and, for $g \in GL^{k+1}$,

$$\begin{aligned} R_g^* D_\theta f_{k+1} &= R_g^* {f_{k+1}}^* \theta^k - R_g^* \theta^k = {f_{k+1}}^* R_g^* \theta^k - R_g^* \theta^k \\ &= \overline{\mathrm{Ad}}(g^{-1}) {f_{k+1}}^* \theta^k - \overline{\mathrm{Ad}}(g^{-1}) \theta^k = \overline{\mathrm{Ad}}(g^{-1}) D_\theta f_{k+1} \end{aligned}$$

Moreover,

D_θ defines a cocycle on the group $\mathrm{Aut}(M_{k+1})$ *with values in* $\Omega^1(M, S_k)$*, with kernel the group of diffeomorphisms of* M*, that is* $D_\theta f_{k+1} = 0$ *iff* $f_{k+1} = j_{k+1}(f_{-1})$.

Indeed, the cocycle relation follows from :

$$\begin{aligned} D_\theta(f_{k+1} \circ g_{k+1}) &= (f_{k+1} \circ g_{k+1})^* \theta^k - \theta^k = {g_{k+1}}^* {f_{k+1}}^* \theta^k - \theta^k \\ &= {g_{k+1}}^* \left({f_{k+1}}^* \theta^k - \theta^k \right) + {g_{k+1}}^* \theta^k - \theta^k \\ &= {g_{k+1}}^* D_\theta f_{k+1} + D_\theta g_{k+1} \end{aligned}$$

Next, we have already shown that $f_{k+1} = j_{k+1}(f)$, $f \in \mathrm{Aut}(M)$, keeps the frame form invariant, i.e. $D_\theta f_{k+1} = 0$. Conversely, suppose $D_\theta f_{k+1} = 0$ i.e. ${f_{k+1}}^* \theta^k = \theta^k$. As

$$\begin{aligned} {f_{k+1}}^* {\theta^k}_{|e_{k+1}} &= {\theta^k}_{|f_{k+1}(e_{k+1})} \circ f_{k+1*} \\ &= f_{k+1}(e_{k+1})^{-1} \pi_{k+1,k*} f_{k+1*} \\ &= f_{k+1}(e_{k+1})^{-1} f_{k*} \pi_{k+1,k*} \end{aligned}$$

the equation ${f_{k+1}}^* \theta^k = \theta^k$ implies $f_{k+1}(e_{k+1})^{-1} f_{k*} \pi_{k+1,k*} = {e_{k+1}}^{-1} \pi_{k+1,k*}$ and so $D_\theta f_{k+1} = 0$ is equivalent to :

$$f_{k+1}(e_{k+1}) = f_{k*} e_{k+1} \tag{27}$$

Then, from equation (27) the result is easily proved by induction on k.

■

All this is summarised in the exact sequence

$$\mathrm{id} \longrightarrow \mathrm{Aut}(M) \xrightarrow{j_{k+1}} \mathrm{Aut}(M_{k+1}) \xrightarrow{D_\theta} \Omega^1(M, S_k) \longrightarrow 0 \tag{28}$$

j_{k+1} being a group morphism and D_θ a group cocycle.

3.2.4 Action of $\mathrm{Aut}(M_{k+1})$ on local fields

It is useful for next purpose to compute the action of an extended diffeomorphism on a local field.

• As a preliminary, take $f_{k+1} \in \mathrm{Aut}(M_{k+1})$, then as $D_\theta f_{k+1} \in \Omega^1(M, S_k)$, we can view it as a function $\tilde{D}_\theta f_{k+1}$ on M_{k+1} with values in $\mathfrak{gl}_{-1} \oplus \cdots \oplus \cdots \mathfrak{gl}_k$ (section 3.1.1). For $X = X_{-1} \oplus \cdots \oplus X_k$, and any e_{k+2} above e_{k+1}, one finds :

$$\begin{aligned} \tilde{D}_\theta f_{k+1|e_{k+1}}(X) &= D_\theta f_{k+1|e_{k+1}}(e_{k+2}(X)) \\ &= {f_{k+1}}^* \theta_{|e_{k+1}}(e_{k+2}(X)) - X \\ &= f_{k+1}(e_{k+1})^{-1} \pi_{k+1,k*} f_{k+1*} e_{k+2}(X) - X \end{aligned}$$

All this proves that the map

$$X \mapsto X + f_{k+1}(e_{k+1})^{-1} \pi_{k+1,k*} f_{k+1*} e_{k+2}(X)$$

that we shall denote $1 + \tilde{D}_\theta f_{k+1}$, is an automorphism of $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$, inducing the identity on the $\mathfrak{gl}_0 \oplus \cdots \oplus \mathfrak{gl}_k$ part, and that this is indeed the equivariant version of $\theta + D_\theta f_{k+1} = {f_{k+1}}^* \theta$.

• Take a local field $\alpha \in \Omega^*(M, S_k)$. Viewing α as a tensorial form on M_{k+1}, the action of f_{k+1} is simply

$$\alpha \to \alpha' = {f_{k+1}}^* \alpha$$

Equivariance of f_{k+1} shows this is consistent.

• View now α as a equivariant function $\tilde{\alpha}$ on M_{k+1} (section 3.1.1). $\tilde{\alpha}$ transforms to $\tilde{\alpha}'$. At e_{k+1}, we have

$$\begin{aligned} \tilde{\alpha}' \circ \theta_{|e_{k+1}} &= \alpha'_{|e_{k+1}} = {f_{k+1}}^* \alpha_{|e_{k+1}} \\ &= \alpha_{|f_{k+1}(e_{k+1})} \circ f_{k+1*|e_{k+1}} \\ &= \tilde{\alpha}_{|f_{k+1}(e_{k+1})} \circ \theta_{|f_{k+1}(e_{k+1})} \circ f_{k+1*|e_{k+1}} \end{aligned}$$

and, by evaluating on $e_{k+2}(X)$, for any e_{k+2} above e_{k+1} :

$$\begin{aligned} \tilde{\alpha}'_{|e_{k+1}}(X) &= \tilde{\alpha}_{|f_{k+1}(e_{k+1})}(\theta_{|f_{k+1}(e_{k+1})}(f_{k+1*} e_{k+2}(X))) \\ &= \tilde{\alpha}_{|f_{k+1}(e_{k+1})}(1 + \tilde{D}_\theta f_{k+1})_{|e_{k+1}} \end{aligned}$$

All this means that, from the equivariant viewpoint, the field $\tilde{\alpha}$ transforms as :

$$\tilde{\alpha} \to \tilde{\alpha}' = {f_{k+1}}^* \tilde{\alpha} \circ (1 + \tilde{D}_\theta f_{k+1}) \tag{29}$$

Note that if f_{k+1} comes from a diffeomorphism, i.e. $D_\theta f_{k+1} = 0$, then the preceding tranformation law is, as expected, $\tilde{\alpha} \to {f_{k+1}}^* \tilde{\alpha}$.

3.3 Deformations

3.3.1 Deformation space

• Consider the group B_k of automorphisms of $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$ which induce the identity on $\mathfrak{gl}_0 \oplus \cdots \oplus \mathfrak{gl}_k$. This group then consists in inversible transformations such that :

$$X_{-1} \oplus X_0 \oplus \cdots \oplus X_k \mapsto X_{-1} + \mu_{-1}(X_{-1}) \oplus X_0 + \mu_0(X_{-1}) \oplus \cdots \oplus X_k + \mu_k(X_{-1})$$

where $\mu_l \in \mathfrak{gl}_{l,1}$. We denote this simply $1 + \tilde{\mu}$. The inverse transformation is

$$\begin{aligned} X_{-1} \oplus \cdots X_0 \oplus \cdots \oplus X_k \quad \mapsto \quad & (1+\tilde{\mu}_{-1})^{-1} X_{-1} \oplus X_0 - \tilde{\mu}_0 (1+\tilde{\mu}_{-1})^{-1} X_{-1} \oplus \\ & \cdots \oplus X_k - \tilde{\mu}_k (1+\tilde{\mu}_{-1})^{-1} X_{-1} \end{aligned}$$

so that $1 + \tilde{\mu} \in B_k$ iff $1 + \mu_{-1} \in B_{-1} = GL_0$. As B_k is a subspace of $\mathfrak{gl}_{-1,1} \oplus \cdots \oplus \mathfrak{gl}_{k,1}$, GL^{k+1} acts on the left with $\overline{\mathrm{Ad}}$ on it (preserving the inversibility property), and we can define the associated fiber bundle

$$\mathcal{B}_k = M_{k+1} \times_{\overline{\mathrm{Ad}}} B_k$$

To each section $\tilde{\mu} \in \Gamma(\mathcal{B}_k)$ seen as a equivariant B_k-valued function on M_{k+1}, we can associate the tensorial one-form $\mu \in \Omega^1(M, S_k)$ defined by (see section 3.1.1) :

$$\mu = \tilde{\mu} \circ \theta_{-1} \tag{30}$$

so that we have the identity

$$(1 + \tilde{\mu}) \circ \theta = \theta + \mu$$

We shall denote by $\Omega'^1(M, S_k)$ the subspace of $\Omega^1(M, S_k)$ constituted of sections of $\mathcal{B}_k$ under the correspondance (30). Then $\omega = \theta + \mu$ obeys the equivariance and horizontality conditions :

$$\begin{aligned} (i) \quad & R_g^* \omega = \overline{\mathrm{Ad}}(g^{-1})\omega, \ g \in GL^{k+1} \\ (ii) \quad & \omega(\hat{X}_0 \oplus \cdots \oplus \hat{X}_k) = X_0 \oplus \cdots \oplus X_k \end{aligned}$$

and, for any e_{k+2} above e_{k+1},

$$(iii) \ \ X_{-1} \oplus \cdots \oplus X_k \mapsto \omega_{|e_{k+1}} (e_{k+2}(X_{-1} \oplus \cdots \oplus X_k)) \ \text{ is inversible}$$

Reciprocally, if ω obeys (i) and (ii), then defining $\mu = \omega - \theta$, we have $i_{\hat{X}} \mu = X - X = 0$ so horizontality, and $R_g^* \mu = \overline{\mathrm{Ad}}(g^{-1})\mu$ by equivariance of θ, so $\mu \in \Omega^1(M, S_k)$, with corresponding $\tilde{\mu}$. Next, as $\omega_{|e_{k+1}} (e_{k+2}(X_{-1} \oplus \cdots \oplus X_k)) = X_{-1} + \tilde{\mu}_{-1} X_{-1} \oplus \cdots \oplus X_k + \mu_k X_{-1}$, (iii) implies that in fact $1 + \tilde{\mu}$ is inversible i.e. $\tilde{\mu} \in \mathcal{B}_k$.

• For μ, ν in $\Gamma(\mathcal{B}_k)$, we can compose the isomorphisms $1+\tilde{\mu}$ and $1+\tilde{\nu}$ of $\mathfrak{gl}_{-1}\oplus\cdots\oplus\mathfrak{gl}_k$, at each e_{k+1}, to obtain $(1+\tilde{\mu})(1+\tilde{\nu}) = 1+\widetilde{\mu.\nu}$ (recall B_k is a group). We have, from the equivariant point of vue

$$\widetilde{\mu.\nu} = \tilde{\mu} + \tilde{\nu} + \tilde{\mu}\circ\tilde{\nu}_{-1}$$

and from the form point of vue

$$\mu.\nu = \mu + \nu + i_\nu \mu$$

where we see μ and ν as 1-forms valued in $\Gamma(TM_k)$ (section 3.1.1) and i_ν is the interior product extended to vector-valued forms.

• Alternatively, we can see the bundle $\mathcal{B}_k$ as some jet space relative to the differential operator D_θ previously defined. Indeed, defining at each e_{k+1}, the equivalence relation :

$$f_{k+1} \sim f'_{k+1} \quad : \quad D_\theta f_{k+1|e_{k+1}} = D_\theta f'_{k+1|e_{k+1}} \quad , \quad f_{k+1}, f'_{k+1} \in \mathrm{Aut}(M_{k+1})$$

and denoting $[D_\theta f_{k+1}]_{|e_{k+1}}$ the resulting class, we build a bundle associated to M_{k+1} by considering the elements $[\tilde{D}_\theta f_{k+1}]$ with equivariance under GL^{k+1} inherited from the tensoriality of $D_\theta f_{k+1}$:

$$[\tilde{D}_\theta f_{k+1}] \rightarrow \overline{\mathrm{Ad}}(g^{-1})\circ[\tilde{D}_\theta f_{k+1}]\circ\overline{\mathrm{Ad}}(g_0)$$

under $e_{k+1}\rightarrow e_{k+1}.g$. This allows us to identify this bundle with $\mathcal{B}_k$.
Now, for $\tilde{\mu}\in\mathcal{B}_k$, written as $\tilde{\mu} = [\tilde{D}_\theta g_{k+1}]$ i.e. $\mu = [D_\theta g_{k+1}]$, the cocycle relation for D_θ passes to the jet equivalence to give :

$$[D_\theta(g_{k+1}\circ f_{k+1})] = {f_{k+1}}^*[D_\theta g_{k+1}] + D_\theta f_{k+1}$$

and induces the following action of $\mathrm{Aut}(M_{k+1})$ on $\Omega'^1(M,S_k)$:

$$\mu \rightarrow {f_{k+1}}^*\mu + D_\theta f_{k+1} \tag{31}$$

Next, for $\tilde{\mu},\tilde{\nu}\in\mathcal{B}_k$, written as $\tilde{\mu} = [\tilde{D}_\theta f_{k+1}], \tilde{\nu} = [\tilde{D}_\theta g_{k+1}]$, that is $\mu = [D_\theta f_{k+1}], \nu = [D_\theta g_{k+1}]$, the same cocycle condition written from the equivariant point of vue (see 3.2.4)

$$\tilde{D}_\theta(f_{k+1}\circ g_{k+1}) = \tilde{D}_\theta f_{k+1}\circ\Big(1+\tilde{D}_\theta g_{k+1}\Big) + \tilde{D}_\theta g$$

and conveniently rewritten as

$$1+\tilde{D}_\theta(f_{k+1}\circ g_{k+1}) = \Big(1+\tilde{D}_\theta f_{k+1}\Big)\Big(1+\tilde{D}_\theta g_{k+1}\Big) \tag{32}$$

where $1+\tilde{D}_\theta f_{k+1}$ is evaluated at the point $g_{k+1}(e_{k+1})$ and $1+\tilde{D}_\theta g_{k+1}$ at e_{k+1} as stated in (29), passes to the jet equivalence, and give us back the composition of deformations :

$$1+\widetilde{\mu.\nu} = 1+[\tilde{D}_\theta(f_{k+1}\circ g_{k+1})] = \left(1+[\tilde{D}_\theta f_{k+1}]\right)\left(1+[\tilde{D}_\theta g_{k+1}]\right) = (1+\tilde{\mu})(1+\tilde{\nu})$$

i.e. the composition in $\mathrm{Aut}(M_{k+1})$ induces at the jet level the composition of deformations.

3.3.2 Deformed frame bundle

Now, we analyse the deformations from another point of vue, perhaps more concrete, and we show how to rederive in this context the results given above, and how it allows to produce new ones.

• For $\tilde{\mu} \in \Gamma(\mathcal{B}_k)$, we notice that, for $l \leq k$, the section $\tilde{\mu}_{-1}\oplus\cdots\oplus\tilde{\mu}_l$ is equivariant under GL^{l+1} and invariant under $GL_{l+2}\ltimes\cdots\ltimes GL_k$, so we can descend $\tilde{\mu}_{-1}\oplus\cdots\oplus\tilde{\mu}_l$ to a section of $\Gamma(\mathcal{B}_l)$ that is we can view it as a equivariant function on M_{l+1}.

• We define $M_{0,\mu}$ as the space of $e_{0,\mu}$'s obtained as

$$e_{0,\mu} = e_0 \circ (1+\tilde{\mu}_{-1})^{-1}_{|e_0}$$

Thus $M_{0,\mu}$ is a GL_0 principal bundle over M which is in fact, here, M_0 (as here $1+\tilde{\mu}_{-1}$ is a gauge transformation). We denote :

$$F_{-1,\mu} : M_0 \to M_{0,\mu},\ e_0 \mapsto e_0\circ(1+\tilde{\mu}_{-1})^{-1}_{|e_0}$$

This map is a principal bundle isomorphism inducing the identity on the base M.

• Next, define $M_{1,\mu}$ as the space of $e_{1,\mu}$'s obtained as :

$$e_{1,\mu} = F_{-1,\mu*}e_1\circ(1+\tilde{\mu}_{-1}\oplus\tilde{\mu}_0)^{-1}_{|e_1}$$

These are linear frame above $M_{0,\mu}$ since

$$\begin{array}{ll}(i) & \pi_{0,-1*}e_{1,\mu}(X_{-1}\oplus X_0) = e_{0,\mu}(X_{-1})\\ (ii) & e_{1,\mu}(X_0) = \hat{X}_0\end{array}$$

where this comes from the fact $F_{-1,\mu}$ is a principal bundle isomorphism. Again, $M_{1,\mu}$ is a GL_1-principal bundle above $M_{0,\mu}$ and a GL^1-principal bundle above M. We define

$$F_{0,\mu} : M_1 \to M_{1,\mu},\ e_1 \mapsto F_{-1,\mu*}e_1\circ(1+\tilde{\mu}_{-1}\oplus\tilde{\mu}_0)^{-1}_{|e_1}$$

This map is a principal bundle isomorphism.

- Recursively, we define $M_{l+1,\mu}$ as the space of $e_{l+1,\mu}$'s obtained as

$$e_{l+1,\mu} = F_{l-1,\mu *} e_{l+1} \circ (1 + \tilde{\mu}_{-1} \oplus \cdots \oplus \tilde{\mu}_l)^{-1}_{|e_{l+1}} \tag{33}$$

Using the fact that the precedingly constructed $F_{l-1,\mu}$ is a principal bundle isomorphism, we show that $e_{l+1,\mu}$ are linear frames above $M_{l,\mu}$, and obtain the principal bundles

$$GL_{l+1} \longrightarrow M_{l+1,\mu} \xrightarrow{\pi'_{l+1,l}} M_{l,\mu} \tag{34}$$

$$GL^{l+1} \longrightarrow M_{l+1,\mu} \xrightarrow{\pi'_{l+1,-1}} M \tag{35}$$

In summary, we have interpreted $\tilde{\mu}$ as providing an iterative fibering encoded in the commutative diagram :

$$\begin{array}{ccccccccccc}
M_{k+1} & \longrightarrow & M_k & \longrightarrow \cdots \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & M_{-1} \\
\downarrow{\scriptstyle F_{k,\mu}} & & \downarrow{\scriptstyle F_{k-1,\mu}} & & \downarrow{\scriptstyle F_{0,\mu}} & & \downarrow{\scriptstyle F_{-1,\mu}} & & \\
M_{k+1,\mu} & \longrightarrow & M_{k,\mu} & \longrightarrow \cdots \longrightarrow & M_{1,\mu} & \longrightarrow & M_{0,\mu} & \longrightarrow & M_{-1}
\end{array}$$

that is $\pi'_{l+1,l} \circ F_{l+1,\mu} = F_{l,\mu} \circ \pi_{l+1,l}$, with commutation of the subsquares (covariance of $F_{l-1,\mu}$) :

$$\begin{array}{ccccc}
GL_{l'+1} \ltimes \cdots \ltimes GL_l & \longrightarrow & M_l & \longrightarrow & M_{l'} \\
\| & & \downarrow{\scriptstyle F_{l-1,\mu}} & & \downarrow{\scriptstyle F_{l'-1,\mu}} \\
GL_{l'+1} \ltimes \cdots \ltimes GL_l & \longrightarrow & M_{l,\mu} & \longrightarrow & M_{l',\mu}
\end{array} \tag{36}$$

Note that if a deformation μ is a ∂-cocycle, i.e. $\partial \tilde{\mu}_{|e_{k+1}} = 0$ at each e_{k+1}, then the induced deformation is simply a gauge transformation $g^{k+1} \in \mathcal{GL}^{k+1}$ of M_{k+1} whose equivariant form $\tilde{g} = \tilde{g}^{k+1}$ satisfies (see section 1.2) : $\overline{\mathrm{Ad}}(\tilde{g}^{-1}) = 1 + \tilde{\mu}$.

3.3.3 Deformed frame form

As each $M_{l+1,\mu}$ is a bundle of $(l+2)$-linear frames above $M_{l,\mu}$, we can dually define the frame form. On $M_{k+1,\mu}$, define the deformed frame form at $e_{k+1,\mu}$ by

$$\theta^k_\mu = {e_{k+1,\mu}}^{-1} \pi'_{k+1,k*} \tag{37}$$

Then, by construction of the $e_{l,\mu}$'s, θ^k_μ satisfies the same properties of equivariance, horizontality, and recursion as the ordinary frame form on M_{k+1}. Moreover we have from (33)

$$\begin{aligned}
\theta^k_\mu &= {e_{k+1,\mu}}^{-1}\pi'_{k+1,k*} \\
&= (1+\tilde{\mu}_{-1}\oplus\cdots\tilde{\mu}_k)_{|e_{k+1}}{e_{k+1}}^{-1}{F_{k-1,\mu}}^{-1}{}_*\pi'_{k+1,k*} \\
&= (1+\tilde{\mu})_{|e_{k+1}}{e_{k+1}}^{-1}\pi_{k+1,k*}{F_{k,\mu}}^{-1}{}_*
\end{aligned}$$

that is the deformed frame form is related to the frame from on M_{k+1} thanks to

$$F_{k,\mu}{}^*\theta^k_\mu = (1+\tilde{\mu})\circ\theta^k = \theta+\mu \qquad (38)$$

The deformed curvature is defined as

$$\Theta^{k-1}_\mu = d\theta^k_\mu + \frac{1}{2}[\theta^k_\mu,\theta^k_\mu] \mod \mathfrak{h}_{k-1}$$

and is null iff the frames $e_{k+1,\mu}$ are indeed jet frames (this being a consequence of section 2.4). Next, computing the deformed curvature from (38), we have :

$$F_{k,\mu}{}^*\Theta^{k-1}_\mu = \mathrm{d}(\theta+\mu)+\frac{1}{2}[\theta+\mu,\theta+\mu] \mod \mathfrak{h}_{k-1}$$

With all this in mind, we define the second Spencer operator as

$$D_\theta\mu = d(\theta+\mu)+\frac{1}{2}[\theta+\mu,\theta+\mu] \mod \mathfrak{h}_{k-1} \qquad (39)$$

Then, without anymore calculations, it becomes clear from the deformed frames point of vue that $D_\theta\mu$ is a $\mathfrak{gl}_{-1}\oplus\cdots\oplus\mathfrak{gl}_{k-1}$-valued tensorial 2-form on M_{k+1}, which is null iff the deformed frame bundle $M_{k+1,\mu}$ is actually the jet frame bundle M_{k+1}.

3.3.4 Extended diffeomorphisms action

We shall now derive, from the deformed bundle point of vue, the transformation of $\tilde{\mu}$ and μ under $\mathrm{Aut}(M_{k+1})$, that is, we explain where does come from the transformation law $\mu\to f_{k+1}{}^*\mu+Df_{k+1}$, equation (31).

Take $f_{k+1}\in\mathrm{Aut}(M_{k+1})$, denote by f_l its projections on $\mathrm{Aut}(M_l)$, and call $\tilde{\mu}'$ the transformed of $\tilde{\mu}$.

• To first order, we define $\tilde{\mu}'$ uniquely from :

$$f_0(e_0)\circ(1+\tilde{\mu}_{-1})^{-1}_{|f_0(e_0)} = f_{-1*}e_0\circ(1+\tilde{\mu}'_{-1})^{-1}_{|e_0}$$

Thus, we have :

$$(1+\tilde{\mu}'_{-1})_{|e_0} = (1+{f_0}^*\tilde{\mu}_{-1})_{|e_0} \circ ((f_0(e_0))^{-1} \circ f_{-1*}e_0)$$

Now, from the section point of vue, $D_\theta f_0$ is such that (see section 3.2.4) :

$$1+\tilde{D}_\theta f_0 = (f_0(e_0))^{-1} \circ f_{-1*}e_0 \tag{40}$$

so that we find

$$1+\tilde{\mu}'_{-1} = (1+{f_0}^*\tilde{\mu}_{-1}) \circ (1+\tilde{D}_\theta f_0) \tag{41}$$

In one word, we have constructed the commutative square

$$\begin{array}{ccc} M_0 & \xrightarrow{f_0} & M_0 \\ \Big\downarrow{\scriptstyle F_{-1,\mu'}} & & \Big\downarrow{\scriptstyle F_{-1,\mu}} \\ M_{0,\mu'} & \xrightarrow{f_{-1*}} & M_{0,\mu} \end{array}$$

since we have $F_{-1,\mu} \circ f_{0|e_0} = f_{0|e_0} \circ (1+\tilde{\mu}_{-1})_{|f_0(e_0)}$. We define the intertwining diffeomorphism

$$f_{0,\mu} = F_{-1,\mu} \circ f_0 \circ {F_{-1,\mu'}}^{-1}$$

as a useful object for later purpose.

• To second order, we define in the same way $\tilde{\mu}'$ from the commutative square (note the appearance of the intertwining diffeomorphism at this level)

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M_1 \\ \Big\downarrow{\scriptstyle F_{0,\mu'}} & & \Big\downarrow{\scriptstyle F_{0,\mu}} \\ M_{1,\mu'} & \xrightarrow{f_{0,\mu*}} & M_{1,\mu} \end{array}$$

that is :

$$F_{-1,\mu*}f_1(e_1) \circ (1+\tilde{\mu}_{-1} \oplus \tilde{\mu}_0)^{-1}_{|f_1(e_1)} = f_{0,\mu*}F_{-1,\mu'*}e_1 \circ (1+\tilde{\mu}'_{-1} \oplus \tilde{\mu}'_0)^{-1}_{|e_1}$$

This is equivalent, from the definition of $f_{0,\mu}$, to

$$f_1(e_1) \circ (1+\tilde{\mu}_{-1} \oplus \tilde{\mu}_0)^{-1}_{|f_1(e_1)} = f_{0*}e_1 \circ (1+\tilde{\mu}'_{-1} \oplus \tilde{\mu}'_0)^{-1}_{|e_1}$$

and, by the same reasoning as for the first order case, this proves

$$(1+\tilde{\mu}'_{-1} \oplus \tilde{\mu}'_0) = (1+{f_1}^*\tilde{\mu}_{-1} \oplus {f_1}^*\tilde{\mu}_0) \circ (1+\tilde{D}_\theta f_1)$$

Note that this is consistent with the first order result since this last equation implies, by invariance of μ_{-1} with respect to GL_1 and graded action of $1+\tilde{D}_\theta f_1$, $1+\tilde{\mu}'_{-1} = (1+{f_0}^*\tilde{\mu}_{-1})\circ(1+\tilde{D}_\theta f_0)$.

• Recursively, if we have defined the action at the M_l level, obtaining the commutative square

$$\begin{array}{ccc} M_l & \xrightarrow{f_l} & M_l \\ \downarrow{\scriptstyle F_{l-1,\mu'}} & & \downarrow{\scriptstyle F_{l-1,\mu}} \\ M_{l,\mu'} & \xrightarrow{f_{l-1,\mu*}} & M_{l,\mu} \end{array}$$

we define the intertwining diffeomorphism $f_{l,\mu} = F_{l-1,\mu}\circ f_l \circ {F_{l-1,\mu'}}^{-1}$, and $\tilde{\mu}'$ by the commutative square at next level

$$\begin{array}{ccc} M_{l+1} & \xrightarrow{f_{l+1}} & M_{l+1} \\ \downarrow{\scriptstyle F_{l,\mu'}} & & \downarrow{\scriptstyle F_{l,\mu}} \\ M_{l+1,\mu'} & \xrightarrow{f_{l,\mu*}} & M_{l+1,\mu} \end{array}$$

This means $F_{l,\mu}(f_{l+1}(e_{l+1})) = f_{l,\mu*}F_{l,\mu'}(e_{l+1})$, that is :

$$F_{l-1,\mu*}f_{l+1}(e_{l+1})\circ(1+\tilde{\mu}_{-1}\oplus\cdots\oplus\tilde{\mu}_l)^{-1}_{|f_{l+1}(e_{l+1})} = f_{l,\mu*}F_{l-1,\mu'*}e_{l+1}\circ(1+\tilde{\mu}'_{-1}\oplus\cdots\oplus\tilde{\mu}'_l)^{-1}_{|e_{l+1}}$$

and, thanks to the definition of $f_{l,\mu}$,

$$f_{l+1}(e_{l+1})\circ(1+\tilde{\mu}_{-1}\oplus\cdots\oplus\tilde{\mu}_l)^{-1}_{|f_{l+1}(e_{l+1})} = f_{l*}e_{l+1}\circ(1+\tilde{\mu}'_{-1}\oplus\cdots\oplus\tilde{\mu}'_l)^{-1}_{|e_{l+1}}$$

Now, using the fact

$$(1+\tilde{D}_\theta f_{l+1})_{|e_{l+1}} = (f_{l+1}(e_{l+1}))^{-1}\circ f_{l*}e_{l+1}$$

we obtain the transformation law

$$(1+\tilde{\mu}'_{-1}\oplus\cdots\oplus\tilde{\mu}'_l)_{|e_{l+1}} = (1+\tilde{\mu}_{-1}\oplus\cdots\oplus\tilde{\mu}_l)_{|f_{l+1}(e_{l+1})}\circ(1+\tilde{D}_\theta f_{l+1})_{|e_{l+1}}$$

Finally, we have obtained the action of $f_{k+1}\in \mathrm{Aut}(M_{k+1})$ on $\tilde{\mu}$ in the form :

$$1+\tilde{\mu}\to 1+\tilde{\mu}' = (1+{f_{k+1}}^*\tilde{\mu})\circ(1+\tilde{D}_\theta f_{k+1}) \qquad (42)$$

In the form language, from section 3.2.4, the equation (42) becomes

$$\theta+\mu' = {f_{k+1}}^*(\theta+\mu)$$

and we recover the transformation law

$$\mu \to \mu' = {f_{k+1}}^* \mu + D_\theta f_{k+1} \tag{43}$$

Note that the intertwining diffeomeorphisms $f_{k,\mu}$ not only depend on the transformation f_{k+1} but also on the deformation μ. Infinitesimally, this difference between f_{k+1} and $f_{k,\mu}$ is reflected, at least in 2D CFT, by 'field dependant ghosts' [6] originally introduced in [2].

3.3.5 Action of deformations on local fields

We now look for the action of deformations on local fields, in the same way as in section 3.2.4. For a deformation μ of M_{k+1}, as the deformed frame bundle $M_{k+1,\mu}$ is also principal, we can speak of the local fields on $M_{k+1,\mu}$, by doing the same construction as in section 3.1.1, with M_{k+1} replaced by $M_{k+1,\mu}$. We shall denote $S_{k,\mu}$ the deformed bundle

$$S'_{k,\mu} = M_{k+1,\mu} \times_{\overline{\mathrm{Ad}}} \mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$$

Take a local field $\alpha' \in \Omega^*(M, S_{k,\mu})$.

• From the tensorial form point of view, the action of a deformation is to read the form α' on M_{k+1} by pullback i.e. :

$$\alpha' \to \alpha = {F_{k,\mu}}^* \alpha' \tag{44}$$

This is consistent since $F_{k,\mu}$ is a principal bundle isomorphism.

• Viewing α' as a equivariant function $\tilde{\alpha}'$ on $M_{k+1,\mu}$, thanks to the formula

$$\alpha' = \tilde{\alpha}' \circ \theta_\mu$$

the transformation (44) now reads :

$$\tilde{\alpha}' \to \tilde{\alpha} = {F_{k,\mu}}^* \tilde{\alpha} \circ (1 + \tilde{\mu}) \tag{45}$$

This is obtained with a calculation similar to that establishing (29).

3.3.6 Synthesis

• We have obtained thus the operator $D_\theta : \Omega'^1(M, S_k) \to \Omega^2(M, S_{k-1})$, acting on deformations as

$$D_\theta \mu = d(\theta + \mu) + \frac{1}{2}[\theta + \mu, \theta + \mu] \mod \mathfrak{h}_{k-1}$$

We can alternatively write, using the structure equation $\Theta^{k-1} = 0$,

$$D_\theta \mu = d_\theta \mu + \frac{1}{2}[\mu, \mu] \mod \mathfrak{h}_{k-1} \tag{46}$$

This is the definition of D_θ we will take.
More generally, for ω a $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_k$-valued 1-form on M_{k+1} satisfying properties $(i), (ii), (iii)$ of a deformation (see section 3.3.1), we define

$$D_\omega \mu = d_\omega \mu + \frac{1}{2}[\mu, \mu] \mod \mathfrak{h}_{k-1}$$

and, for technical purpose

$$\underline{D}_\omega \mu = d(\omega + \mu) + \frac{1}{2}[\omega + \mu, \omega + \mu] \mod \mathfrak{h}_{k-1}$$

Note that we then have

$$\underline{D}_\omega \mu = d_\omega \omega + D_\omega \mu = d\omega + \frac{1}{2}[\omega, \omega] + D_\omega \mu \mod \mathfrak{h}_{k-1} \tag{47}$$

Then $D_\omega \mu$ and $\underline{D}_\omega \mu$ are still tensorial i.e. in $\Omega^2(M, S_{k-1})$. We shall call the quantities $D_\omega \mu$ and $\underline{D}_\omega \mu$ torsion or curvature, as these concepts are not to be distinguished in Cartan geometry. These definitions can also be used on any of the deformed frame bundles.
As for the symmetries, the properties of D_θ are summarised in :

D_θ defines a cocycle on the space of deformations $\Gamma(\mathcal{B}_k) \simeq \Omega'^1(M, S_k)$, seen as a group, with values in $\Omega^2(M, S_{k-1})$ (see eq. (49) hereafter for the explicit cocycle law). Its kernel contains the deformations induced by $\mathrm{Aut}(M_{k+1})$, that is $D_\theta \mu = 0$, for $\mu = D_\theta f_{k+1}$.

• We prove first the cocycle property. To a deformation μ seen as an equivariant function $\tilde{\mu}$, we associate the deformation $\tilde{\mu}' = \tilde{\mu} \circ {F_{k,\nu}}^{-1}$ on the deformed frame bundle $M_{k+1,\nu}$, and so the corresponding $\mu' = \tilde{\mu} \circ \theta_\nu$.
Then, from the point of vue of $M_{k+1,\nu}$, the curvature is obtained as :

$$\underline{D}_{\theta_\nu} \mu' = d(\theta_\nu + \mu') + \frac{1}{2}[\theta_\nu + \mu', \theta_\nu + \mu'] \mod \mathfrak{h}_{k-1} \tag{48}$$

Next, this curvature form is read on M_{k+1} via the pullback ${F_{k,\nu}}^* \underline{D}_{\theta_\nu} \mu'$. We have :

$$\begin{aligned} {F_{k,\nu}}^*(\underline{D}_{\theta_\nu} \mu') &= {F_{k,\nu}}^* \left(d(\theta_\nu + \mu') + \frac{1}{2}[\theta_\nu + \mu', \theta_\nu + \mu'] \right) \mod \mathfrak{h}_{k-1} \\ &= d(\theta + \nu + {F_{k,\nu}}^* \mu') + \frac{1}{2}[\theta + \nu + {F_{k,\nu}}^* \mu', \theta + \nu + {F_{k,\nu}}^* \mu'] \mod \mathfrak{h}_{k-1} \end{aligned}$$

Now, ${F_{k,\nu}}^*\mu'$ is the deformation μ deformed by ν, since (compare with equation (44))

$$
\begin{aligned}
{F_{k,\nu}}^*\mu' &= \mu' \circ F_{k,\nu *} \\
&= \tilde{\mu}' \circ \theta_\nu \circ F_{k,\nu *} \\
&= \tilde{\mu} \circ {F_{k,\nu}}^*\theta_\nu \\
&= \tilde{\mu} \circ (\theta + \nu) \\
&= \mu + i_\nu \mu
\end{aligned}
$$

So, we obtain :

$$\underline{D_\theta}(\mu.\nu) = {F_{k,\nu}}^* \underline{D_{\theta_\nu}} \mu'$$

Then, this last equation can be rewritten thanks to (47) as a cocycle law for D_θ (recall the action of deformations (44)) :

$$D_\theta(\mu.\nu) = {F_{k,\nu}}^* D_{\theta_\nu} \mu' + D_\theta \nu \tag{49}$$

• Now, we prove the nilpotency. For $\mu = D_\theta f_{k+1}$, we have thanks to the structure equation

$$
\begin{aligned}
D_\theta D_\theta f_{k+1} &= d({f_{k+1}}^*\theta^k) + \frac{1}{2}[{f_{k+1}}^*\theta^k, {f_{k+1}}^*\theta^k] \quad \text{mod } \mathfrak{h}_{k-1} \\
&= {f_{k+1}}^*\Theta^{k-1} \\
&= 0
\end{aligned}
$$

■

• All this is summarised in the sequence :

$$\mathrm{Aut}(M_{k+1}) \xrightarrow{D_\theta} \Omega'^1(M, S_k) \xrightarrow{D_\theta} \Omega^2(M, S_{k-1}) \tag{50}$$

We have, as stated in [9] :

The non linear complex (50) is locally exact i.e. on a suitable open cover (U_i) of M, the equation $D_\theta \mu = 0$ on U_i implies

$$\mu = D_\theta f_{k+1,i} \ \text{ for } \ f_{k+1,i} \in \mathrm{Aut}(U_{i,k+1})$$

A proof of this in local coordinate form is given in [9]. Here, we shall indicate another way to see this, using Cartan geometry [10]. We work on a chart (U_i, φ_i) of M, with (invertible) maps $\varphi_i : U_i \to \mathbb{R}^n$, and U_i contractible. Thanks to the 'fundamental theorem of calculus' of [10], the condition $D_\theta \mu = 0$, written $d\omega + \frac{1}{2}[\omega, \omega] = 0 \mod \mathfrak{h}_{k-1}, \ , \omega = \theta + \mu$, proves that there exists locally on U_i, a map

$$\phi_{k+1,i} : U_{i,k+1} \to {\mathbb{R}^n}_{k+1}$$

such that

$$\omega = \theta + \mu = {\phi_{k+1,i}}^* \tilde{\theta}$$

where $\tilde{\theta}$ is the frame form on ${\mathbb{R}^n}_{k+1}$. Then, equivariance of $\theta + \mu$ and $\tilde{\theta}$, and evaluation on frames, proves that $\phi_{k+1,i}$ is indeed a principal bundle isomorphism, locally defined above U_i.

Moreover, the prolongation $\varphi_{k+1,i} : U_{i,k+1} \to {\mathbb{R}^n}_{k+1}$, which satisfies by construction ${\varphi_{k+1,i}}^* \tilde{\theta} = \theta$, enables us to define

$$f_{k+1,i} = {\varphi_{k+1,i}}^{-1} \circ \phi_{k+1,i}$$

such that $f_{k+1,i} \in \mathrm{Aut}(M_{k+1})$. In this way, we obtain :

$$\begin{aligned} \mu &= {\phi_{k+1,i}}^* \tilde{\theta} - \theta \\ &= (\varphi_{k+1,i} \circ f_{k+1,i})^* \tilde{\theta} - \theta \\ &= {f_{k+1,i}}^* {\varphi_{k+1,i}}^* \tilde{\theta} - \theta \\ &= {f_{k+1,i}}^* \theta - \theta \\ &= D_\theta f_{k+1,i} \end{aligned}$$

This means the sequence (50) is locally exact at $\Omega'^1(M, S_k)$. This construction can be summarised in the commutative square where each arrow is a principal bundle morphism :

$$\begin{array}{ccc} U_{i,k+1} & \xrightarrow{\phi_{k+1,i}} & {\mathbb{R}^n}_{k+1} \\ \downarrow{\scriptstyle f_{k+1,i}} & & \| \\ U_{i,k+1} & \xrightarrow{\varphi_{k+1,i}} & {\mathbb{R}^n}_{k+1} \end{array}$$

The map $\phi_{k+1,i}$ is a development map [10], here adapted to the diffeomorphism symmetry.

■

3.4 Synthesis

3.4.1 Symmetries and deformations

• The study of symmetries and deformations in the language of linear frames reveals that they have the same structure, as shown in the covariant and commutative diagrams :

$$\begin{array}{ccccccccccc} M_{k+1} & \longrightarrow & M_k & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & M_{-1} \\ \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f_k} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} \\ M_{k+1} & \longrightarrow & M_k & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & M_{-1} \end{array}$$

for the symmetries, and similarly

$$\begin{array}{ccccccccccc} M_{k+1} & \longrightarrow & M_k & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0 & \longrightarrow & M_{-1} \\ \downarrow{\scriptstyle F_{k,\mu}} & & \downarrow{\scriptstyle F_{k-1,\mu}} & & & & \downarrow{\scriptstyle F_{0,\mu}} & & \downarrow{\scriptstyle F_{-1,\mu}} & & \downarrow{\scriptstyle \mathrm{id}} \\ M_{k+1,\mu} & \longrightarrow & M_{k,\mu} & \longrightarrow & \cdots & \longrightarrow & M_{1,\mu} & \longrightarrow & M_{0,\mu} & \longrightarrow & M_{-1} \end{array}$$

for the deformations. From a gravity point of vue, the similarity between these two structures is natural as one can understand them in term of a generalised equivalence principle : the *gravitational fields* μ of the second diagram are 'locally' equivalent, i.e. in fact at the level of jets (see section 3.3.1), to the *general changes of coordinate frame* f_{k+1} of the first diagram (see e.g. [7] for the use of Cartan geometry in gravity). Alternatively, one can also think of the deformations μ as generalised Beltrami differentials [6], the equation $\mu = D_\theta f_{k+1}$ being then a generalised Beltrami equation, with integrability conditions $D_\theta \mu = 0$. This fact will be further studied elsewhere. The interesting fact here is that both symmetries, i.e. $\mathrm{Aut}(M_{k+1})$, and fields, i.e. deformations $\Omega'^1(M, S_k)$, appear on the same footing.

• Alternatively, as $\mathrm{Aut}(M_{k+1})$ acts on $\Omega'^1(M, S_k) \simeq \Gamma(\mathcal{B}_k)$, and as $\Omega'^1(M, S_k)$ is a group, we can consider the group semi-direct product

$$\Omega'^1(M, S_k) \rtimes \mathrm{Aut}(M_{k+1}) \tag{51}$$

as encoding the preceding two diagrams in a unified manner. The group law is explicitly given by

$$(\mu, f_{k+1}).(\mu', f'_{k+1}) = (\mu.(f_{k+1}.\mu'), f'_{k+1} \circ f_{k+1}) \tag{52}$$

In this equation, $f.\mu' = {f_{k+1}}^* \mu' + D_\theta f_{k+1}$ is the (right) action of f_{k+1} on μ', and $\mu.\nu$ with $\nu = f_{k+1}.\mu'$ denotes the composition of deformations. We have $f'_{k+1} \circ f_{k+1}$ on the r.h.s. because of pull-back law.

This structure is roughly speaking some non linear analogue to the one in [8] used for treating diffeomorphisms. Maybe one could use this to derive, as in [8], some cohomological structure related to the BRS one. In this respect, as it is natural to view the space $\Omega'^1(M,S_k)$ as a classifying space for $\mathrm{Aut}(M_{k+1})$ by analogy with gauge theory, we can also view the product (51) as giving rise to the equivariant cohomology type quotient :

$$\mathrm{Aut}(M_{k+1}) \longrightarrow \mathrm{Aut}(M_{k+1}) \times \Omega'^1(M,S_k) \longrightarrow \mathrm{Aut}(M_{k+1}) \times_{\mathrm{Aut}(M_{k+1})} \Omega'^1(M,S_k)$$

where $\mathrm{Aut}(M_{k+1})$ acts on both sides of the product as in (52) with $\mu = 0$.

3.4.2 Non linear Spencer sequences

• The two sequences (28) and (50) enable us to construct the non linear Spencer sequence of [9] as :

$$\mathrm{id} \longrightarrow \mathrm{Aut}(M) \xrightarrow{j_{k+1}} \mathrm{Aut}(M_{k+1}) \xrightarrow{D_\theta} \Omega'^1(M,S_k) \xrightarrow{D_\theta} \Omega^2(M,S_{k-1}) \longrightarrow 0 \qquad (53)$$

This sequence is then globally exact at $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M_{k+1})$, and locally exact at $\Omega'^1(M,S_k)$. This sequence embodies all the structure necessary for gravity theories : from left to right, we have the base space symmetry, then the frame space symmetry, then the gravity potentials (deformations), and finally the gravity field strenghts (curvatures).
For any deformation μ, we also have Bianchi type identities in the form

$$\begin{aligned} d_{\theta+\mu} D_\theta \mu &= d_{\theta+\mu} d_{\theta+\mu}(\theta+\mu) \\ &= 0 \end{aligned}$$

This fact indicates that if we want to prolongate the non linear Spencer sequence (53) we have to intertwine the differential operators involved with μ fields, such as $d_{\theta+\mu} : \Omega^2(M,S_{k-1}) \to \Omega^3(M,S_{k-2})$ here. This means one cannot extend the non linear Spencer sequence to forms of degree > 2 without introducing more fields, in analogy with the fact that one cannot extend non abelian Čech sequences (see section 3.4.3) to cochains of degree > 2 without introducing, e.g., gerbes.

We now study the covariance properties of the subsequences (28) and (50), this will give rise to a refined version of (53), called second Spencer sequence in [9].

• First, we study the covariance of (28) with respect to the structure group $\mathcal{GL}_{k+1}$ of the principal bundle

$$\mathcal{GL}_{k+1} \longrightarrow \mathrm{Aut}(M_{k+1}) \longrightarrow \mathrm{Aut}(M_k)$$

For a gauge transformation $g_{k+1} \in \mathcal{GL}_{k+1}$ ($k > -1$ otherwise we get nothing), we have

$$\begin{aligned} D_\theta(f_{k+1} \circ g_{k+1}) &= {g_{k+1}}^* D_\theta f_{k+1} + D_\theta g_{k+1} \\ &= \overline{\mathrm{Ad}}(\tilde{g}_{k+1}^{-1}) D_\theta f_{k+1} + \overline{\mathrm{Ad}}(\tilde{g}_{k+1}^{-1})\theta^k - \theta^k \\ &= \overline{\mathrm{Ad}}(\tilde{g}_{k+1}^{-1}) D_\theta f_{k+1} + \tilde{\mu}_k \circ \theta^k \\ &= D_\theta f_{k+1} + \tilde{\mu}_k \circ {f_{k+1}}^* \theta^k \end{aligned} \tag{54}$$

where $\tilde{g}_{k+1}$ is the equivariant function corresponding to g_{k+1}, and $\tilde{\mu}_k$ is the section of $M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1} \subset M_{k+1} \times_{\overline{\mathrm{Ad}}} \mathfrak{gl}_{k,1}$ such that at each point e_{k+1} (see sections 1.2 and 3.3.1): $\overline{\mathrm{Ad}}(\tilde{g}_{k+1})X = X - \tilde{\mu}_k X_{-1}$ (the minus sign is taken because the gauge transformation $\tilde{g}_{k+1}$ is the particular deformation $(1 + \tilde{\mu}_k)^{-1} = 1 - \tilde{\mu}_k$ for $k > -1$), with $\partial \tilde{\mu}_k = 0$ i.e. $\tilde{\mu}_{k|e_{k+1}} \in \mathfrak{gl}_{k+1} \simeq GL_{k+1}$. The covariance law (54), which is just the composition of the deformations $\tilde{D}_\theta f_{k+1}$ and $\tilde{\mu}_k$, is rewritten from the equivariant viewpoint as :

$$\tilde{D}_\theta(f_{k+1} \circ g_{k+1}) = \tilde{D}_\theta f_{k+1} + \tilde{\mu}_k \circ (1 + \tilde{D}_\theta f_{k+1}) \tag{55}$$

where all quantities are evaluated at the same e_{k+1}, contrary to equation (32). This suggests to define the quotient bundle

$$\overline{\mathcal{B}}_k = \mathcal{B}_k / (M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1}) \simeq M_{k+1} \times_{\overline{\mathrm{Ad}}} (B_k / \mathfrak{gl}_{k+1})$$

where GL^{k+1} acts naturally on $B_k/\mathfrak{gl}_{k+1}$, and denote by $\Omega'^1(M, \overline{S}_k)$ its space of sections, which satisfies

$$\Omega'^1(M, \overline{S}_k) = \Omega'^1(M, S_k) / \Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1})$$

Note that, at the fiber level we have $B_k/\mathfrak{gl}_{k+1} \simeq B_{k-1} \ltimes (\mathfrak{gl}_{k,1}/\mathfrak{gl}_{k+1})$. The calculations above then show that the operator

$$\begin{aligned} \overline{D}_\theta \ &: \ \mathrm{Aut}(M_k) \to \Omega'^1(M, \overline{S}_k) \\ & f_k \mapsto \overline{D}_\theta f_k = \tilde{D}_\theta f_{k+1} \circ (1 + \tilde{D}_\theta f_{k+1})^{-1} \circ \theta \quad \mathrm{mod}\ \Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1}) \end{aligned}$$

is well defined for any f_{k+1} above f_k.
Note that the projection map $\Omega'^1(M, S_k) \to \Omega'^1(M, \overline{S}_k)$ is, in relation with the definition of $\overline{D}_\theta$,

$$\tilde{\mu} \to \tilde{\mu}(1 + \tilde{\mu})^{-1} \quad \mathrm{mod}\ \mathfrak{gl}_{k+1}$$

as we have (see section 3.3.1) $\tilde{\mu} = [\tilde{D}_\theta f_{k+1}]$, and the action of $\tilde{\nu}_k \in \Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1})$ on $\Omega'^1(M, S_k)$ defining the quotient is :

$$\tilde{\mu} \to \tilde{\mu} + \tilde{\nu}_k \circ (1 + \tilde{\mu})$$

which keeps invariant the class of μ.
We can summarise this construction in the exact commutative diagram

$$\begin{array}{ccccc}
\mathrm{id} & \longrightarrow & \mathcal{GL}_{k+1} & \xrightarrow{D_\theta \simeq \overline{\mathrm{Ad}}} & \Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Aut}(M) & \xrightarrow{j_{k+1}} & \mathrm{Aut}(M_{k+1}) & \xrightarrow{D_\theta} & \Omega'^1(M, S_k) \\
\| & & \downarrow & & \downarrow \\
\mathrm{Aut}(M) & \xrightarrow{j_k} & \mathrm{Aut}(M_k) & \xrightarrow{\overline{D}_\theta} & \Omega'^1(M, \overline{S}_k)
\end{array}$$

where the first line corresponds to the covariance law under $\mathcal{GL}^{k+1}$ and the central row encodes the symmetry we started from. This results in the sequence of the last line, which is the projected version of (28). By construction, we then end with the exact sequence :

$$\mathrm{id} \longrightarrow \mathrm{Aut}(M) \xrightarrow{j_k} \mathrm{Aut}(M_k) \xrightarrow{\overline{D}_\theta} \Omega'^1(M, \overline{S}_k)$$

• Second, we study the covariance of (50) with respect to the structure group $\Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1})$ of the principal bundle (which is the third row of the preceding diagram):

$$\Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1}) \longrightarrow \Omega'^1(M, S_k) \longrightarrow \Omega'^1(M, \overline{S}_k)$$

and more generally under the group $\Gamma(M_{k+1} \times_{\overline{\mathrm{Ad}}} \mathfrak{gl}_{k,1}) \subset \Omega'^1(M, S_k) \simeq \Gamma(\mathcal{B}_k)$. Inspired by the preceding point, the action of a maximal degree deformation $\nu_k \in \Gamma(M_{k+1} \times_{\overline{\mathrm{Ad}}} \mathfrak{gl}_{k,1})$ for $k > -1$ is given by :

$$\tilde{\mu} \to \tilde{\mu} + \tilde{\nu}_k \circ (1 + \tilde{\mu})$$

that is

$$\mu \to \mu + \nu_k + i_\mu \nu_k$$

in form language. Next, a direct calculation gives (this is another version of the cocycle law (49))

$$\begin{aligned}
D_\theta(\mu + \nu_k + i_\mu \nu_k) &= D_\theta \mu + [\theta + \mu, \tilde{\nu}_k \circ (\theta + \mu)] \mod \mathfrak{h}_{k-1} \\
&= D_\theta \mu + \partial \tilde{\nu}_k \circ (\theta + \mu)
\end{aligned} \tag{56}$$

From the equivariant viewpoint, the covariance law (56) reads (compare with equation (55))

$$\begin{aligned}
\tilde{D}_\theta(\mu . \nu_k) &= \tilde{D}_\theta \mu + \partial \tilde{\nu}_k \circ (1 + \tilde{\mu}) \\
&= \tilde{D}_\theta \mu + \partial \tilde{\nu}_k \circ (1 + \tilde{\mu}_{-1})
\end{aligned} \tag{57}$$

This suggests to define the quotient bundle

$$\Lambda^2(M, S_{k-1})/(M_k \times_{\overline{\mathrm{Ad}}} \partial\mathfrak{gl}_{k,1}) \simeq M_k \times_{\overline{\mathrm{Ad}}} (\mathfrak{gl}_{-1,2} \oplus \cdots \oplus \mathfrak{gl}_{k-1,2})/\partial\mathfrak{gl}_{k,1}$$

whose space of sections, denoted $\Omega^2(M, \overline{S}_{k-1})$, satisfies :

$$\Omega^2(M, \overline{S}_{k-1}) = \Omega^2(M, S_{k-1})/\Gamma(M_k \times_{\overline{\mathrm{Ad}}} \partial\mathfrak{gl}_{k,1})$$

The preceding calculations then proves that if ν_k is a deformation in the structure group $\Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1})$, i.e. $\partial\tilde{\nu}_k = 0$, then $D_\theta\mu$ is left invariant under its action, and that the operator

$$\begin{aligned} \overline{D}_\theta \quad : \quad & \Omega'^1(M, \overline{S}_k) \to \Omega^2(M, \overline{S}_{k-1}) \\ & \overline{\mu} \mapsto \overline{D}_\theta\overline{\mu} = \tilde{D}_\theta\mu \circ (1+\tilde{\mu})^{-1} \circ \theta \mod \Gamma(M_k \times_{\overline{\mathrm{Ad}}} \partial\mathfrak{gl}_{k,1}) \end{aligned}$$

is well defined for any μ above $\overline{\mu} \in \Omega'^1(M, \overline{S}_k)$.

The construction is summarised in the commutative diagram

$$\begin{array}{ccccc} \mathcal{GL}_{k+1} & \xrightarrow[\simeq]{D_\theta \simeq \overline{\mathrm{Ad}}} & \Gamma(M_{k+1} \times_{\mathrm{Ad}} \mathfrak{gl}_{k+1}) & \xrightarrow{D_\theta \simeq \partial} & \Gamma(M_k \times_{\overline{\mathrm{Ad}}} \partial\mathfrak{gl}_{k,1}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Aut}(M_{k+1}) & \xrightarrow{D_\theta} & \Omega'^1(M, S_k) & \xrightarrow{D_\theta} & \Omega^2(M, S_{k-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Aut}(M_k) & \xrightarrow{\overline{D}_\theta} & \Omega'^1(M, \overline{S}_k) & \xrightarrow{\overline{D}_\theta} & \Omega^2(M, \overline{S}_{k-1}) \end{array}$$

where the first line corresponds to the covariance under maximal degree deformations, and the second row encodes the symmetry we started from. This gives the projected version of (50), that is the sequence :

$$\mathrm{Aut}(M_k) \xrightarrow{\overline{D}_\theta} \Omega'^1(M, \overline{S}_k) \xrightarrow{\overline{D}_\theta} \Omega^2(M, \overline{S}_{k-1})$$

• Putting things altogether, we obtain thus the non linear second Spencer sequence :

$$\mathrm{id} \longrightarrow \mathrm{Aut}(M) \xrightarrow{j_k} \mathrm{Aut}(M_k) \xrightarrow{\overline{D}_\theta} \Omega'^1(M, \overline{S}_k) \xrightarrow{\overline{D}_\theta} \Omega^2(M, \overline{S}_{k-1})$$

This is the projected form of (53).

• Finally, note that the linearised version of the first Spencer sequence (53) is (we still denote j_{k+1} the linearised version)

$$0 \longrightarrow \mathrm{aut}(M) \xrightarrow{j_{k+1}} \mathrm{aut}(M_{k+1}) \xrightarrow{d_\theta} \Omega^1(M, S_k) \xrightarrow{d_\theta} \Omega^2(M, S_{k-1})$$

$\mathrm{aut}(M) \simeq \Gamma(TM)$ is the Lie algebra of $\mathrm{Aut}(M)$ i.e. the vector fields on M which satisfies

$$\mathrm{aut}(M) \simeq \Omega^0(M, S_{-1})$$

$\mathrm{aut}(M_{k+1})$ is the Lie algebra of $\mathrm{Aut}(M_{k+1})$ i.e. the right invariant vector fields on M_{k+1} which satisfies :

$$\mathrm{aut}(M_{k+1}) \simeq \Omega^0(M, S_{k+1})$$

So, this linearised sequence contains the beginning of the linear sequence (23). Putting these together, we obtain the linear Spencer sequence :

$$0 \longrightarrow \Omega^0(M, S_{-1}) \xrightarrow{j_{k+1}} \Omega^0(M, S_{k+1}) \xrightarrow{d_\theta} \Omega^1(M, S_k) \xrightarrow{d_\theta}$$
$$\cdots \longrightarrow \Omega^n(M, S_{k+1-n}) \xrightarrow{d_\theta} 0$$

This sequence is locally exact [9].

3.4.3 Lagrangian and Čech formulations

• On the differentiable n-manifold M, we consider the lagrangian

$$\mathcal{L}(\beta, \mu) = \mathrm{tr}\ \beta \wedge D_\theta \mu \tag{58}$$

for $\mu \in \Omega'^1(M, S_k)$, and $\beta \in \Omega^{n-2}(M, S^*_{k-1})$. tr is the coupling between $\mathfrak{gl}_{-1} \oplus \cdots \oplus \mathfrak{gl}_{k-1}$ and its dual, and S^*_{k-1} is the dual vector bundle of S_{k-1}. This lagrangian is analogue to the *bc* models of 2D CFT and to the BF models of gauge theory [4].

The lagrangian $\mathcal{L}$ has $\mathrm{Aut}(M_{k+1})$ symmetry :

$$\mu \to {f_{k+1}}^*\mu + D_\theta f_{k+1},\ \ \beta \to {f_{k+1}}^*\beta \implies {f_{k+1}}^*\mathcal{L} = \mathcal{L} \tag{59}$$

since $D_\theta\mu \to {f_{k+1}}^* D_\theta \mu$ under the action of $f_{k+1} \in \mathrm{Aut}(M_{k+1})$. The equations of motions are :

$$D_\theta \mu = 0,\ \ d^*_{\theta+\mu}\beta = 0 \tag{60}$$

Here the dual d^*_ω of d_ω, $\omega = \theta + \mu$, is defined by

$$d\,\mathrm{tr}\ \beta \wedge \alpha = \mathrm{tr}\ d^*_\omega \beta \wedge \alpha + (-1)^{n-2}\mathrm{tr}\ \beta \wedge d_\omega \alpha$$

for all $\alpha \in \Omega^1(M, S_k)$.

We see that (60) corresponds to the fact that the lagrangian $\mathcal{L}$ computes non linear Spencer cocycles and (59) corresponds to the covariance property of the non linear Spencer sequence under $\mathrm{Aut}(M_{k+1})$. Both combined proves that $\mathcal{L}$ is indeed computing non linear Spencer cohomology at the $\Omega'^1(M, S_k)$ level. Of course, one can similarly define a lagrangian model relative to the linear Spencer sequence.

• Now, we shall end by a calculation emphasizing the analogy between $\mathcal{L}$ and BF gauge theory models [4], that is between k-frames and gauge theory.
Either from the lagrangian, or from the Spencer sequence point of vue, the equation of motion for the deformation

$$D_\theta \mu = 0$$

is locally solved by

$$\mu = D_\theta f_{k+1,i} \tag{61}$$

for $f_{k+1,i} \in \mathrm{Aut}(U_{i,k+1})$ above a open subset $U_i \subset M$. The U_i's are chosen as in section 3.3.6. As μ is globally defined, equation (61) implies that, above $U_{ij} = U_i \cap U_j$, we have $D_\theta f_{k+1,i} = D_\theta f_{k+1,j}$, so the element $f_{k+1,ij} = f_{k+1,i} \circ {f_{k+1,j}}^{-1} \in \mathrm{Aut}(U_{ij,k+1})$ satisfies, thanks to the cocycle property of D_θ :

$$D_\theta f_{k+1,i} = D_\theta(f_{k+1,ij} \circ f_{k+1,j}) = {f_{k+1,j}}^* D_\theta f_{k+1,ij} + D_\theta f_{k+1,j} \implies D_\theta f_{k+1,ij} = 0$$

so we have $f_{k+1,ij} = j_{k+1}(f_{-1,ij})$ (exactness of (53)) where $f_{-1,ij} = f_{ij}$ is a diffeomorphism of U_{ij}. Next, we also have

$$f_{k+1,ij} \circ f_{k+1,jk} \circ f_{k+1,ki} = \mathrm{id} \ , \ \text{above } U_{ijk} = U_i \cap U_j \cap U_k$$

so, as j_{k+1} is a morphism,

$$j_{k+1}(f_{ij} \circ f_{jk} \circ f_{ki}) = \mathrm{id} \ , \ \text{above } U_{ijk}$$

Now, as j_{k+1} is injective (exactness of (53) again), this last equality is equivalent to

$$f_{ij} \circ f_{jk} \circ f_{ki} = \mathrm{id} \ , \ \text{on } U_{ijk}$$

Consequently we have associated to μ a Čech 1-cocycle (f_{ij}) with values in the diffeomorphisms of M.

Note that the same type of calculation proves that $f_{k+1,i}$ is defined up to the transformation

$$f_{k+1,i} \to j_{k+1}(f'_{-1,i}) \circ f_{k+1,i} \ , \ \text{for } f'_{-1,i} \in \mathrm{Aut}(U_i)$$

because of the cocycle property :

$$D_\theta(j_{k+1}(f'_{-1,i}) \circ f_{k+1,i}) = f_{k+1,i}{}^* D_\theta(j_{k+1}(f'_{-1,i})) + D_\theta f_{k+1,i} = D_\theta f_{k+1,i}$$

Under such a transformation, the Čech cochains transform as

$$\begin{array}{rcl} f_{k+1,ij} & \to & j_{k+1}(f'_{-1,i}) \circ f_{k+1,ij} \circ j_{k+1}(f'_{-1,j}{}^{-1}) \\ f_{-1,ij} & \to & f'_{-1,i} \circ f_{-1,ij} \circ f'_{-1,j}{}^{-1} \end{array}$$

These covariance properties are the Čech version of the covariance under $\mathrm{Aut}(M)$ of the non linear Spencer sequence, or alternatively of the space (51).

• All these facts suggest that the (differential) cohomology of the non linear Spencer sequence is related to the (combinatorial and non abelian) cohomology of diffeomorphisms Čech type sequences. Recall what are the Čech cochains for the diffeomorphisms. 0-cochains are $(f_i) \in C^0(\mathrm{Aut}(M))$ where f_i is a diffeomorphism of U_i, 1-cochains are $(f_{ij}) \in C^1(\mathrm{Aut}(M))$ where f_{ij} is a diffeomorphism of U_{ij} with $f_{ji} = f_{ij}{}^{-1}$, and 2-cochains are $(f_{ijk}) \in C^2(\mathrm{Aut}(M))$ where f_{ijk} is a diffeomorphism of U_{ijk}. The Čech differential δ is defined as usual, respectively on 0-cochains and 1-cochains by :

$$\begin{array}{rcl} (\delta f)_{ij} & = & f_i \circ f_j{}^{-1} \\ (\delta f)_{ijk} & = & f_{ij} \circ f_{jk} \circ f_{ki} \end{array}$$

With this, using holonomy/homotopy type arguments, we expect that the cohomology of the Čech sequence (the second arrow being the restriction map)

$$\mathrm{id} \longrightarrow \mathrm{Aut}(M) \longrightarrow C^0(\mathrm{Aut}(M)) \xrightarrow{\delta} C^1(\mathrm{Aut}(M)) \xrightarrow{\delta} C^2(\mathrm{Aut}(M))$$

is isomorphic to the Spencer non linear cohomology.

• Of course, the interest in the lagrangian $\mathcal{L}$ is as limited as those of BF type in gauge theory : it only encodes topological information on the space M equiped with a background differential structure. Nevertheless, we formally expect, as in [4] for gauge theory, that the quantum theory corresponding to $\mathcal{L}$ is encoded in some sort of non abelian intersection theory between 1-cycles (sources of the μ field) and $(n-2)$-cycles (sources of the β field) in M, the cycles being here understood in the sense of some non abelian singular homology.

• The theory of linear frames, in all the aspects described here, as well as another ones like e.g. flag structures [3], can be modified (reduction of frame bundles) or extended (definition of graded type frames) to embody all kind of gravitational type structures. The gravitationnal field is then a Cartan connection, [3, 5, 7, 10], which can be thought as a μ field, or the inverse of some k-frame, with $k = 2$ for Riemannian gravity, $k = 3$ for conformal [7] or projective gravity, $k = \infty$ for Kodaira-Spencer gravity [6].

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