Partitioning powers of traceable or hamiltonian graphs*

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Abstract

A graph $G = (V, E)$ is arbitrarily partitionable (AP) if for any sequence $\tau = (n_1, \ldots, n_p)$ of positive integers adding up to the order of G, there is a sequence of mutually disjoints subsets of V whose sizes are given by τ and which induce connected graphs. If, additionally, for given k, it is possible to prescribe $l = \min\{k, p\}$ vertices belonging to the first l subsets of τ, G is said to be AP+k.

The paper contains the proofs that the k^{th} power of every traceable graph of order at least k is AP+$(k-1)$ and that the k^{th} power of every hamiltonian graph of order at least $2k$ is AP+$(2k-1)$, and these results are tight.

Keywords: arbitrarily partitionable graph, power of a graph, hamiltonian graph, traceable graph.

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1 Introduction

Consider a simple graph $G = (V, E)$ of order n. A sequence $\tau = (n_1, \ldots, n_p)$ of positive integers is called *admissible* for G if it is a *partition* of n, *i.e.*, $n_1 + \ldots + n_p = n$. If additionally there exists a partition $(V_1, \ldots, V_p)$ of the vertex set V such that each V_i induces a connected subgraph of order n_i in G, then we say that τ is *realizable* in G, while $(V_1, \ldots, V_p)$ is called a *realization* of τ in G. If every admissible sequence is also realizable in G, then we say that this graph is *arbitrarily partitionable* (or *arbitrarily vertex decomposable*) and we call it an *AP graph* for short.

The notion of AP graphs was first introduced by Barth, Baudon and Puech in [2], and motivated by the following problem in computer science. Consider a network connecting different computing resources; such a network is modelled by a graph. Suppose there are p different users, where the i^{th} one needs n_i resources from our network. The subgraph induced by the set of resources attributed to each user should be connected and each resource should be attributed to one user. So we are seeking a realization of the sequence $\tau = (n_1, ..., n_p)$ in this graph. Suppose that we want to do it for any number of users and any sequence of request. Thus, such a network should be an AP graph.

Independently (see [7] or [9]), this problem was also considered as a natural analogy of the similar notion in which vertices are replaced by edges (see for instance [1] or [8]).

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers. Obviously, a graph needs to be connected in order to be AP. The investigation of AP trees gained lots of attention in this context, since a connected graph is AP if one of its spanning trees is AP. It turned out, however, that the structure of AP trees is not obvious in general (see for instance [3], [4], [5] or [14]).

Since each traceable (*i.e.* containing a hamiltonian path) graph is evidently AP, each condition implying the existence of a hamiltonian path in a graph also implies that the graph is AP. So, AP graphs may be considered as a generalization of traceable (or hamiltonian) graphs (see for instance [10]).

Suppose now that as managers of the computer network we have a number of at most k specially privileged clients (users), so called *vip's*, each of whom may choose one computing resource which must be attributed to their connected subnetwork. It might be a powerful or conveniently located computer, which may serve our vip as an administrative center for managing the subnetwork. Then we naturally obtain the following modification of our model on graphs: Let $G = (V, E)$ be a graph of order n and let $n > k$. The graph G is said to be *AP+k* if for any partition $\tau = (n_1, \ldots, n_p)$ of

n and any sequence $(u_1,\ldots,u_{k'})$ of k' pairwise distinct vertices of G with $k' \leq \min\{k,p\}$, there exists a realization $(V_1,\ldots,V_p)$ of τ in G such that $u_1 \in V_1,\ldots,u_{k'} \in V_{k'}$.

Observe that we have adopted the convention that the numbers representing the sizes of subnetworks attributed to vip's are listed in the beginning of the sequence τ.

If the number of subnetworks (users) is limited, say by r, *i.e.* we can realize in G each sequence $\tau = (n_1,\ldots,n_p)$ with $p \leq r$, we say that G is r-AP. So, a graph is AP if it is r-AP for $r = 1,2,\ldots$ (see [12], [13] and [15] for algorithmic approach for small k).

If additionally for a given $s \leq r$, each of the first s' users for any $s' \leq \min\{s,p\}$ is allowed to choose a vertex belonging to their subnetwork, then the corresponding graph G of order $n > r$ is called r-AP+s.

The most significant result concerning these notions is the following famous result on k-AP+k graphs by Győri [6] and, independently, Lovász [11].

Theorem 1 *Every k-connected graph G is k-AP+k.* ■

It is straightforward to notice that the converse is also true. Indeed, removal of $k-1$ vertices $v_1,\ldots,v_{k-1}$ cannot disconnect a k-AP+k graph G, since otherwise there would not exist a realization $(V_1,\ldots,V_k)$ of an admissible sequence $(1,\ldots,1,n-k+1)$ in G such that $v_1 \in V_1,\ldots,v_{k-1} \in V_{k-1}$.

Analogously, by analyzing an admissible sequence $(1,\ldots,1,n-k)$, one can easily see that the following observation holds.

Observation 2 *Every AP+k graph has to be $(k+1)$-connected.* ■

It is worth noting that if we change the requirement concerning the number of parts we partition our network into (from bounded to arbitrary case), this may have dramatical consequences. For instance, consider the complete bipartite graph $K_{k,k}$. Since it is k-connected, then by Theorem 1, it is also k-AP+k. On the other hand, if we remove two vertices on the "same side" of $K_{k,k}$, we obtain the graph $K_{k,k-2}$, which evidently does not contain a perfect matching. In other words, with the above choice of two vip's, the sequence $(1,1,2,\ldots,2)$ is not realizable. In consequence, the graph $K_{k,k}$ is not even AP+2.

Given a graph $G=(V,E)$, its k^{th} power G^k is the graph obtained from G by adding the edge between every pair of vertices with distance at most k in G. In this paper we prove that k^{th} powers of traceable graphs are AP+$(k-1)$, see Corollary 7, and that k^{th} powers of hamiltonian graphs are AP+$(2k-1)$, see Corollary 9. These results are sharp.

2 Results

Given a path P_n (or a cycle C_n), its consecutive vertices $v_1, v_2, \ldots, v_n$ define a natural *orientation* of the path (or the cycle). We shall call them also the *consecutive* vertices of its k^{th} power P_n^k (or C_n^k). Similarly, v_1 and v_n will be called the *first* and the *last* vertices of P_n^k (C_n^k), respectively.

In both cases, for a vertex x, we shall also use the notation x^+ and x^- in order to denote the next or the previous vertex to x, respectively, with respect to the natural orientation. For two vertices a and b of the cycle C_n, we denote by aC_nb the set of all consecutive vertices of C_n starting from a and ending at b with respect to the natural orientation of the cycle.

First, we prove that k^{th} powers of paths are AP+$(k-1)$. We shall use Lemma 5 below, which is even stronger than required for this purpose. The both results however will be then necessary to show that k^{th} powers of cycles are AP+$(2k-1)$. Since the property of being AP+k is monotone with respect to adding edges, the results for paths and cycles immediately imply the corresponding properties for traceable and hamiltonian graphs, *i.e.*, Corollaries 7 and 9. Note here also that our results for paths (hence also for the family of traceable graphs) and for cycles (thus for hamiltonian graphs) are tight, since the connectivity of the k^{th} power of a path P_n, $n \geq k+1$, is k, and the connectivity of the k^{th} power of a cycle C_n, $n \geq 2k+1$, is $2k$. This is obvious for paths, while for cycles it is sufficient to notice that so that we could disconnect two vertices u, v of C_n^k, these must be at distance more than k in C_n. Then we have to remove (at least) k consecutive vertices from each of the two paths joining u and v in C_n.

Below we state two basic observations concerning the operation of removing a vertex from a graph $G = P_n^k$ being the k^{th} power of a path P_n. Let $v_1, \ldots, v_n$ be the consecutive vertices of P_n. By a graph obtained by removing the first (respectively, the last) vertex of G we mean the graph $G \setminus \{v_1\}$ (respectively, $G \setminus \{v_n\}$) with consecutive vertices given by $v_2, \ldots, v_n$ or $v_1, \ldots, v_{n-1}$, respectively. By a graph obtained by removing other than the first or the last vertex of G, say x, we mean the graph $G \setminus \{x\}$ with consecutive vertices given by $v_1, \ldots, x^-, x^+, \ldots, v_n$.

Observation 3 *A graph obtained by removing the first or the last vertex of any k^{th} power of a path is also a k^{th} power of a path.* ■

Observation 4 *A graph obtained by removing from P_n^k, $k \geq 2$, a vertex subset whose vertices are pairwise at distance at least k in the underlying path P_n, contains a spanning $P_{n'}^{(k-1)}$ for some $n' < n$. Power of a path. Moreover, if we do not remove the last vertex of P_n^k, then it is also the last vertex of the obtained $(k-1)^{th}$ power of a path.*

Proof. Suppose $v_1, \ldots, v_n$ are the consecutive vertices of P_n. Then the result is obvious, since for each vertex v_i which has not been removed, all but at most one of its neighbours v_j from P_n^k with $j < i$ ($j > i$) belong to the obtained graph. ∎

Lemma 5 *Let $G = (V, E)$ be a k^{th} power of a path P_n with consecutive vertices $v_1, v_2, \ldots, v_n$, $n \geq k$. For every partition $\tau = (n_1, \ldots, n_p)$ of n into $p \geq k$ parts and every list of k vertices $v_{i_1}, \ldots, v_{i_k} \in V$ with $i_1 < i_2 < \ldots < i_k$, we have: if $i_k = n$ (or $i_1 = 1$), then there exists a realization $(V_1, \ldots, V_p)$ of τ in G such that $v_{i_1} \in V_1, \ldots, v_{i_k} \in V_k$.*

Proof. First observe that without loss of generality, we may suppose that $i_k = n$, for, if $i_k = 1$ we can change the orientation of P_n. We prove the theorem by induction with respect to k. For $k = 1$ the result is obvious. Assume then that $k \geq 2$ and that the theorem holds for $(k-1)^{th}$ powers of paths.

Denote by $r_1, \ldots, r_{k-1}$ the residues modulo k of $i_1, i_2, \ldots, i_{k-1}$, respectively, and let r be an unused residue, *i.e.*, any element of the non-empty set $\{0, 1, \ldots, k-1\} \setminus \{r_1, \ldots, r_{k-1}\}$. We shall construct a sequence $\sigma = (v_{j_1}, \ldots, v_{j_q})$ of pairwise distinct vertices of our P_n^k with the following properties:

(1) $v_{j_1} = v_{i_1}$ and $v_{i_2}, \ldots, v_{i_k}$ do not belong to σ,

(2) any *initial* block of σ induces a connected subgraph in G,

(3) after removing any *initial* subsequence of vertices of σ from G, the remaining graph contains a $(k-1)^{th}$ power of a path as a spanning subgraph with the last vertex v_n,

(4) each vertex of G is either a neighbour of some vertex from σ or belongs to σ.

First we choose every k^{th} vertex from the sequence $v_1, \ldots, v_{i_1}$ starting from v_{i_1} and then "jumping back" as long as we can, *i.e.*, we set $v_{j_1} = v_{i_1}, v_{j_2} = v_{i_1-k}, v_{j_3} = v_{i_1-2k}, \ldots, v_{j_a} = v_{i_1-(a-1)k}$, where $i_1 - (a-1)k \in [1, k]$. Note that so far rule (2) and, by Observation 4, rule (3) are fulfilled. Then one after another we choose the consecutive yet not chosen vertices from the sequence $v_1, \ldots, v_{i_1}$ as elements of σ starting from the one with the lowest index, *i.e.* v_1 or v_2. By Observation 3, rule (3) (and obviously rule (2)) has not been broken this way. Note also that the remaining vertices induce now a k^{th} power of a path in G. Then to finalize the construction of σ we make a "short jump forward" from the last vertex of the subsequence of σ

constructed so far to v_b, where b is the smallest index with residue r modulo k which is greater than i_1 and smaller than n (if such b exists), followed by choosing every k^{th} element of the sequence $v_b, \ldots, v_{n-1}$ starting from v_b, *i.e.*, we set $v_{j_{i_1+1}} = v_b, v_{j_{i_1+2}} = v_{b+k}, v_{j_{i_1+3}} = v_{b+2k}, \ldots, v_{j_{i_1+c}} = v_{b+(c-1)k}$, where $b + (c-1)k \in [n-k, n-1]$. By Observation 4, rule (3) (and rule (2)) is obeyed. Moreover, by the choice of r and our construction, property (1) also holds. Finally, since the constructed sequence σ contains "every k^{th}" vertex from the sequence $v_1, \ldots, v_{n-1}$, rule (4) is also satisfied.

Now if n_1 is at most as big as the number of vertices in σ, *i.e.* $n_1 \leq q$, then the set $V_1 = \{v_{j_1}, \ldots, v_{j_{n_1}}\}$ has n_1 vertices including $v_{i_1} = v_{j_1}$. By rule (2), this set induces a connected subgraph in G. Then we let $G' = G[V \setminus V_1]$ and $\tau' = (n_2, \ldots, n_p)$. By rule (1), $v_{i_2}, \ldots, v_{i_k} \in V(G')$, and by rule (3), G' contains a $(k-1)^{th}$ power of a path as a spanning subgraph with v_n being its last vertex. By induction we therefore may find a realization $(V_2, \ldots, V_p)$ of τ' in G' such that $v_{i_2} \in V_2, \ldots, v_{i_k} \in V_k$.

On the other hand, if $n_1 > q$, then we set $V_1' = \{v_{j_1}, \ldots, v_{j_q}\}$, $G'' = G[V \setminus V_1']$ and $\tau'' = (n_2, \ldots, n_p, n_1 - q)$. Then analogously as above we may find a realization $(V_2, \ldots, V_p, V_1'')$ of τ'' in G'' such that $v_{i_2} \in V_2, \ldots, v_{i_k} \in V_k$ by induction. Then by rules (2) and (4), the set $V_1 := V_1' \cup V_1''$ induces a connected subgraph in G, $v_{i_1} \in V_1$.

In both cases we obtain a desired realization of τ in G. ■

Corollary 6 *Every P_n^k with $n \geq k$ is AP+$(k-1)$.*

Proof. Assume that $v_1, \ldots, v_n$ are the consecutive vertices of our graph. For partitions into at most k parts, the result follows by Theorem 1. Consider then a partition $\tau = (n_1, \ldots, n_p)$ of n with $p > k$, together with associated vertices $v_{j_1}, \ldots, v_{j_{k-1}}$, $j_1 < j_2 < \ldots < j_{k-1}$. Since $n \geq k$, it is possible to find an increasing sequence $(i_1, \ldots, i_k)$ of integers with $i_k = n$, which contains all terms of the sequence $(j_1, \ldots, j_{k-1})$.

■

Corollary 7 *For every traceable graph G with at least k vertices, G^k is AP+$(k-1)$.*

Theorem 8 *Every C_n^k with $n \geq 2k$ is AP+$(2k-1)$.*

Proof. We assume that $k \geq 2$, since C_n is obviously AP+1. Let C_n be a cycle, $n \geq 2k$, with consecutive vertices denoted by $v_0, v_1, \ldots, v_{n-1}$ and consider its k^{th} power $G = (V, E) = C_n^k$. Since G is a $2k$-connected graph (for $n > 2k$), then by Theorem 1, it is sufficient to consider partitions into more than $2k$ parts. Assume then that $\tau = (n_1, \ldots, n_p)$ is a partition of n

with $p > 2k$, and $v_{i_1}, \ldots, v_{i_{2k-1}}$, where $i_1 < i_2 < \ldots < i_{2k-1}$, are the vertices associated with the first $2k-1$ elements of this partition, respectively. (We allow the situation where the $i_{2k-1} = 0$.)

For each such vertex v_{i_j}, $j = 1, \ldots, 2k-1$, denote by D_j the set of vertices which are between $v_{i_{j-1}}$ and v_{i_j} along the cycle C_n together with v_{i_j}, *i.e.*, $D_j = v^+_{i_{j-1}} C_n v_{i_j}$ for $j \geq 2$ and $D_1 = v^+_{i_{2k-1}} C_n v_{i_1}$, where the indices (here and further) should be understood modulo n, and let $d_j = |D_j|$ denote the distance between $v_{i_{j-1}}$ and v_{i_j} (or, if $j = 1$, between $v_{i_{2k-1}}$ and v_{i_1}) along the cycle C_n (according to its orientation). Let further

$$s_j := d_{j+1} + \ldots + d_{j+k-1} \quad \text{and} \quad m_j := n_{j+1} + \ldots + n_{j+k-1}$$

for $j = 1, \ldots, 2k-1$, where the indices are counted modulo $2k-1$. Note that $d_1 + \ldots + d_{2k-1} = n$, and since $p > 2k$, then $n_1 + \ldots + n_{2k-1} < n$. Therefore, there must exist j for which $m_j < s_j$, since otherwise we would obtain the following contradiction:

$$(k-1)n > (k-1)\sum_{j=1}^{2k-1} n_j = \sum_{j=1}^{2k-1} m_j \geq \sum_{j=1}^{2k-1} s_j = (k-1)\sum_{j=1}^{2k-1} d_j = (k-1)n.$$

Set $W := \{v_{i_1}, v_{i_2}, \ldots, v_{i_{2k-1}}\}$ and assume first that there exists some j' such that $m_{j'} \geq s_{j'}$. Without loss of generality we may assume that $j' = 1$ and $j = 2k-1$ (*i.e.*, $m_{2k-1} < s_{2k-1}$ and $m_1 \geq s_1$), and $v^+_{i_{2k-1}} = v_1$. We thus have:

$$\begin{aligned} n_1 + \ldots + n_{k-1} \leq \quad & d_1 + \ldots + d_{k-1} - 1 && = |\{v_1, v_2, \ldots, v^-_{i_{k-1}}\}| \quad \text{and} \\ n_1 + n_2 + \ldots + n_k \geq \quad & 1 + d_2 + \ldots + d_k && = |\{v_{i_1}, v_{i_1+1}, \ldots, v_{i_k}\}|. \end{aligned}$$

Then there exists t, $1 \leq t \leq i_1$ such that for $U := \{v_t, v_{t+1}, \ldots, v_{i_k}\}$ we have:

$$n_1 + \ldots + n_{k-1} \leq |U| - 1 \quad \text{and} \tag{1}$$

$$n_1 + \ldots + n_k \geq |U|. \tag{2}$$

Note that $U \cap W = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$. Thus if $|U| = n_1 + \ldots + n_k$, then by Lemma 5 and Corollary 6 we may find a realization $(V_1, \ldots, V_k)$ of $(n_1, \ldots, n_k)$ in the k^{th} power of a path induced in G by U, and a realization $(V_{k+1}, \ldots, V_p)$ of $(n_{k+1}, \ldots, n_p)$ in the remaining part of G in such a way that $v_{i_1} \in V_1, \ldots, v_{i_{2k-1}} \in V_{2k-1}$. If on the other hand $n_1 + \ldots + n_k > |U|$, then by (1) there exist positive integers n'_k, n''_k such that $n_1 + \ldots + n_{k-1} + n'_k = |U|$ and $n'_k + n''_k = n_k$. Let G', G'' be the k^{th} powers of paths induced, respectively, by $U, V \setminus U$ in G, and let v_a be the first vertex after v_{i_k} (according to the orientation of the cycle C_n) such that $v_a \in (V \setminus U) \setminus W$. Note that since $|(V \setminus U) \cap W| = k-1$, then v_a must be a neighbour of v_{i_k} in G. By Lemma 5 there exist realizations $(V_1, \ldots, V_{k-1}, V'_k)$, $(V''_k, V_{k+1}, \ldots, V_p)$ of $(n_1, \ldots, n_{k-1}, n'_k)$, $(n''_k, n_{k+1}, \ldots, n_p)$ in G', G'', respectively, such that $v_{i_1} \in V_1, \ldots v_{i_{k-1}} \in V_{k-1}, v_{i_k} \in V'_k$ and $v_a \in V''_k, v_{i_{k+1}} \in V_{k+1}, \ldots, v_{i_{2k-1}} \in V_{2k-1}$. Then $(V_1, \ldots V_{k-1}, V'_k \cup V''_k, V_{k+1}, \ldots, V_p)$ is a desired realization of τ in G.

Assume now that $m_j < s_j$ for every $j \in \{1, \ldots, 2k-1\}$. Thus, in particular, there is no k consecutive vertices of the cycle C_n in W. Note also that since $n_2 + \ldots + n_k = m_1 < s_1 = d_2 + \ldots + d_k$, then there must exist $i' \in \{2, \ldots, k\}$ such that $n_{i'} < d_{i'}$. Without loss of generality we may assume that $i' = k$ and $v_{i_{2k-1}} = v_0$. Then $V_k := \{v_{i_k - n_k + 1}, v_{i_k - n_k + 2} \ldots, v_{i_k}\}$, then $|V_k| = n_k$ and $V_k \cap W = \{v_{i_k}\}$. Moreover, the sets $U_1 := \{v_1, v_2, \ldots, v_{i_k - n_k}\}$, $U_2 := V \setminus (U_1 \cup V_k)$ induce k^{th} powers of paths G_1, G_2 in G such that $W \cap U_1 = \{v_{i_1}, v_{i_2} \ldots, v_{i_{k-1}}\}$, $W \cap U_2 = \{v_{i_{k+1}}, v_{i_{k+2}} \ldots, v_{i_{2k-1}}\}$ and

$$n_1 + \ldots + n_{k-1} = m_{2k-1} < s_{2k-1} = d_1 + \ldots + d_{k-1} < |U_1|,$$

$$n_{k+1} + \ldots + n_{2k-1} = m_k < s_k = d_{k+1} + \ldots + d_{2k-1} = |U_2|.$$

If we then are able to divide the remaining elements $n_{2k}, \ldots, n_p$ of τ into two groups, *i.e.* fix I_1, I_2 with $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{2k, 2k+1, \ldots, p\}$, such that $\sum_{i=1}^{k-1} n_i + \sum_{i \in I_1} n_i = |U_1|$ and $\sum_{i=k+1}^{2k-1} n_i + \sum_{i \in I_2} n_i = |U_2|$, then the result follows by Corollary 6. Otherwise, there exist $r \in \{2k, \ldots, p\}$ and two integers $n'_r, n''_r \geq 1$ such that $n_r = n'_r + n''_r$ and $\sum_{i=1}^{k-1} n_i + \sum_{i=2k}^{r-1} n_i + n'_r = |U_1|$. Let v_c be the first vertex of U_1 that does not belong to W, and let v_d be the last vertex of U_2 that does not belong to W. Since W cannot contain k consecutive vertices of C_n, then v_c and v_d are neighbours in G. By Lemma 5 there exist realizations $(V_1, \ldots, V_{k-1}, V'_r, V_{2k}, \ldots, V_{r-1})$, $(V_{k+1}, \ldots, V_{2k-1}, V''_r, V_{r+1}, \ldots, V_p)$ of $(n_1, \ldots, n_{k-1}, n'_r, n_{2k}, \ldots, n_{r-1})$, $(n_{k+1}, \ldots, n_{2k-1}, n''_r, n_{r+1}, \ldots, n_p)$, respectively, such that $v_{i_1} \in V_1, \ldots, v_{i_{k-1}} \in V_{k-1}, v_c \in V'_r$ and $v_{i_{k+1}} \in V_{k+1}, \ldots, v_{i_{2k-1}} \in V_{2k-1}, v_d \in V''_r$. Then $(V_1, \ldots, V_{r-1}, V'_r \cup V''_r, V_{r+1}, \ldots, V_p)$ is a desired realization of τ. ■

Corollary 9 *For every hamiltonian graph G with at least $2k$ vertices, G^k is AP+$(2k-1)$.*

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