EXISTENCE AND DECAY OF SOLUTIONS OF A NONLINEAR VISCOELASTIC PROBLEM WITH A MIXED NONHOMOGENEOUS CONDITION

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Abstract. We study the initial-boundary value problem for a nonlinear wave equation given by

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u_t|^{q-2}u_t = f(x,t,u),$$
$$u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = g(t),$$
$$u(x,0) = u_0(x), u_t(x,0) = u_1(x),$$

where $\eta \geq 0$, $q \geq 2$ are given constants and u_0, u_1, g, k, f are given functions. In this paper, we consider two main parts. In Part 1, under a certain local Lipschitzian condition on f with $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$; $k, g \in H^1(0,T)$, $\eta \geq 0$; $q \geq 2$, a global existence and uniqueness theorem is proved. The proof is based on the paper [10] associated to a contraction mapping theorem and standard arguments of density. In Part 2, the asymptotic behavior of the solution u as $t \to +\infty$ is studied, under more restrictive conditions, namely $g = 0$, $f(x,t,u) = -|u|^{p-2}u + F(x,t)$, $p \geq 2$, $F \in L^1(\mathbb{R}_+; L^2) \bigcap L^2(\mathbb{R}_+; L^2)$, $\int_0^{+\infty} e^{\sigma t}\|F(t)\|^2 dt < +\infty$, with $\sigma > 0$, and $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$, $k \in H^1(\mathbb{R}_+)$, and some others ($\|\cdot\|$ denotes the $L^2(0,1)$ norm). It is proved that under these conditions, a unique solution $u(t)$ exists on $\mathbb{R}_+$ such that $\|u^{/}(t)\| + \|u_x(t)\|$ decay exponentially to 0 as $t \to +\infty$. Finally, we present some numerical results.

1. Introduction

In this paper we will consider the following initial and boundary value problem:

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u_t|^{q-2}u_t = f(x,t,u), 0 < x < 1; 0 < t < T, \tag{1.1}$$

$$u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = g(t), \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{1.3}$$

where $\eta \geq 0$, $q \geq 2$ are given constants and u_0, u_1, g, k, f are given functions satisfying conditions specified later.

2000 *Mathematics Subject Classification.* 35L05, 35L15, 35L70, 37B25.
Key words and phrases. Faedo-Galerkin method; Global existence; contraction mapping theorem; nonlinear wave equation; Viscoelastic; Exponential decay.

In a recent paper [1], Berrimia and Messaoudi considered the problem

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds = |u|^{p-2}u, x \in \Omega, t > 0, \tag{1.4}$$

$$u = 0, \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

$$u(x,0) = \widetilde{u}_0(x), u_t(x,0) = \widetilde{u}_1(x), x \in \Omega, \tag{1.6}$$

where $p > 2$ is a constant, k is a given positive function, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. This type of problems have been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established. In this regard, Cavalcanti et al. [3] studied the following equation

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds + |u|^{p-2}u + a(t)u_t = 0, \quad \text{in} \quad \Omega \times (0,\infty), \tag{1.7}$$

for $a : \Omega \to \mathbb{R}_+$, a function, which may be null on a part of the domain Ω. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$-\zeta_1 k(t) = k^{/}(t) = -\zeta_2 k(t), t \geq 0, \tag{1.8}$$

the authors established an exponential rate of decay.

In [2] Bergounioux, Long and Dinh studied problem (1.1), (1.3) with $k = 0, q = 2, f(x,t,u) = -Ku + F(x,t)$, and the mixed boundary conditions (1.2) standing for

$$u_x(0,t) = g(t) + hu(0,t) - \int_0^t H(t-s)u(0,s)ds, \tag{1.9}$$

$$u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \tag{1.10}$$

where $h \geq 0$, $K, \lambda, K_1, \lambda_1$ are given constants and g, H are given functions.

In [7], Long, Dinh and Diem obtained the unique existence, regularity and asymptotic expansion of the problem (1.1), (1.3), (1.9) and (1.10) in the case of $k = 0$, $f(x,t,u) = -K|u|^{p-2}u + F(x,t)$, with $p \geq 2$, $q \geq 2$; K, λ are given constants.

In [8], Long, Ut and Truc gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)- (1.3) when $k = 0$, $q = 2$, $f(x,t,u) = -Ku + F(x,t)$ and $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$. In this case, the problem (1.1)- (1.3) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

In [9], Long and Giai obtained the unique existence and asymptotic expansion for the solution of problem (1.1), (1.3) when $k = 0$, $q = 2$, $f(x,t,u) = -Ku + F(x,t)$ and $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$, and the mixed boundary conditions (1.2) standing for

$$\begin{aligned} u_x(0,t) =& g(t) + K_1|u(0,t)|^{\alpha-2}u(0,t) + \lambda_1|u_t(0,t)|^{\beta-2}u_t(0,t) \\ &- \int_0^t H(t-s)u(0,s)ds, \end{aligned} \tag{1.11}$$

$$u(1,t) = 0, \tag{1.12}$$

where K, λ, K_1, λ_1, α, β are given constants and g, H are given functions. In this case, the problem (1.1), (1.3), (1.11), (1.12) is the mathematical model describing a shock problem involving a nonlinear viscoelastic bar.

In [10], Long and Truong obtained the unique existence and asymptotic expansion for the solution of problem (1.1) -(1.3) when $f(x,t,u) = -K|u|^{p-2}u + F(x,t)$,

$(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$; $F, F_t \in L^2(Q_T)$, $k \in W^{2,1}(0,T)$, $g \in H^2(0,T)$; K, $\eta \geq 0$, $\eta_0 > 0$; $p, q \geq 2$.

In this paper, we consider two main parts. In Part 1, under a certain local Lipschitzian condition on f with $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$; $k, g \in H^1(0,T)$,$\lambda > 0$, $\eta_0 > 0$; $\eta \geq 0$; $q \geq 2$, a global existence and uniqueness theorem is proved. The proof is based on the paper [10] associated to a contraction mapping theorem and standard arguments of density. In Part 2, the asymptotic behavior of the solution u as $t \to \infty$ is studied, under more restrictive conditions, namely $f(x,t,u) = -|u|^{p-2}u + F(x,t)$, $p \geq 2$, $F \in L^1(\mathbb{R}_+; L^2) \bigcap L^2(\mathbb{R}_+; L^2)$, $\int_0^{+\infty} e^{\sigma t} \|F(t)\|^2 dt < +\infty$, with $\sigma > 0$, and $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$, $g = 0$, $k \in H^1(\mathbb{R}_+$, and some others ($\| \cdot \|$ denotes the $L^2(0,1)$ norm). It is proved that under these conditions, a unique solution $u(t)$ exists on $\mathbb{R}_+$ such that $\|u'(t)\| + \|u_x(t)\|$ decay exponentially to 0 as $t \to +\infty$. The results obtained here relatively are in part generalizations of those in [1-3, 6-10]. Finally, we present some numerical results.

2. Preliminary Results

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of usual function spaces: $C^m(\overline{\Omega})$, $L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$ The norm in L^2 is denoted by $\| \cdot \|$. We also denote by $\langle \cdot, \cdot \rangle$ the scalar product in L^2 or pair of dual scalar product of continuous linear functional with an element of a function space. We denote by $\| \cdot \|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^p dt \Big)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, and $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, and $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

Without loss of generality, we can suppose that $\eta_0 = \lambda = 1$. For every $\eta \geq 0$, we put

$$a_\eta(u,v) = \int_0^1 u_x(x) v_x(x) dx + u(0)v(0) + \eta u(1)v(1), \forall u, v \in H^1, \tag{2.1}$$

$$\|v\|_\eta = (a_\eta(v,v))^{1/2}. \tag{2.2}$$

On H^1 we shall use the following equivalent norm

$$\|v\|_1 = \left(v^2(0) + \int_0^1 |v_x(x)|^2 dx \right)^{1/2} \tag{2.3}$$

Then we have the following lemmas.

Lemma 2.1. *The imbedding $V \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leq \|v\|_V, \text{ for all } v \in V. \tag{2.4}$$

Lemma 2.2. *Let $\eta \geq 0$. Then, the symmetric bilinear form $a_\eta(\cdot,\cdot)$ defined by* (2.1) *is continuous on $H^1 \times H^1$ and coercive on H^1, i.e.,*
(i) $|a_\eta(u,v)| = C_\eta \|u\|_1 \|v\|_1$, for all $u, v \in H^1$,
(ii) $a_\eta(v,v) = \|v\|_1^2$, for all $v \in H^1$,
where $C_\eta = 1 + 2\eta$.

The proofs of these lemmas are straightforward, and we omit the details.
We also note that on H^1, $\|v\|_1, \|v\|_{H^1} = \left(\|v\|^2 + \|v^/\|^2\right)^{1/2}$, $\|v\|_\eta = \sqrt{(a_\eta(v,v))}$ are three equivalent norms.

$$\|v\|_1^2 \leq \|v\|_\eta^2 \leq C_\eta \|v\|_1^2, \quad \text{for all} \quad v \in H^1, \tag{2.5}$$

$$\frac{1}{3}\|v\|_{H^1}^2 \leq \|v\|_1^2 \leq 3\|v\|_{H^1}^2, \quad \text{for all} \quad v \in H^1, \tag{2.6}$$

3. The Existence and uniqueness theorem of the solution

In this section we study the global existence of solutions for problem (1.1)-(1.3). For this purpose, we consider, first, a related nonlinear problem. Then, we use the well-known Banach's fixed point theorem to prove the existence of solutions to the nonlinear problem (1.1)-(1.3).

We make the following assumptions:

(H1) $\eta \geq 0, q \geq 2$,
(H2) $k, g \in H^1(0,T)$,
(H3) $\widetilde{u}_0 \in H^1$ and $\widetilde{u}_1 \in L^2$,
(H4) $f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R})$ satisfies the conditions $D_2 f, D_3 f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R})$.

For each $T > 0$, we put

$$W(T) = v \in L^\infty(0,T;H^1) : v_t \in L^\infty(0,T;L^2) \bigcap L^q(Q_T). \tag{3.1}$$

Then $W(T)$ is a Banach space with respect to the norm (see[5]):

$$\|v\|_{W(T)} = \|v\|_{L^\infty(0,T;H^1)} + \|v_t\|_{L^\infty(0,T;L^2)} + \|v_t\|_{L^q(Q_T)}, v \in W(T). \tag{3.2}$$

For each $v \in W(T)$, we associate with the problem (1.1)-(1.3) the following variational problem.
Find $u \in W(T)$ which satisfies the variational problem

$$\begin{aligned} &< u^{//}(t), w > + a_\eta(u(t), w) - \int_0^t k(t-s) a_\eta(u(s), w) ds + < \psi_q(u^/(t)), w > \\ &= g_1(t) w(1) + < f(\cdot, t, v(\cdot,t)), w > \quad \text{for all} \quad w \in H^1, \end{aligned} \tag{3.3}$$

$$u(0) = \widetilde{u}_0, u_t(0) = \widetilde{u}_1, \tag{3.4}$$

where

$$\psi_q(z) = |z|^{q-2} z, g_1(t) = g(t) - \int_0^t k(t-s) g(s) ds. \tag{3.5}$$

Then, we have the following theorem

Theorem 3.1. *Let (H1)-(H4) hold. Then, for every $T > 0$ and $v \in W(T)$, problem* (3.3)- (3.5) *has a unique solution $u \in W(T)$ and such that*

$$u^{//}, u_{xx} \in L^{q^/}(0,T;(H^1)^/), \quad \text{where} \quad q^/ = q/(q-1). \tag{3.6}$$

Furthermore, we have

$$\|u^{/}(t)\|^2 + \|u(t)\|_\eta^2 + 2\int_0^t \|u^{/}(s)\|_{L^q}^q ds \le C_{1T}\exp(TC_{2T}), \forall t \in [0,T], \tag{3.7}$$

where

$$\begin{aligned} C_{1T} = C_{1T}(v,\widetilde{u}_0,\widetilde{u}_1,k,g) = 2\Big[\|\widetilde{u}_1\|^2 + \|\widetilde{u}_0\|_\eta^2 + 2|g_1(0)\widetilde{u}_0(1)| \\ +6\|g_1\|_{L^\infty(0,T)}^2 + 2\|g_1^{/}\|_{L^2(0,T)}^2 + \int_0^T \|f(\cdot,s,v(s))\|^2 ds\Big], \end{aligned} \tag{3.8}$$

$$C_{2T} = C_{2T}(k) = 2\left[3 + 2|k(0)| + 6\|k\|_{L^2(0,T)}^2 + T\|k^{/}\|_{L^2(0,T)}^2\right], \tag{3.9}$$

and

$$g_1(t) = g(t) - \int_0^t k(t-s)g(s)ds. \tag{3.10}$$

Proof of theorem 3.1. The proof consists of steps two steps
a. The existence of solution. We approximate $\widetilde{u}_0$, $\widetilde{u}_1$, k, g by sequences $\{u_{0m}\} \subset C_0^\infty\left(\overline{\Omega}\right)$, $u_{1m} \subset C_0^\infty(\Omega)$, $k_m, g_m \subset C_0^\infty([0,T])$, respectively, such that

$$\begin{aligned} u_{0m} &\to \widetilde{u}_0 \quad \text{strongly in} \quad H^1, \\ u_{1m} &\to \widetilde{u}_1 \quad \text{strongly in} \quad L^2, \\ k_m &\to k \quad \text{strongly in} \quad H^1(0,T), \\ g_m &\to g \quad \text{strongly in} \quad H^1(0,T). \end{aligned} \tag{3.11}$$

Then we consider the following variational problem: Find $u_m \in W(T)$ which satisfies the variational problem

$$\begin{aligned} &< u_m^{//}(t), w > + a_\eta(u_m(t), w) - \int_0^t k_m(t-s)a_\eta(u_m(s),w)ds \\ &+ < \psi_q(u_m^{/}(t)), w >= g_{1m}(t)w(1) + < f(\cdot,t,v(\cdot,t)), w >, \forall w \in H^1, \end{aligned} \tag{3.12}$$

$$u(0) = u_{0m}, u^{/}(0) = u_{1m}, \tag{3.13}$$

and

$$u_m \in L^\infty(0,T;H^2), u_m^{/} \in L^\infty(0,T;H^1), u_m^{//} \in L^\infty(0,T;L^2), \tag{3.14}$$

where

$$g_{1m}(t) = g_m(t) - \int_0^t k_m(t-s)g_m(s)ds. \tag{3.15}$$

The existence of a sequence of solutions u_m satisfying (3.12)-(3.15) is a direct result of the theorem 2.1 in [10]. We shall prove that u_m is a Cauchy sequence in $W(T)$.
(i) A priori estimates.

We take $w = u_m^{/}(t)$ in (3.12), afterwards integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned}\sigma_m(t) =&\sigma_m(0) - 2g_{1m}(0)u_{0m}(1) + 2g_{1m}(t)u_m(1,t)\\ &- 2\int_0^t g_{1m}^{/}(r)u_m(1,r)dr - 2k_m(0)\int_0^t \|u_m(r)\|_\eta^2 dr\\ &+ 2\int_0^t k_m(t-s)a_\eta(u_m(s),u_m(t))ds\\ &- 2\int_0^t dr\int_0^r k_m^{/}(r-s)a_\eta(u_m(s),u_m(r))ds\\ &+ 2\int_0^t < f(\cdot,s,v(\cdot,s)), u_m^{/}(s) > ds,\end{aligned} \tag{3.16}$$

where

$$\sigma_m(t) = \|u_m^{/}(t)\|^2 + \|u_m(t)\|_\eta^2 + 2\int_0^t ||u_m^{/}(s)||_{L^q}^q ds. \tag{3.17}$$

Proving in the same manner as in [10], we have the following results:

$$\sigma_m(t) = C_{1T}(m) + C_{2T}(m)\int_0^t \sigma_m(s)ds, \forall t \in [0,T], \tag{3.18}$$

where

$$\begin{aligned}C_{1T}(m) =&2\Big[\|u_{1m}\|^2 + \|u_{0m}\|_\eta^2 + 2|g_{1m}(0)u_{0m}(1)| + 6\|g_{1m}\|_{L^\infty(0,T)}^2\\ &+2\|g_{1m}^{/}\|_{L^2(0,T)}^2 + \int_0^T \|f(\cdot,s,v(s))\|^2 ds\Big],\end{aligned} \tag{3.19}$$

$$C_{2T}(m) = 2\left[3 + 2|k_m(0)| + 6\|k_m\|_{L^2(0,T)}^2 + T\|k_m^{/}\|_{L^2(0,T)}^2\right]. \tag{3.20}$$

From the assumptions (H1)-(H4), afterwards using Gronwall's lemma, we deduce from (3.11), that

$$\sigma_m(t) \le \widetilde{C}_T, \quad \text{for all} \quad m \quad \text{and} \quad t \in [0,T], \tag{3.21}$$

where C_T is a constant independent of m.

On the other hand, we deduce from (3.12), (3.21), that, for all $w \in H^1$, we have

$$\begin{aligned}| < u_m^{//}(t), w > | \le&\|u_m(t)\|_\eta\|w\|_\eta + \int_0^t |k_m(t-s)|\|u_m(s)\|_\eta\|w\|_\eta ds\\ &+ \|\psi_q(u_m^{/})\|_{L^{q^{/}}(\Omega)}\|w\|_{L^q(\Omega)} + |g_{1m}(t)|\|w\|_\eta\\ &+ \|f(\cdot,t,v(\cdot,t))\|\|w\|\\ &\le C_T\sqrt{(3C_\eta)}\left[1 + \|\psi_q(u_m^{/})\|_{L^{q^{/}}(\Omega)}\right]\|w\|_{H^1}.\end{aligned} \tag{3.22}$$

This implies that

$$\begin{aligned}\|u_m^{//}(t)\|_{(H^1)^{/}} =& \sup_{0\ne w\in H^1}\frac{\left|< u_m^{//}(t), w >\right|}{\|w\|_{H^1}}\\ &\le C_T\sqrt{3C_\eta}\left[1 + \|\psi_q(u_m^{/})\|_{L^{q^{/}}(\Omega)}\right].\end{aligned} \tag{3.23}$$

Hence

$$\begin{aligned}\|u_m^{//}\|^{q/}_{L^{q/}(0,T;(H^1)^/)} &= \int_0^T \|u_m^{//}(t)\|^{q/}_{(H^1)^/}dt \\ &\le \left(C_T\sqrt{3C_\eta}\right)^{q/} 2^{q/-1}\int_0^T \left[1+\|u_m^/(t)\|^q_{L^q(\Omega)}\right]dt \\ &\le C_T,\end{aligned} \tag{3.24}$$

where C_T always indicating a constant depending on T.

(ii) The convergence of sequence $\{u_m\}$

We shall prove that u_m is a Cauchy sequence in $W(T)$. Let $\widehat{u} = u_m - u_\mu$. Then $\widehat{u}$ satisfies the variational problem

$$\begin{aligned}&< u^{//}(t), w > + a_\eta(\widehat{u}(t), w) - \int_0^t k_m(t-s)a_\eta(\widehat{u}(s), w)ds \\ &- \int_0^t \widehat{k}(t-s)a_\eta(u_\mu(s), w)ds + < \psi_q(u_m^/(t)) - \psi_q(u_\mu^/(t)), w > \\ &= g_1(t)w(1) \quad \text{for all} \quad w \in H^1,\end{aligned} \tag{3.25}$$

$$\widehat{u}(0) = \widehat{u}_0, \widehat{u}^/(0) = \widehat{u}1, \tag{3.26}$$

where

$$\begin{aligned}\widehat{u}_0 = u_{0m} - u_{0\mu}, \widehat{u}_1 = u_{1m} - u_{1\mu}, \\ \widehat{k} = k_m - k_\mu, \widehat{g} = g_m - g_\mu, \widehat{g}_1 = g_{1m} - g_{1\mu}, \\ \widehat{g}_1(t) = \widehat{g}(t) - \int_0^t k_m(t-s)\widehat{g}(s)ds - \int_0^t \widehat{k}(t-s)g_\mu(s)ds.\end{aligned} \tag{3.27}$$

We take $w = u^/(t)$ in (3.25), after integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned}Z(t) =& Z(0) - 2\widehat{g}_1(0)\widehat{u}_0(1) + 2\widehat{g}_1(t)\widehat{u}(1,t) - 2\int_0^t \widehat{g}1^/(r)\widehat{u}(1,r)dr \\ &- 2k_m(0)\int_0^t \|\widehat{u}(r)\|^2_\eta dr + 2\int_0^t k_m(t-s)a_\eta(\widehat{u}(s), \widehat{u}(t))ds \\ &- 2\int_0^t dr\int_0^r k_m^/(r-s)a_\eta(\widehat{u}(s), \widehat{u}(r))ds \\ &- 2\widehat{k}(0)\int_0^t a_\eta(u_\mu(s), \widehat{u}(s))ds + 2\int_0^t \widehat{k}(t-s)a_\eta(u_\mu(s), \widehat{u}(t))ds \\ &- 2\int_0^t dr\int_0^r \widehat{k}^/(r-s)a_\eta(u_\mu(s), \widehat{u}(r))ds,\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}Z(t) =& \|\widehat{u}^/(t)\|^2 + \|\widehat{u}(t)\|^2_\eta \\ &+ 2\int_0^t < \psi_q(u_m^/(s)) - \psi_q(u_\mu^/(s)), u_m^/(s) - u_\mu^/(s) > ds.\end{aligned} \tag{3.29}$$

Using the following inequality

$$\forall q \ge 2, \exists C_q > 0 : (|x|^{q-2}x - |y|^{q-2}y)(x-y) \ge C_q|x-y|^q, \forall x, y \in \mathbb{R}, \tag{3.30}$$

it follows from (3.29) that

$$Z(t) \geq \|\widehat{u}^{/}(t)\|^2 + \|\widehat{u}(t)\|_\eta^2 + 2C_q \int_0^t \|\widehat{u}^{/}(s)\|_{L^q}^q ds. \tag{3.31}$$

Using the inequality

$$2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2, \forall a, b \in \mathbb{R}, \forall \epsilon > 0, \tag{3.32}$$

and the following inequalities

$$|a_\eta(u,v)| \leq \|u\|_\eta \|v\|_\eta, \forall u, v \in H^1, \tag{3.33}$$

$$|\widehat{u}(1,t)| \leq \|\widehat{u}(t)\|_{C^0(\Omega)} \leq \sqrt{2}\|\widehat{u}(t)\|_1 \leq \sqrt{2}\|\widehat{u}(t)\|_\eta \leq \sqrt{2Z(t)}, \tag{3.34}$$

we shall estimate respectively the following terms on the right-hand side of (3.28) as follows

$$\begin{aligned} Z(0) - 2\widehat{g}_1(0)\widehat{u}_0(1) \leq & \|u_{1m} - u_{1\mu}\|^2 + \|u_{0m} - u_{0\mu}\|_\eta^2 \\ & + 2|g_{1m}(0) - g_{1\mu}(0)||u_{0m}(1) - u_{0\mu}(1)|, \end{aligned} \tag{3.35}$$

$$2\widehat{g}_1(t)\widehat{u}(1,t) \leq 8\|\widehat{g}_1\|_{L^\infty(0,T)}^2 + \frac{1}{4} Z(t), \quad \text{with} \quad \epsilon = \frac{1}{8}, \tag{3.36}$$

$$-2\int_0^t \widehat{g}_1^{/}(r)\widehat{u}(1,r)dr \leq 2\|\widehat{g}_1^{/}\|_{L^2(0,T)}^2 + \int_0^t Z(r)dr, \tag{3.37}$$

$$2\int_0^t k_m(t-s)a_\eta(\widehat{u}(s), \widehat{u}(t))ds \leq \frac{1}{8} Z(t) + 8\|k_m\|_{L^2(0,T)}^2 \int_0^t Z(s)ds, \tag{3.38}$$

$$-2k_m(0)\int_0^t \|\widehat{u}(r)\|_\eta^2 dr \leq 2|k_m(0)| \int_0^t Z(r)dr, \tag{3.39}$$

$$-2\int_0^t dr \int_0^r k_m^{/}(r-s)a_\eta(\widehat{u}(s), \widehat{u}(r))ds \leq \left(1 + T\|k_m^{/}\|_{L^2(0,T)}^2\right) \int_0^t Z(s)ds, \tag{3.40}$$

$$2\int_0^t \widehat{k}(t-s)a_\eta(u_\mu(s), \widehat{u}(t))ds \leq \frac{1}{8} Z(t) + 8\widetilde{C}_T \|\widehat{k}\|_{L^1(0,T)}^2, \tag{3.41}$$

$$-2\widehat{k}(0)\int_0^t a_\eta(u_\mu(s), \widehat{u}(s))ds \leq T\widetilde{C}_T |\widehat{k}(0)|^2 + \int_0^t Z(s)ds, \tag{3.42}$$

$$-2\int_0^t dr \int_0^r \widehat{k}^{/}(r-s)a_\eta(u_\mu(s), \widehat{u}(r))ds \leq T^2 \widetilde{C}_T \|\widehat{k}^{/}\|_{L^2(0,T)}^2 + \int_0^t Z(s)ds. \tag{3.43}$$

Combining (3.28), (3.29), (3.31) and (3.35)-(3.43), we obtain

$$Z(t) \leq \rho_{1T}(m,\mu) + \rho_{2T}(m) \int_0^t Z(s)ds, \forall t \in [0,T], \tag{3.44}$$

where

$$\begin{aligned} \rho_{1T}(m,\mu) = 2 \Big[& \|\widehat{u}_1\|^2 + \|\widehat{u}_0\|_\eta^2 + 2|\widehat{g}_1(0)\widehat{u}_0(1)| + 8\|\widehat{g}_1\|_{L^\infty(0,T)}^2 \\ & + 2\|\widehat{g}_1^{/}\|_{L^2(0,T)}^2 + 8\widetilde{C}_T\|\widehat{k}\|_{L^1(0,T)}^2 + T\widetilde{C}_T|\widehat{k}(0)|^2 + T^2\widetilde{C}_T\|\widehat{k}^{/}\|_{L^2(0,T)}^2 \Big], \\ \rho_{2T}(m) = 2 \Big[& 4 + 2|k_m(0)| + 8\|k_m\|_{L^2(0,T)}^2 + T\|k_m^{/}\|_{L^2(0,T)}^2 \Big]. \end{aligned} \tag{3.45}$$

By Gronwall's lemma, we deduce from (3.31), (3.44), (3.45), that

$$\begin{aligned}&\|\widehat{u}^{/}(t)\|^2+\|\widehat{u}(t)\|_\eta^2+2C_q\int_0^t\|u^{/}(s)\|_{L^q}^q ds\le Z(t)\\&\le\rho_{1T}(m,\mu)exp(T\rho_{2T}(m)),\quad\text{for all}\quad t\in[0,T].\end{aligned}\tag{3.46}$$

By (3.11), (3.27) and (3.45), we obtain $\rho_{1T}(m,\mu)exp(T\rho_{2T}(m))\to 0$ as $m,\mu\to+\infty$. Hence, it follows from (3.46) that $\{u_m\}$ is a Cauchy sequence in $W(T)$. Therefore there exists $u\in W(T)$ such that

$$u_m\to u\quad\text{strongly in}\quad W(T).\tag{3.47}$$

On the other hand, by (3.47) and the continuity of ψ_q, we obtain

$$\psi_q(u_m^{/})\to\psi_q(u^{/})\quad\text{a.e.}\quad(x,t)\in Q_T.\tag{3.48}$$

By means of (3.21), it follows that

$$\|\psi_q(u_m^{/})\|_{L^{q^{/}}(Q_T)}^{q^{/}}=\|u_m^{/}\|_{L^q(Q_T)}^q\le\frac{1}{2}\widetilde{C}_T,\tag{3.49}$$

for all m. By Lions's lemma [5, Lemma 1.3, p. 12], it follows from (3.48) and (3.49) that

$$\psi_q(u_m^{/})\to\psi_q(u^{/})\quad\text{in}\quad L^{q^{/}}(Q_T)\quad\text{weakly.}\tag{3.50}$$

Noticing $(3.11)_3$ and (3.47) we have

$$\begin{aligned}&\left|\int_0^T dt\int_0^t k_m(t-s)a_\eta(u_m(s),w(t))ds\right.\\&\left.-\int_0^T dt\int_0^t k(t-s)a_\eta(u(s),w(t))ds\right|\\&\le\left|\int_0^T dt\int_0^t k_m(t-s)a_\eta(u_m(s)-u(s),w(t))ds\right|\\&+\left|\int_0^T dt\int_0^t[k_m(t-s)-k(t-s)]a_\eta(u(s),w(t))ds\right|\\&\le 3C_\eta\|w\|_{L^1(0,T;H^1)}\left[\|k_m\|_{L^1(0,T)}\|u_m-u\|_{L^\infty(0,T;H^1)}\right.\\&\left.+\|k_m-k\|_{L^1(0,T)}\|u\|_{L^\infty(0,T;H^1)}\right]\to 0\end{aligned}\tag{3.51}$$

for all $w\in L^1(0,T;H^1)$.

On the other hand, by $(3.11)_{3,4}$ and (3.15), we also obtain

$$g_{1m}\to g_1\quad\text{strongly in}\quad H^1(0,T).\tag{3.52}$$

From (3.24), we deduce the existence of a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$u_m^{//}\to u^{//}\quad\text{in}\quad L^{q^{/}}(0,T;(H^1)^{/})\quad\text{weak.}\tag{3.53}$$

Passing to the limit in (3.12), (3.13) by (3.47) and (3.50)-(3.53) we have u satisfying the equation

$$\begin{aligned}&\left\langle u^{//}(t),w\right\rangle+a_\eta(u(t),w)-\int_0^t k(t-s)a_\eta(u(s),w)ds+\left\langle\psi_q(u^{/}(t)),w\right\rangle\\&=g_1(t)w(1)+<f(\cdot,t,v(\cdot,t)),w>,\forall w\in H^1,\quad\text{in}\quad L^{q^{/}}(0,T)\quad\text{weak,}\end{aligned}\tag{3.54}$$

and

$$u(0) = \widetilde{u}_0, u^{/}(0) = \widetilde{u}_1. \tag{3.55}$$

On the other hand, we deduce from (3.54), that

$$u_{xx}(t) - \int_0^t k(t-s)u_{xx}(s)ds = \phi(t), \tag{3.56}$$

where

$$\phi(t) = u^{//}(t) + |u^{/}|^{q-2}u^{/} - f(\cdot, t, v(\cdot, t)) \in L^{q^{/}}(0, T; (H^1)^{/}). \tag{3.57}$$

Hence, it follows from (3.56) and (3.57), that

$$\begin{aligned}
\|u_{xx}(t)\|^{q^{/}}_{(H^1)^{/}} &\leq \left(\|\phi(t)\|_{(H^1)^{/}} + \int_0^t |k(t-s)| \|u_{xx}(s)\|_{(H^1)^{/}} ds \right)^{q^{/}} \\
&\leq 2^{q^{/}-1} \left[\|\phi(t)\|^{q^{/}}_{(H^1)^{/}} + \left(\int_0^t |k(t-s)| \|u_{xx}(s)\|_{(H^1)^{/}} ds \right)^{q^{/}} \right] \\
&\leq 2^{q^{/}-1} \left[\|\phi(t)\|^{q^{/}}_{(H^1)^{/}} + \|k\|^{q^{/}}_{L^q(0,T)} \left(\int_0^t \|u_{xx}(s)\|^{q^{/}}_{(H^1)^{/}} ds \right) \right].
\end{aligned} \tag{3.58}$$

This implies that

$$\begin{aligned}
\int_0^r \|u_{xx}(t)\|^{q^{/}}_{(H^1)^{/}} dt \leq & 2^{q^{/}-1} \|\phi\|^{q^{/}}_{L^{q^{/}}(0,T;(H^1)^{/})} \\
& + 2^{q^{/}-1} \|k\|^{q^{/}}_{L^q(0,T)} \int_0^r dt \int_0^t \|u_{xx}(s)\|^{q^{/}}_{(H^1)^{/}} ds
\end{aligned} \tag{3.59}$$

Using Gronwall's lemma, we obtain

$$\int_0^r \|u_{xx}(t)\|^{q^{/}}_{(H^1)^{/}} dt \leq 2^{q^{/}-1} \|\phi\|^{q^{/}}_{L^{q^{/}}(0,T;(H^1)^{/})} exp\left(2^{q^{/}-1} \|k\|^{q^{/}}_{L^q(0,T)} r \right) \leq C_T, \tag{3.60}$$

where C_T always indicating a constant depending on T
Thus

$$u_{xx} \in L^{q^{/}}(0, T; (H^1)^{/}). \tag{3.61}$$

On ther other hand, the estimate (3.7) hold by means of (3.11), (3.18), (3.19), (3.20), (3.47). The existence of the theorem is proved completely.

b. Uniqueness of the solution. First, we shall now require the following lemma.

Lemma 3.2. *Let u be the weak solution of the following problem*

$$\begin{aligned}
& u^{//} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds = \Phi, 0 < x < 1, 0 < t < T, \\
& u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = 0, \\
& u(x,0) = \widetilde{u}_0(x), u^{/}(x,0) = \widetilde{u}_1(x), \\
& u \in L^\infty(0,T;H^1), u^{/} ? L^\infty(0,T;L^2), \\
& k \in H^1(0,T), \Phi \in L^2(Q_T).
\end{aligned} \tag{3.62}$$

Then we have

$$\frac{1}{2}\|u^{/}(t)\|^2 + \frac{1}{2}\|u(t)\|_\eta^2 = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|u_0\|_\eta^2 - k(0)\int_0^t \|u(r)\|_\eta^2 dr$$
$$+ \int_0^t k(t-s)a(u(s),u(t))ds - \int_0^t dr\int_0^r k^{/}(r-s)a(u(s),u(r))ds \quad (3.63)$$
$$+ \int_0^t < \Phi(s), u^{/}(s) > ds, \quad a.e. \quad t\in[0,T].$$

Furthermore, if $u_0 = u_1 = 0$ *there is equality in* (3.63).

The idea of the proof is the same as in [4, Lemma 2.1, p. 79].

We now return to the proof of the uniqueness of a solution of the problem (3.3)-(3.5). Let u_1, u_2 be two weak solutions of problem (3.3)-(3.5), such that

$$u_i \in W(T), u_i^{//}, u_{ixx} \in L^{q^{/}}(0,T;(H^1)^{/}), i = 1,2. \quad (3.64)$$

Then $u = u_1 - u_2$ is the weak solution of the following problem

$$\begin{aligned} &u^{//} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + \psi_q(u_1^{/}) - \psi_q(u_2^{/}) = 0,\\ &u_x(0,t) - u(0,t) = u_x(1,t) + \eta u(1,t) = 0,\\ &u(0) = u^{/}(0) = 0,\\ &u \in W(T), u^{//}, u_{xx} \in L^{q^{/}}(0,T;(H^1)^{/}). \end{aligned} \quad (3.65)$$

By using Lemma 3.2 with $u_0 = u_1 = 0$, $\Phi = -\psi_q(u_1^{/}) + \psi_q(u_2^{/})$, we have

$$\begin{aligned}\sigma(t) =& 2\int_0^t k(t-s)a(u(s),u(t))ds - 2k(0)\int_0^t \|u(r)\|_\eta^2 dr\\ &- 2\int_0^t dr\int_0^r k^{/}(r-s)a(u(s),u(r))ds,\end{aligned} \quad (3.66)$$

where

$$\sigma(t) = \|u^{/}(t)\|^2 + \|u(t)\|_\eta^2 + 2\int_0^t \left\langle \psi_q(u_1^{/}(s)) - \psi_q(u_2^{/}(s)), u^{/}(s)\right\rangle ds. \quad (3.67)$$

By using the same computations as in the above part we obtain from (3.66) that

$$\sigma(t) = 2\left(1 + 2\|k\|_{L^2(0,T)}^2 + 2|k(0)| + \|k^{/}\|_{L^1(0,T)}^2\right)\int_0^t \sigma(r)dr. \quad (3.68)$$

By Gronwall's lemma, we deduce that $\sigma(t) = 0$ and Theorem 3.1 is completely proved. □

Theorem 3.3. *Let* $T > 0$ *and* $(H1) - (H4)$ *hold. Then there exists* $T_1 \in (0,T)$ *such that problem* (1.1)- (1.3) *has a unique weak solution* $u \in W(T_1)$ *and such that*

$$u^{//}, u_{xx} \in L^{q^{/}}(0,T_1;(H^1)^{/}). \quad (3.69)$$

Proof. For each $T_1 > 0$, we put

$$W_1(T_1) = \left\{v \in L^\infty(0,T_1;H^1) : v_t \in L^\infty(0,T_1;L^2)\right\}. \quad (3.70)$$

Then $W_1(T_1)$ is a Banach space with respect to the norm (see[5]):

$$\|v\|_{W_1(T_1)} = \|v\|_{L^\infty(0,T_1;H^1)} + \|v_t\|_{L^\infty(0,T_1;L^2)}, v \in W_1(T_1). \quad (3.71)$$

For $M>0$ and $T_1>0$, we put

$$B(M,T_1)=\left\{v\in W_1(T_1):\|v\|_{W_1(T_1)}\le M\right\}. \tag{3.72}$$

We also define the operator $\mathcal{F}$ from $B(M,T_1)$ into $W(T_1)$ by $u=\mathcal{F}(v)$, where u is the unique solution of problem (3.3)- (3.5). We would like to show that $\mathcal{F}$ is a contraction operator from $B(M,T_1)$ into itself. Applying the contraction mapping theorem, the operator $\mathcal{F}$ has a fixed point in $B(M,T_1)$ that is also a weak solution of the problem (1.1)- (1.3).

First, by Theorem 3.1, we note that the unique solution of problem (3.3)- (3.5) satisfies (3.7), (3.8), (3.9). On the other hand, it follows from $(H3)$, that

$$\begin{aligned}\int_0^t\|f(\cdot,s,v(s))\|^2ds&\le 2\int_0^t\|f(\cdot,s,v(s))-f(\cdot,s,0)\|^2ds\\&\quad+2\int_0^t\|f(\cdot,s,0)\|^2ds\\&\le 2T_1K_1^2M^2+2\int_0^T\|f(\cdot,s,0)\|^2ds,\end{aligned}\tag{3.73}$$

where

$$\begin{aligned}K_1&=K_1(M,T,f)\\&=sup\left\{|D_3f(x,t,u)|:0\le x\le 1,0\le t\le T,|u|\le\sqrt{2M}\right\}.\end{aligned}\tag{3.74}$$

It follows from (3.7)-(3.10) and (3.73) that

$$\begin{aligned}&\|u^{/}(t)\|^2+\|u(t)\|_\eta^2+2\int_0^t\|u^{/}(s)\|_{L^q}^q ds\\&\le\left(C_{1T}+2T_1K_1^2M^2\right)exp(T_1C_{2T}),\forall t\in[0,T_1],\end{aligned}\tag{3.75}$$

where

$$\begin{aligned}C_{1T}&=C_{1T}\left(\widetilde{u}_0,\widetilde{u}_1,k,g\right)=2\Big[\|\widetilde{u}_1\|^2+\|\widetilde{u}_0\|_\eta^2+2\left|g_1(0)\widetilde{u}_0(1)\right|\\&\quad+6\|g_1\|^2_{L^\infty(0,T)}+2\|g_1^{/}\|^2_{L^2(0,T)}+2\int_0^T\|f(\cdot,s,0)\|^2ds\Big],\\C_{2T}&=C_{2T}(k)=2\left[3+2|k(0)|+6\|k\|^2_{L^2(0,T)}+T\|k^{/}\|^2_{L^2(0,T)}\right].\end{aligned}\tag{3.76}$$

By choosing $M>0$ large enough so that $C_{1_T}=\frac{1}{4}M^2$, then T_1 sufficiently small so that

$$\left(\frac{1}{4}M^2+2T_1K_1^2M^2\right)exp(T_1C_{2T})\le\frac{1}{2}M^2, \tag{3.77}$$

and

$$2\sqrt{2T_1}K_1exp\left[T_1\left(2+2|k(0)|+2\|k\|^2_{L^2(0,T)}+\|k^{/}\|^2_{L^1(0,T)}\right)\right]<1. \tag{3.78}$$

From (3.75), (3.77) we have $\|u\|_{W_1(T_1)}\le M$, hence $u\in B(M,T_1)$. This shows that $\mathcal{F}$ maps $B(M,T_1)$ into itself.

Next, we verify that $\mathcal{F}$ is a contraction. Let $u_1=\mathcal{F}(v_1)$, $u_2=\mathcal{F}(v_2)$, where $v_1,v_2\in B(M,T_1)$. Put $U=u_1-u_2$ and $V=v_1-v_2$. Then U is the weak solution of the following problem

$$\begin{aligned}
&U^{//} - U_{xx} + \int_0^t k(t-s)U_{xx}(s)ds + \psi_q(u_1^/) - \psi_q(u_2^/) \\
&= f(x,t,v_1(t)) - f(x,t,v_2(t)), 0 < x < 1, 0 < t < T_1, \\
&U_x(0,t) - U(0,t) = U_x(1,t) + \eta U(1,t) = 0, \\
&U(0) = U^/(0) = 0, \\
&U \in W(T_1); U^{//}, U_{xx} \in L^{q^/}(0,T_1;(H^1)^/).
\end{aligned} \tag{3.79}$$

By using Lemma 3.2 with $\widetilde{u}_0 = \widetilde{u}_1 = 0$, $\Phi = -\psi_q(u_1^/) + \psi_q(u_2^/) + f(x,t,v_1(t)) - f(x,t,v_2(t))$, we have

$$\begin{aligned}
\delta(t) = &- 2k(0)\int_0^t \|U(r)\|_\eta^2 dr + 2\int_0^t k(t-s)a(U(s),U(t))ds \\
&- 2\int_0^t dr \int_0^r k^/(r-s)a(U(s),U(r))ds \\
&+ 2\int_0^t < f(\cdot,s,v_1(s)) - f(\cdot,s,v_2(s)), U^/(s) > ds, \text{a.e.} \quad t \in [0,T1],
\end{aligned} \tag{3.80}$$

where

$$\begin{aligned}
\delta(t) &= \|U^/(t)\|^2 + \|U(t)\|_\eta^2 + 2\int_0^t \left\langle \psi_q(u_1^/) - \psi_q(u_2^/), U^/(s) \right\rangle ds \\
&\geq \|U^/(t)\|^2 + \|U(t)\|_\eta^2 + 2C_q \int_0^t \|U^/(s)\|_{L^q}^q ds.
\end{aligned} \tag{3.81}$$

By the assumption $(H4)$, we have

$$\begin{aligned}
&2\int_0^t \left\langle f(\cdot,s,v_1(s)) - f(\cdot,s,v_2(s)), U^/(s) \right\rangle ds \\
&\leq \int_0^t \|U^/(s)\|^2 ds + \int_0^t \|f(\cdot,s,v_1(s)) - f(\cdot,s,v_2(s))\|^2 ds \\
&\leq \int_0^t \|U^/(s)\|^2 ds + 2T_1 K_1^2 \|V\|_{W_1(T_1)}^2,
\end{aligned} \tag{3.82}$$

Therefore, we can prove in a similar manner as above that

$$\begin{aligned}
\delta(t) \leq & 2T_1 K_1^2 \|V\|_{W_1(T_1)}^2 \\
&+ 2\left(2 + 2|k(0)| + 2\|k\|_{L^2(0,T)}^2 + \|k^/\|_{L^1(0,T)}^2\right) \int_0^t \delta(s)ds.
\end{aligned} \tag{3.83}$$

By Gronwall's lemma, we obtain from (3.83) that

$$\delta(t) = 2\left(\rho_1(k,K_1,T,T_1)\|V\|_{W_1(T_1)}\right)^2, \tag{3.84}$$

where

$$\begin{aligned}
&\rho_1(k,K_1,T,T_1) = \\
&\sqrt{2T_1}K_1 exp\left[T_1\left(2 + 2|k(0)| + 2\|k\|_{L^2(0,T)}^2 + \|k^/\|_{L^1(0,T)}^2\right)\right].
\end{aligned} \tag{3.85}$$

It follows from (3.81), (3.84) and (3.85) that

$$\|U\|_{W_1(T_1)} \leq 2\rho_1(k,K_1,T,T_1)\|V\|_{W_1(T_1)}, \tag{3.86}$$

where

$$2\rho_1(k, K_1, T, T_1) < 1, \tag{3.87}$$

since (3.78) and (3.85).

Hence, (3.86) shows that $\mathcal{F} : B(M,T_1) \to B(M,T_1)$ is a contraction. Applying the contraction mapping theorem, the operator $\mathcal{F}$ has a fixed point in $B(M,T_1)$ that is also a weak solution of the problem (1.1)- (1.3).

The solution of the problem (1.1)- (1.3) is unique, that can be showed using the same arguments as in the proof of Theorem 3.1. The proof of Theorem 3.3 is completed. □

Remark 3.4. *In the case of $\lambda = 0$, $f(x,t,u) = |u|^{p-2}u$, $p > 2$, $k \in W^{2,1}(\mathbb{R}_+)$, $k \geq 0$, $k(0) > 0$, $0 < \int_0^{+\infty} k(t)dt < 1$, $k'(t) + \zeta k(t) \leq 0$ for all $t \geq 0$, with $\zeta > 0$, and the boundary condition $u(0,t) = u(1,t) = 0$ standing for (1.2), S. Berrimia, S. A. Messaoudi [1] has obtained a global existence and uniqueness theorem.*

4. Decay of solution

In this part, we will consider the problem of global existence and asymptotic behavior for $t \to +\infty$. We assume that $g(t) = 0$, $f(x,t,u) = F(x,t) - |u|^{p-2}u$, $p \geq 2$ and consider the following problem

$$\begin{aligned} &u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u|^{p-2}u + |u_t|^{q-2}u_t \\ &= F(x,t), 0 < x < 1, t > 0, \\ &u_x(0,t) = u(0,t), u_x(1,t) + \eta u(1,t) = 0, \\ &u(x,0) = \widetilde{u}_0(x), u_t(x,0) = \widetilde{u}_1(x), \end{aligned} \tag{4.1}$$

We make the following assumptions:

- $(\widetilde{H}1)$ $\eta \geq 0, p, q \geq 2$,
- $(\widetilde{H}2)$ $k \in W^{2,1}(\mathbb{R}_+)$, $k \geq 0$, satisfying
  - (i) $k(0) > 0$, $0 < 1 - \int_0^{+\infty} k(t)dt = k_\infty < 1$,
  - (ii) there exists a positive constant ζ such that $k'(t) + \zeta k(t) \leq 0$ for all $t \geq 0$,
- $(\widetilde{H}3)$ $\widetilde{u}_0 \in H^2$ and $\widetilde{u}_1 \in H^1$,
- $(\widetilde{H}4)$ $F \in L^1(0,\infty;L^2) \cap L^2(0,\infty;L^2)$, $F_t \in L^1(0,\infty;L^2)$,
- $(\widetilde{H}5)$ There exists a constant $\sigma > 0$ such that $\int_0^\infty e^{st}\|F(t)\|^2 dt < +\infty$.

Under assumptions $(\widetilde{H}1)$-$(\widetilde{H}4)$ and let $T > 0$, by theorem 2.3, the problem (4.1) has a unique weak solution $u(t)$ such that

$$u \in L^\infty(0,T;H^2), u_t \in L^\infty(0,T;H^1), u_{tt} \in L^\infty(0,T;L^2). \tag{4.2}$$

Then, we have the following

Lemma 4.1. *Suppose that $(\widetilde{H}1) - (\widetilde{H}4)$ hold. Then there is a unique solution $u(t)$ of problem (4.1) defined on $\mathbb{R}_+$. Moreover*

$$\|u'(t)\| + \|u(t)\|_\eta \leq C \quad \textit{for all} \quad t \geq 0, \tag{4.3}$$

where C is a positive constant depending only on $\widetilde{u}_0$, $\widetilde{u}_1$, F, k_∞ and p.

Proof. By multiplying the equation $(4.1)_1$ by u_t and integrate over $(0,1)\times(0,t)$ we obtain

$$\begin{aligned} E(t) &+ 2\int_0^t \|u^{/}(s)\|_{L^q}^q ds + \int_0^t k(s)\|u(s)\|_\eta^2 ds \\ &- \int_0^t dr \int_0^r k^{/}(r-s)\|u(s)-u(r)\|_\eta^2 ds \\ &= E(0) + 2\int_0^t < F(s), u^{/}(s) > ds, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} E(t) =& \|u^{/}(t)\|^2 + \left(1 - \int_0^t k(s)ds\right)\|u(t)\|_\eta^2 + \frac{2}{p}\|u(t)\|_{L^p}^p \\ &+ \int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds. \end{aligned} \tag{4.5}$$

On the other hand, by $(\widetilde{H}4)$ and the Cauchy's inequality, we obtain

$$\begin{aligned} 2\int_0^t \left\langle F(s), u^{/}(s)\right\rangle ds &\le \int_0^t \|F(s)\|ds + \int_0^t \|F(s)\|\|u^{/}(s)\|^2 ds \\ &\le \int_0^{+\infty} \|F(s)\|ds + \int_0^t \|F(s)\|E(s)ds. \end{aligned} \tag{4.6}$$

By Gronwall's lemma, we obtain from (4.4) and (4.6) that

$$\begin{aligned} E(t) &\le \left(E(0) + \int_0^{+\infty}\|F(s)\|ds\right) exp\left(\int_0^t \|F(s)\|ds\right) \\ &\le \left(E(0) + \int_0^{+\infty}\|F(s)\|ds\right) exp\left(\int_0^{+\infty} \|F(s)\|ds\right) = C, \forall t \ge 0. \end{aligned} \tag{4.7}$$

By $(\widetilde{H}3, i)$, we have

$$E(t) \ge \|u^{/}(t)\|^2 + \left(1 - \int_0^t k(s)ds\right)\|u(t)\|_\eta^2 \ge \|u^{/}(t)\|^2 + k_\infty\|u(t)\|_\eta^2. \tag{4.8}$$

Then we obtain (4.3) from (4.7) and (4.8). This completes the proof of Lemma 4.1. □

In this section we state and prove decay result.

Theorem 4.2. *Suppose that $(\widetilde{H}1)-(\widetilde{H}5)$ hold. Then the solution $u(t)$ of problem* (4.1) *decays exponentially to zero as $t \to +\infty$ in the following sense: there exist the positive constants N and γ such that*

$$\|u^{/}(t)\| + \|u(t)\|_\eta \le Ne^{-\gamma t} \quad \textit{for all} \quad t \ge 0. \tag{4.9}$$

Proof. We use the following functional

$$\Gamma(t) = \Gamma(\varepsilon_1, \varepsilon_2, t) = E(t) + \varepsilon_1 E_1(t) + \varepsilon_2 E_2(t), \tag{4.10}$$

where

$$E_1(t) =< u(t), u^{/}(t) >, \tag{4.11}$$

$$E_2(t) = -\int_0^t k(t-s)\left\langle u^{/}(t), u(t)-u(s)\right\rangle ds. \tag{4.12}$$

Estimating $\Gamma(t)$.
By (2.3), (2.4), we obtain from $(\widetilde{H}2, i)$ that

$$|E_1(t)| = \left| < u(t), u^{/}(t) > \right| \le \frac{1}{2}\|u^{/}(t)\|^2 + \|u(t)\|_\eta^2, \tag{4.13}$$

$$\begin{aligned} |E_2(t)| &= \left| \int_0^t k(t-s) \left\langle u^{/}(t), u(t) - u(s) \right\rangle ds \right| \\ &\le \frac{1}{2}\|u^{/}(t)\|^2 + \frac{1}{2}\left(\int_0^t k(t-s)\|u(t) - u(s)\| ds \right)^2 \\ &\le \frac{1}{2}\|u^{/}(t)\|^2 + (1 - k_\infty) \int_0^t k(t-s)\|u(t) - u(s)\|_\eta^2 ds. \end{aligned} \tag{4.14}$$

Hence, it follows from (4.10)-(4.14) that for ε_1, ε_2 small enough, there exist two positive constants α_1, α_2, such that

$$\alpha_1 E(t) \le \Gamma(t) \le \alpha_2 E(t). \tag{4.15}$$

Estimating $\Gamma^{/}(t)$.
Now differentiating (4.4) with respect to t, we have

$$\begin{aligned} E^{/}(t) &= -2\|u^{/}(t)\|_{L^q}^q + \int_0^t k^{/}(t-s)\|u(s) - u(t)\|_\eta^2 ds \\ &\quad - k(t)\|u(t)\|_\eta^2 + 2\left\langle F(t), u^{/}(t) \right\rangle \\ &\le -2\|u^{/}(t)\|_{L^q}^q + \int_0^t k^{/}(t-s)\|u(s) - u(t)\|_\eta^2 ds + 2\left\langle F(t), u^{/}(t) \right\rangle, \end{aligned} \tag{4.16}$$

since $k(t) \ge 0$.

By multiplying the equation $(4.1)_1$ by u and integrate over $(0, 1)$ we obtain

$$\begin{aligned} E_1^{/}(t) &= \|u^{/}(t)\|^2 - \|u(t)\|_\eta^2 - \|u(t)\|_{L^p}^p + \langle F(t), u(t) \rangle \\ &\quad + \int_0^t k(t-s)a(u(s), u(t))ds - \left\langle |u^{/}(t)|^{q-2}u^{/}(t), u(t) \right\rangle \\ &= \|u^{/}(t)\|^2 - \|u(t)\|_\eta^2 - \|u(t)\|_{L^p}^p + \langle F(t), u(t) \rangle + I_1(t) + I_2(t). \end{aligned} \tag{4.17}$$

We now estimate the last two terms in the right side of (4.17) as follows

Estimating $I_1(t)$.
Using the inequality

$$ab \le \frac{\delta}{r}a^r + \frac{r-1}{r}\delta^{\frac{-r}{r-1}} b^{\frac{r}{r-1}}, \forall a, b \ge 0, \forall r > 1, \forall \delta > 0, \tag{4.18}$$

we have

$$\begin{aligned}
I_1(t) &= \int_0^t k(t-s)a(u(s),u(t))ds \\
&= \int_0^t k(t-s)a\left(u(s)-u(t),u(t)\right)ds + \int_0^t k(t-s)\|u(t)\|_\eta^2 ds \\
&\le \delta_1\|u(t)\|_\eta^2 + \frac{1}{4\delta_1}\left(\int_0^t k(s)ds\right)\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds\right) \\
&+ \left(\int_0^t k(s)ds\right)\|u(t)\|_\eta^2 \\
&\le \delta_1\|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_1}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds \\
&+ (1-k_\infty)\|u(t)\|_\eta^2 \\
&\le (\delta_1+1-k_\infty)\|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_1}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds,
\end{aligned} \tag{4.19}$$

for all $\delta_1 > 0$.
Estimating $I_2(t)$.
We again use inequality (4.18) we obtain from (4.3) that

$$\begin{aligned}
I_2(t) &= -\left\langle |u'(t)|^{q-2}u'(t),u(t)\right\rangle \le \|u'(t)\|_{L^q}^{q-1}\|u(t)\|_{L^q} \\
&\le \frac{\delta_1^q}{q}\|u(t)\|_{L^q}^q + \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}\|u'(t)\|_{L^q}^q \\
&\le 2\frac{\delta_1^q}{q}\left(\sqrt{2}C\right)^{q-2}\|u(t)\|_\eta^2 + \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}\|u'(t)\|_{L^q}^q,
\end{aligned} \tag{4.20}$$

for all $\delta_1 > 0$.
By combining (4.17), (4.19) and (4.20), we obtain

$$\begin{aligned}
E_1'(t) &\le -\|u(t)\|_{L^p}^p + \|u'(t)\|^2 - \left(k_\infty - \delta_1 - 2\frac{\delta_1^q}{q}\left(\sqrt{2}C\right)^{q-2}\right)\|u(t)\|_\eta^2 \\
&+ \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}\|u'(t)\|_{L^q}^q + \frac{1-k_\infty}{4\delta_1}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds \\
&+ \langle F(t),u(t)\rangle .
\end{aligned} \tag{4.21}$$

Then, we can always choose the constant $\delta_1 > 0$ such that

$$\gamma_1 = k_\infty - \delta_1 - 2\frac{\delta_1^q}{q}\left(\sqrt{2}C\right)^{q-2} > 0. \tag{4.22}$$

This implies that

$$\begin{aligned}
E_1'(t) \le -\|u(t)\|_{L^p}^p + \|u'(t)\|^2 - \gamma_1\|u(t)\|_\eta^2 + \gamma_2\|u'(t)\|_{L^q}^q \\
+ \gamma_3\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds + \langle F(t),u(t)\rangle ,
\end{aligned} \tag{4.23}$$

where

$$\gamma_2 = \frac{q-1}{q}\delta_1^{\frac{-q}{q-1}}, \gamma_3 = \frac{1-k_\infty}{4\delta_1}. \tag{4.24}$$

Direct calculations give

$$\begin{aligned} E_2^{/}(t) &= -\left(\int_0^t k(s)ds\right)\|u^{/}(t)\|^2 - \int_0^t k^{/}(t-s)\left\langle u^{/}(t), u(t)-u(s)\right\rangle ds \\ &+ \int_0^t k(t-s)a(u(t),u(t)-u(s))ds \\ &- \int_0^t k(t-s)a\left(\int_0^t k(t-\tau)u(\tau)d\tau, u(t)-u(s)\right)ds \\ &+ \int_0^t k(t-s)\left\langle |u(t)|^{p-2}u(t), u(t)-u(s)\right\rangle ds \\ &+ \int_0^t k(t-s)\left\langle |u^{/}(t)|^{q-2}u^{/}(t), u(t)-u(s)\right\rangle ds \\ &- \int_0^t k(t-s)\left\langle F(t), u(t)-u(s)\right\rangle ds = \sum_{i=1}^{7} J_i(t). \end{aligned} \tag{4.25}$$

Similarly to (4.17), we estimate respectively the following terms on the right-hand side of (4.25) as follows.
Estimating $J_1(t)$.
Since k is continuous and $k(0)>0$ then there exists $t_0>0$, such that

$$\int_0^t k(s)ds \geq \int_0^{t_0} k(s)ds = k_0 > 0 \quad \text{for all} \quad t \geq t_0. \tag{4.26}$$

Hence,

$$J_1(t) = -\left(\int_0^t k(s)ds\right)\|u^{/}(t)\|^2 \leq -k_0\|u^{/}(t)\|^2 \quad \text{for all} \quad t\geq t_0. \tag{4.27}$$

Estimating $J_2(t)$.

$$\begin{aligned} J_2(t) &= -\int_0^t k^{/}(t-s)\left\langle u^{/}(t), u(t)-u(s)\right\rangle ds \\ &\leq \delta_2\|u^{/}(t)\|^2 + \frac{1}{4\delta_2}\left(\int_0^t |k^{/}(t-s)|ds\right)\left(\int_0^t |k^{/}(t-s)|\|u(s)-u(t)\|^2 ds\right) \\ &\leq \delta_2\|u^{/}(t)\|^2 + \frac{1}{2\delta_2}\left(\int_0^t |k^{/}(t-s)|ds\right)\left(\int_0^t |k^{/}(t-s)|\|u(s)-u(t)\|_\eta^2 ds\right) \\ &\leq \delta_2\|u^{/}(t)\|^2 - \frac{k(0)}{2\delta_2}\int_0^t k^{/}(t-s)\|u(s)-u(t)\|_\eta^2 ds. \end{aligned} \tag{4.28}$$

Estimating $J_3(t)$.

$$\begin{aligned} J_3(t) &= \int_0^t k(t-s)a\left(u(t),u(t)-u(s)\right)ds \\ &\leq \delta_2\|u(t)\|_\eta^2 + \frac{1}{4\delta_2}\left(\int_0^t k(s)ds\right)\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds\right) \\ &\leq \delta_2\|u(t)\|_\eta^2 + \frac{1-k_\infty}{4\delta_2}\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds \end{aligned} \tag{4.29}$$

Estimating $J_4(t)$.

$$\begin{aligned}
J_4(t) &= -\int_0^t k(t-s)a\left(\int_0^t k(t-\tau)u(\tau)d\tau, u(t)-u(s)\right) ds \\
&\le \int_0^t k(t-\tau)\|u(\tau)\|_\eta d\tau \int_0^t k(t-s)\|u(s)-u(t)\|_\eta ds \\
&\le \delta_2 \left(\int_0^t k(t-\tau)\|u(\tau)\|_\eta d\tau\right)^2 \\
&+ \frac{1}{4\delta_2}\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta ds\right)^2 \\
&\le 2\delta_2 \left(\int_0^t k(t-\tau)\|u(\tau)\|_\eta d\tau\right)^2 \\
&+ \left(2\delta_2 + \frac{1}{4\delta_2}\right)\left(\int_0^t k(t-s)\|u(s)-u(t)\|_\eta ds\right)^2 \\
&\le 2\delta_2(1-k_\infty)^2\|u(t)\|_\eta^2 \\
&+ \left(2\delta_2 + \frac{1}{4\delta_2}\right)(1-k_\infty)\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2 ds.
\end{aligned} \tag{4.30}$$

Estimating $J_5(t)$.

$$\begin{aligned}
&J_5(t) = \int_0^t k(t-s)\left\langle |u(t)|^{p-2}u(t), u(t)-u(s)\right\rangle ds \\
&\le 2\left(\sqrt{2}C\right)^{p-2}\int_0^t k(t-s)\|u(t)\|_\eta\|u(t)-u(s)\|_\eta ds \\
&\le 2\left(\sqrt{2}C\right)^{p-2}\left[\delta_2\|u(t)\|_\eta^2 + \frac{1}{4\delta_2}\left(\int_0^t k(t-s)\|u(t)-u(s)\|_\eta ds\right)^2\right] \\
&\le 2\left(\sqrt{2}C\right)^{p-2}\left[\delta_2\|u(t)\|_\eta^2 + \frac{1}{4\delta_2}(1-k_\infty)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds\right].
\end{aligned} \tag{4.31}$$

Estimating $J_6(t)$. We again use inequality (4.18) with $r = q$, $\delta = \delta_2$, we obtain from (4.3) that

$$\begin{aligned}
&\left\langle |u'(t)|^{q-2}u'(t), u(t)-u(s)\right\rangle \le \|u'(t)\|_{L^q}^{q-1}\|u(t)-u(s)\|_{L^q} \\
&\le \frac{\delta_2^q}{q}\|u(t)-u(s)\|_{L^q}^q + \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}\|u'(t)\|_{L^q}^q \\
&\le 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2}\|u(t)-u(s)\|_\eta^2 + \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}\|u'(t)\|_{L^q}^q.
\end{aligned} \tag{4.32}$$

It follows from (4.32) that

$$\begin{aligned} J_6(t) &= \int_0^t k(t-s)\left\langle |u'(t)|^{q-2}u'(t), u(t)-u(s)\right\rangle ds \\ &\leq 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2}\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds \\ &+ \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}\|u'(t)\|_{L^q}^q\int_0^t k(t-s)ds \\ &\leq 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2}\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds \\ &+ \frac{q-1}{q}\delta_2^{\frac{-q}{q-1}}(1-k_\infty)\|u'(t)\|_{L^q}^q \end{aligned} \tag{4.33}$$

Estimating $J_7(t)$

$$\begin{aligned} J_7(t) &= -\int_0^t k(t-s)\langle F(t), u(t)-u(s)\rangle ds \\ &\leq \int_0^t k(t-s)\|F(t)\|\|u(t)-u(s)\|ds \\ &\leq \frac{1}{4\delta_2}\|F(t)\|^2 + \delta_2\left(\int_0^t k(t-s)ds\right)\left(\int_0^t k(t-s)\|u(t)-u(s)\|^2 ds\right) \\ &\leq \frac{1}{4\delta_2}\|F(t)\|^2 + 2\delta_2(1-k_\infty)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds. \end{aligned} \tag{4.34}$$

By combining (4.25), (4.27)-(4.31), (4.33) and (4.34), we obtain

$$\begin{aligned} E_2'(t) &= -\left(k_0-\delta_2\right)\|u'(t)\|^2 + \delta_2\widehat{\gamma}_1\|u(t)\|_\eta^2 + \widehat{\gamma}_2\|u'(t)\|_{L^q}^q \\ &+ \widehat{\gamma}_3\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2 ds \\ &- \widehat{\gamma}_4\int_0^t k'(t-s)\|u(s)-u(t)\|_\eta^2 ds + \frac{1}{4\delta_2}\|F(t)\|^2, \end{aligned} \tag{4.35}$$

where

$$\begin{aligned} \widehat{\gamma}_1 &= 1 + 2(1-k_\infty)^2 + 2\left(\sqrt{2}C\right)^{p-2}, \\ \widehat{\gamma}_2 &= \frac{q}{q-1}\delta_2^{\frac{-q}{q-1}}(1-k_\infty), \\ \widehat{\gamma}_3 &= 2\frac{\delta_2^q}{q}\left(2\sqrt{2}C\right)^{q-2} + (1-k_\infty)\left[\frac{1}{2\delta_2}\left(\sqrt{2}C\right)^{p-2} + \left(4\delta_2 + \frac{1}{2\delta_2}\right)\right], \\ \widehat{\gamma}_4 &= \frac{k(0)}{2\delta_2}. \end{aligned} \tag{4.36}$$

Combining of (4.10), (4.16), (4.23) and (4.35), we obtain

$$\begin{aligned}
&\Gamma^{/}(t)+\varepsilon_1\|u(t)\|_{L^p}^p+((k_0-\delta_2)\varepsilon_2-\varepsilon_1)\,\|u^{/}(t)\|^2\\
&\quad+(\varepsilon_1\gamma_1-\varepsilon_2\delta_2\widehat{\gamma}_1)\,\|u(t)\|_\eta^2+(2-\varepsilon_1\gamma_2-\varepsilon_2\widehat{\gamma}_2)\,\|u^{/}(t)\|_{L^q}^q\\
&\quad-(\varepsilon_1\gamma_3+\varepsilon_2\widehat{\gamma}_3)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2ds\\
&\quad-(1-\varepsilon_2\widehat{\gamma}_4)\int_0^t k^{/}(t-s)\|u(s)-u(t)\|_\eta^2ds\\
&\quad\leq\left\langle F(t),2u^{/}(t)+\varepsilon_1u(t)\right\rangle+\frac{\varepsilon_2}{4\delta_2}\|F(t)\|^2.
\end{aligned}\tag{4.37}$$

Whence δ_1 is fixed, choosing

$$\delta_2=\frac{1}{2}\frac{k(0)\gamma_1}{\gamma_1+\widehat{\gamma}_1},\varepsilon_2=\frac{2}{k_0}\varepsilon_1,\quad\text{where}\quad\varepsilon_1>0\quad\text{is arbitrary,}\tag{4.38}$$

we deduce from (4.37) and (4.38) that

$$\begin{aligned}
&\Gamma^{/}(t)+\varepsilon_1\|u(t)\|_{L^p}^p+\frac{\varepsilon_1\widehat{\gamma}_1}{\gamma_1+\widehat{\gamma}_1}\|u^{/}(t)\|^2+\frac{\varepsilon_1\gamma_1^2}{\gamma_1+\widehat{\gamma}_1}\|u(t)\|_\eta^2\\
&\quad+\left(2-\varepsilon_1\left(1+\frac{2}{k(0)}\widehat{\gamma}_2\right)\right)\|u^{/}(t)\|_{L^q}^q\\
&\quad-\varepsilon_1\left(\gamma_3+\frac{2}{k(0)}\widehat{\gamma}_3\right)\int_0^t k(t-s)\|u(t)-u(s)\|_\eta^2ds\\
&\quad-\left(1-\frac{2}{k(0)}\varepsilon_1\widehat{\gamma}_4\right)\int_0^t k^{/}(t-s)\|u(s)-u(t)\|_\eta^2ds\\
&\quad\leq\left\langle F(t),2u^{/}(t)+\varepsilon_1u(t)\right\rangle+\frac{\varepsilon_1}{k_0^2}\left(1+\frac{\widehat{\gamma}_1}{\gamma_1}\right)\|F(t)\|^2.
\end{aligned}\tag{4.39}$$

Next, we choose $\varepsilon_1>0$, with

$$\varepsilon_1<min\left\{\frac{\zeta}{\gamma_3+\frac{2}{k_0}\widehat{\gamma}_3+\frac{2}{k_0}\widehat{\gamma}_4\zeta},\frac{2}{1+\frac{2}{k_0}\widehat{\gamma}_2}\right\}$$

and (4.15) is satisfied, then by the assumption $(\widetilde{H}2,ii)$, we deduce that

$$\begin{aligned}
&\Gamma^{/}(t)+\varepsilon_1\|u(t)\|_{L^p}^p+\frac{\varepsilon_1\widehat{\gamma}_1}{\gamma_1+\widehat{\gamma}_1}\|u^{/}(t)\|^2+\frac{\varepsilon_1\gamma_1^2}{\gamma_1+\widehat{\gamma}_1}\|u(t)\|_\eta^2+k_1\|u^{/}(t)\|_{L^q}^q\\
&\quad+k_2\int_0^t k(t-s)\|u(s)-u(t)\|_\eta^2ds\\
&\quad\leq\left\langle F(t),2u^{/}(t)+\varepsilon_1u(t)\right\rangle+k_3\|F(t)\|^2,
\end{aligned}\tag{4.40}$$

where

$$\begin{aligned}
k_1&=2-\varepsilon_1\left(1+\frac{2}{k_0}\widehat{\gamma}_2\right)>0,\\
k_2&=\zeta\left(1-\frac{2}{k_0}\varepsilon_1\widehat{\gamma}_4\right)-\varepsilon_1\left(\gamma_3+\frac{2}{k_0}\widehat{\gamma}_3\right)>0,\\
k_3&=\frac{\varepsilon_1}{k_0^2}\left(1+\frac{\widehat{\gamma}_1}{\gamma_1}\right).
\end{aligned}\tag{4.41}$$

By combining (4.5), (4.15) and (4.40), we can always choose the constant $\widetilde{\gamma} > 0$ is independent of t such that

$$\Gamma^{/}(t) + 2\widetilde{\gamma}\Gamma(t) \leq \left\langle F(t), 2u^{/}(t) + \varepsilon_1 u(t)\right\rangle + k_3\|F(t)\|^2, \tag{4.42}$$

for all $t \geq t_0$.

On ther other hand,

$$\left\langle F(t), 2u^{/}(t) + \varepsilon_1 u(t)\right\rangle + k_3\|F(t)\|^2 \leq \widetilde{N}\|F(t)\|^2 + \widetilde{\gamma}\Gamma(t), \tag{4.43}$$

for some constant $\widetilde{N} > 0$. Therefore

$$\Gamma^{/}(t) + \widetilde{\gamma}\Gamma(t) \leq \widetilde{N}\|F(t)\|^2 \quad \text{for all} \quad t \geq t_0. \tag{4.44}$$

Putting $\gamma = \frac{1}{2}min\{\sigma, \widetilde{\gamma}\}$. A simple integration of (4.44) over (t_0, t) gives

$$\Gamma(t) \leq \left[\Gamma(t_0)e^{\sigma t_0} + \widetilde{N}\int_{t_0}^{+\infty} e^{\sigma s}\|F(s)\|^2 ds\right] e^{-2\gamma t} = N_1 e^{-2\gamma t}, \tag{4.45}$$

for all $t \geq t_0$.

By the boundedness of $\Gamma(t)$ on $[0, t_0]$, we deduce from (4.45) that

$$\Gamma(t) = \|\Gamma\|_{L^\infty(0,t_0)} e^{-2\gamma(t-t_0)} + N_1 e^{-2\gamma t} = N_2 e^{-2\gamma t}, \tag{4.46}$$

for all $t \geq 0$.

By (4.15), it follows from (4.46) that

$$E(t) \leq \frac{1}{\alpha_1}\Gamma(t) \leq \frac{1}{\alpha_1}N_2 e^{-2\gamma t}, \quad \text{for all} \quad t \geq 0. \tag{4.47}$$

This completes the proof of Theorem 4.2. □

Remark 4.3. *The estimate* (4.9) *holds for any regular solution corresponding to* $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$. *This remains holds for solutions corresponding to* $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$ *by simple density argument.*

5. Numerical results

Consider the following problem:

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + u_t^3 = u^2 + F(x,t), 0 < x < 1, 0 < t < T, \tag{5.1}$$

with boundary conditions

$$u_x(0,t) = u(0,t), u_x(1,t) + u(1,t) = 0, \tag{5.2}$$

and initial conditions

$$u(x,0) = \widetilde{u}_0(x), u_t(x,0) = \widetilde{u}_1(x), \tag{5.3}$$

where

$$\widetilde{u}_0(x) = -x^2 + x + 1, \widetilde{u}_1(x) = -\widetilde{u}_0(x), k(t) = \frac{1}{2}e^{-t}, \tag{5.4}$$

$$F(x,t) = (2-t)e^{-t} + U_{ex}(1 - U_{ex} - U_{ex}^2), \tag{5.5}$$

where

$$U_{ex}(x,t) = (-x^2 + x + 1)e^{-t}. \tag{5.6}$$

The exact solution of the problem (5.1)-(5.3) with $\widetilde{u}_0(x)$, $\widetilde{u}_1(x)$, $k(t)$ and $F(x,t)$ defined in (5.4) and (5.5) respectively, is the function U_{ex} given in (5.6). To solve problem (5.1)-(5.3) numerically, we consider the differential system for the unknowns $u_j(t) = u(x_j, t)$, $v_j(t) = \frac{du_j}{dt}(t)$, with $x_j = jh$, $h = \frac{1}{N}$, $j = 0, 1, ..., N$:

$$
\begin{gathered}
\frac{du_j}{dt}(t) = v_j(t), j = 0, 1, ..., N, \\
\frac{dv_0}{dt}(t) = \frac{1}{h^2}\left[-(1+h)u_0(t) + u_1(t)\right] \\
-\frac{1}{h^2}\int_0^t k(t-s)\left[-(1+h)u_0(s) + u_1(s)\right] ds - v_0^3(t) + u_0^2(t) + F(x_0, t), \\
\frac{dv_j}{dt}(t) = \frac{1}{h^2}\left[u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\right] \\
-\frac{1}{h^2}\int_0^t k(t-s)\left[u_{j-1}(s) - 2u_j(s) + u_{j+1}(s)\right] ds \\
-v_j^3(t) + u_j^2(t) + F(x_j, t), j = 1, 2, ..., N-1, \\
\frac{dv_N}{dt}(t) = \frac{1}{h^2}\left[u_{N-1}(t) - (1+h)u_N(t)\right] \\
-\frac{1}{h^2}\int_0^t k(t-s)\left[u_{N-1}(s) - (1+h)u_N(s)\right] ds - v_N^3(t) + u_N^2(t) + F(x_N, t), \\
u_j(0) = \widetilde{u}_0(x_j), v_j(0) = \widetilde{u}_1(x_j), j = 0, 1, ..., N.
\end{gathered} \tag{5.7}
$$

To solve the nonlinear differential system (5.7), we use the following linear recursive scheme generated by the nonlinear term

$$
\begin{aligned}
&\frac{du_j^{(n)}}{dt}(t) = v_j^{(n)}(t), j = 0,1,...,N, \\
&\frac{dv_0^{(n)}}{dt}(t) = \frac{1}{h^2}\left[-(1+h)u_0^{(n)}(t) + u_1^{(n)}(t)\right] \\
&-\frac{\Delta t}{h^2}\sum_{i=1}^{N_1-1} k(t-i\Delta t)\left[-(1+h)u_0^{(n)}(i\Delta t) + u_1^{(n)}(i\Delta t)\right] \\
&-\left(v_0^{(n-1)}(t)\right)^3 + \left(u_0^{(n-1)}(t)\right)^3 + F(x_0,t), \\
&\frac{dv_j^{(n)}}{dt}(t) = \frac{1}{h^2}\left[u_{j-1}^{(n)}(t) - 2u_j^{(n)}(t) + u_{j+1}^{(n)}(t)\right] \\
&-\frac{\Delta t}{h^2}\sum_{i=1}^{N_1-1} k(t-i\Delta t)\left[u_{j-1}^{(n)}(i\Delta t) - 2u_j^{(n)}(i\Delta t) + u_{j+1}^{(n)}(i\Delta t)\right] \\
&-\left(v_j^{(n-1)}(t)\right)^3 + \left(u_j^{(n-1)}(t)\right)^2 + F(x_j,t), j = 1,2,...,N-1, \\
&\frac{dv_N^{(n)}}{dt}(t) = \frac{1}{h^2}\left[u_{N-1}^{(n)}(t) - (1+h)u_N^{(n)}(t)\right] \\
&-\frac{\Delta t}{h^2}\sum_{i=1}^{N_1-1} k(t-i\Delta t)\left[u_{N-1}^{(n)}(i\Delta t) - (1+h)u_N^{(n)}(i\Delta t)\right] \\
&-\left(v_N^{(n-1)}(t)\right)^3 + \left(u_N^{(n-1)}(t)\right)^2 + F(x_N,t), \\
&u_j^{(n)}(0) = \widetilde{u}_0(x_j), v_j^{(n)}(0) = \widetilde{u}_1(x_j), j = 0,1,...,N,
\end{aligned}
\tag{5.8}
$$

and where $u_j^{(n)}(i\Delta t),\ i = 1,...,N_1-1,\ j = 0,1,...,N,$ of the system (5.8) being calculated at the time $t = N_1\Delta t$.

The latter system is solved by a spectral method and since the matrix of this system is very ill-conditioned so we have to regularize it by adding to the diagonal terms a small parameter in order to have a good accuracy of the convergence.

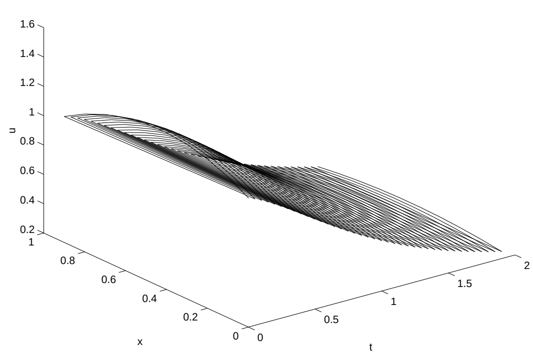

Figure 1

In fig.1 we have drawn the approximated solution of the problem (5.1)-(5.5) while fig.2 represents his corresponding exact solution (5.6).

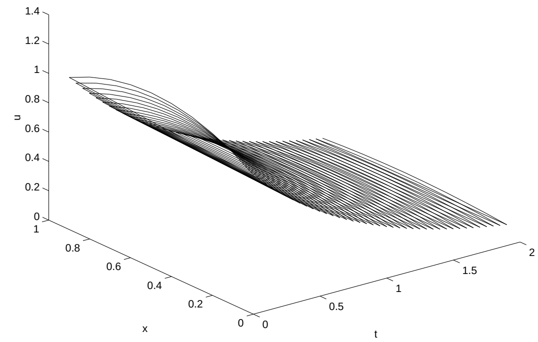

Figure 2

The fig.3 corresponds to the surface $(x,t) \mapsto u(x,t)$ approximated solution in the case where $F(x,t) = 0$. So in both cases we notice the very good decay of these surfaces from $T = 0$ to $T = 2$.

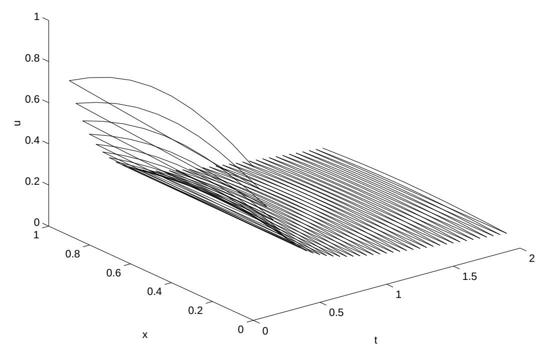

Figure 3

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