Weakly dependent random fields with infinite interactions

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Abstract

We introduce new models of stationary random fields, solutions of

$$X_t = F\left((X_{t-j})_{j\in\mathbb{Z}^d\setminus\{0\}}; \xi_t\right),$$

the input random field ξ is stationary, *e.g.* ξ is independent and identically distributed (iid). Such models extend most of those used in statistics. The (nontrivial) existence of such models is based on a contraction principle and Lipschitz conditions are needed; those assumptions imply Doukhan and Louhichi (1999)'s [6] weak dependence conditions. In contrast to the concurrent ones, our models are not set in terms of conditional distributions. Various examples of such random fields are considered. We also use a very weak notion of causality of independent interest: it allows to relax the boundedness assumption of inputs for several new heteroscedastic models, solutions of a nonlinear equation.

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1 Introduction

Description of random fields is a difficult task, a very deep reference is Georgii's (1988) book [11]; a synthetic presentation is given by Föllmer (1985) [9]. The usual way to describe interactions makes use of conditional distributions with respect to large sets of indices. This presentation is natural for discrete valued random fields as in Comets *et alii* (2002), [1]. The existence of conditional densities is a more restrictive assumption for continuous state spaces. The existence of random fields is often based on conditional specifications, see Föllmer (1985) ([9], pages 109-119) and Dobrushin (1970) [4], through Feller continuity assumptions. The uniqueness of Gibbs measures is often based on projective conditional arguments; it follows with a mixing type argument. Such conditions rely on the regularity of conditional distributions; applications to resampling exclude such hypotheses. Various applications to image, geography, agronomy, physic, astronomy, electromagnetism may for instance be considered, see [11] or [13].

We omit here any assumption relative to the conditional distributions. Our idea is to define random fields through more algebraic and analytic arguments. We present here the new models of stationary random fields subject to the relation:

$$X_t = F\left((X_{t-j})_{j\in\mathbb{Z}^d\setminus\{0\}};\xi_t\right) \tag{1}$$

where $\xi = (\xi_t)_{t\in\mathbb{Z}^d}$ is an independent identically distributed (iid) random field. The independence of inputs ξ may also be relaxed to a stationarity assumption.
For the models with infinite interactions (1), the existence and uniqueness rely on the contraction principle. Lipschitz type conditions are thus needed, they are closely related to weak dependence, see [6]. Analogue weak dependence conditions are already proved in Shashkin (2005) for spin systems, [14]. A causal version of such models, random processes solutions of an equation $X_t = F(X_{t-1}, X_{t-2}, \ldots;\xi_t)$ $(t\in\mathbb{Z})$ is considered in Doukhan and Wintenberger (2006), [8]; in this paper the results are proved in a completely different way fitting to coupling arguments. Our results state existence and uniqueness of a solution of (1) as a Bernoulli shift $X_t = H((\xi_{t-s})_{s\in\mathbb{Z}^d})$ as well as the weak dependence properties of this solution.
Our models are not necessarily Markov, neither linear or homoskedastic. Moreover the inputs do not need additional distributional assumptions (like for Gibbs random fields). They extend on ARMA random fields which are special linear random fields (see [13] or [10]). A forthcoming paper will be aimed at developing statistical issues of those models. Identification and estimation of random fields with integer values is considered in [5].

The paper is organized as follows.
We first recall weak dependence from [6] in § 2. General results are then stated for stationary (non necessarily independent) inputs. Those results imply heavy restrictions on the innovations in some cases: a convenient notion of causality is thus used. A last subsection addresses the problem of simulating such models.
A following section details examples of such models. They are natural extensions of the standard times series models. We shall especially consider LARCH(∞) and doubly stochastic linear random fields for which this causality allows to relax the boundedness assumptions.
Proofs are rejected at in a last section of the paper.

2 Main results

In order to state our dependence results, we first introduce the concepts of weak dependence. Our main results will be stated in the following subsection. After this, causality will be proved to imply other powerful results. A last subsection is aimed at describing a way to simulate those very general random fields.

2.1 Weak dependence

We recall here the weak dependence conditions introduced in Doukhan & Louhichi (1999). They may replace heavy mixing assumptions.

Definition 1 *Set $\|(s_1,\ldots,s_d)\| = \max\{|s_1|,\ldots,|s_d|\}$ for $s_1,\ldots,s_d\in\mathbb{Z}$. The $E=\mathbb{R}^k-$valued random field $(X_t)_{t\in\mathbb{Z}^d}$ is weakly dependent if for a sequence $(\epsilon(r))_{r\in\mathbb{N}}$ with limit 0*

$$|Cov\left(f(X_{s_1},\ldots,X_{s_u})\,,\,g\left(X_{t_1},\ldots,X_{t_v}\right)\right| \le \psi(u,v,Lip\ f, Lip\ g)\epsilon(r),$$

where indices $s_1,\dots,s_u$, $t_1,\dots,t_v\in\mathbb{Z}^d$ *are such that* $\|s_k-t_l\|\geq r$ *for* $1\leq k\leq u$ *and* $1\leq l\leq v$. *Moreover, the real valued functions* f,g *defined on* $\left(\mathbb{R}^k\right)^u$ *and* $\left(\mathbb{R}^k\right)^v$, *satisfy* $\|f\|_\infty$, $\|g\|_\infty\leq 1$ *and* $Lip\ f, Lip\ g<\infty$ *where a norm* $\|\cdot\|$ *is given on* $\mathbb{R}^k$ *and,*

$$Lip\ f=\sup_{(x_1,\dots,x_u)\neq(y_1,\dots,y_u)}\frac{|f(x_1,\dots,x_u)-f(y_1,\dots,y_u)|}{\|x_1-y_1\|+\cdots+\|x_u-y_u\|}.$$

If $\psi(u,v,a,b)=au+vb$, *this is denoted as* $\eta-$*dependence and the sequence* $\epsilon(r)$ *will be written* $\eta(r)$.
If $\psi(u,v,a,b)=abuv$, *this is denoted as* $\kappa-$*dependence and the sequence* $\epsilon(r)$ *will be written* $\kappa(r)$.
If $\psi(u,v,a,b)=au+vb+abuv$, *this is denoted as* $\lambda-$*dependence and the sequence* $\epsilon(r)$ *will be written* $\lambda(r)$.

2.2 Random fields with infinite interactions

Let $\xi=(\xi_t)_{t\in\mathbb{Z}^d}$ be a stationary random field with values in E' (usually $E'=\mathbb{R}^{k'}$ for some $k'\geq 1$ but in some cases E' is a denumerable tensor product of such sets). We shall consider stationary $E=\mathbb{R}^k$ valued random fields driven by the implicit equation (1). For a topological space S, $\mathcal{B}(S)$ denote the Borel σ-algebra on S.
We denote $I=\mathbb{Z}^d\setminus\{0\}$. In the sequel, $F:\left(E^{(I)}\times E',\mathcal{B}(E^I)\otimes\mathcal{B}(E')\right)\to\left(E,\mathcal{B}(E)\right)$ denotes a measurable function defined for each sequence with a finite number of non-vanishing arguments ([1]). In this paper $\|\cdot\|$ will be arbitrary norms on E (or E' when needed). We will always use the suppremum norm on $\mathbb{Z}^d$ and this norm will be also denoted by $\|\cdot\|$. We prove that simple assumptions entail existence of a unique solution as a Bernoulli shift

$$X_t=H\left((\xi_{t-j})_{j\in\mathbb{Z}^d}\right)$$

Let μ denote ξ's distribution; this is a probability measure on the measurable space $\left(E'^{\mathbb{Z}^d},\mathcal{B}\left(E'^{\mathbb{Z}^d}\right)\right)$. For some $m\geq 1$, we denote $\|\cdot\|_m$ the usual norm of $\mathbb{L}^m$ and the space of μ-measurable $H:\left(E'^{\mathbb{Z}^d},\mathcal{B}\left(E'^{\mathbb{Z}^d}\right)\right)\to(E,\mathcal{B}(E))$ with finite moments is denoted

$$\mathbb{L}^m(\mu)=\{H/\ \mathbb{E}\|H(\xi)\|^m<\infty\}.$$

We shall use the assumptions:

(H1) $\|F(0;\xi_0)\|_m<\infty$.

(H2) There exist constants $a_j\geq 0$, for $j\in\alpha>0$ with, for each $\forall z,z'\in E^{(\mathbb{Z}^d\setminus\{0\})}$,

$$\begin{aligned}\|F(z;\xi_0)-F(z';\xi_0)\| &\leq \sum_{j\in\mathbb{Z}^d\setminus\{0\}}a_j\|z_j-z'_j\|,\quad a.s. \qquad (2)\\ \sum_{j\in\mathbb{Z}^d\setminus\{0\}}a_j &= e^{-\alpha}<1.\end{aligned}$$

[1] If V denotes a vector space and B an arbitrary set then $V^{(B)}\subset V^B$ denotes the set of $v=(v_b)_{b\in B}$ such that there is some finite subset $B_1\subset B$ with $v_b=0$ for each $b\notin B_1$.

We now extend the function F to the trajectories of a stationary random field:

Lemma 1 *Assume* **(H1)** *and* **(H2)**. *Let X and X' be two $E-$valued stationnary random fields in $\mathbb{L}^m$, then:*

1) $\lim_{p\to\infty} F\big((X_j \mathbf{1}_{0<\|j\|\le p})_{j\neq 0};\xi_0\big)$ *exists in $\mathbb{L}^m$ and a.s., we denote it* $F\left((X_j)_{j\in\mathbb{Z}^d\setminus\{0\}};\xi_0\right)$.

2) $\left\|F\big((X_j)_{j\in\mathbb{Z}^d\setminus\{0\}};\xi_0\big) - F\big((X'_j)_{j\in\mathbb{Z}^d\setminus\{0\}};\xi_0\big)\right\|_m \le \sum_{j\in\mathbb{Z}^d\setminus\{0\}} a_j \left\|X_j - X'_j\right\|_m.$

Theorem 1 *Assume that ξ is stationary and* **(H1)** *and* **(H2)** *hold. Then there exists a unique stationary solution of equation (1). This solution writes $X_t = H\left((\xi_{t-j})_{j\in\mathbb{Z}^d}\right)$ for some $H\in\mathbb{L}^m(\mu)$.*

Lemma 8 below, will also provide us with an approximation of this solution with finitely many interactions.

2.2.1 Weak dependence of the solution (iid inputs)

In the general case we shall restrict to independent inputs to derive $\eta-$weak dependence of the previous solution.

Theorem 2 *Assume that ξ is iid and* **(H1)** *and* **(H2)** *hold. Then the stationary solution of equation (1) obtained in theorem 1 is $\eta-$weakly dependent and there exists a constant $C>0$ with*

$$\eta(r) \le C\cdot \inf_{p\in\mathbb{N}^*}\Big\{e^{-\alpha\frac{r}{2p}} + \sum_{\|i\|>p} a_i\Big\}. \tag{3}$$

If $a_i = 0$ for $\|i\| > p$ then

$$\eta(r) \le C\cdot e^{-\alpha\frac{r}{2p}}.$$

Explicit (sub-geometric) rates are now derived from more specific decay rates of the coefficients:

Lemma 2 (Geometric decays) *If $a_i \le Ce^{-\beta\|i\|}$ there exists a constant $C'>0$ with*

$$\eta(r) \le C' r^{\frac{d-1}{2}} e^{-\sqrt{\alpha\beta r/2}}.$$

Lemma 3 (Riemanian decays) *If $a_i \le C\|i\|^{-\beta}$ for some $\beta > d$ then there exists $C'>0$ such that*

$$\eta(r) \le C'\left(\frac{r}{\ln r}\right)^{d-\beta}.$$

Thus a large range of decay rates may be considered for such models of random fields.

2.2.2 Weak dependence of the solution (dependent inputs)

If ξ is either η or $\lambda-$dependent it may be proved in specific examples that weak dependence is hereditary. Here follows a general result. The following assumption will be necessary:

(H2') There exist a subset $\Xi \subset E'$ with $\mathbb{P}(\xi_0 \in \Xi) = 1$, nonnegative constants with $\sum_{j\in\mathbb{Z}^d\setminus\{0\}} a_j = e^{-\alpha} < 1$ and a constant $b > 0$ such that

$$\|F(x;u) - F(x';u')\| \le \sum_{j\in\mathbb{Z}^d\setminus\{0\}} a_j \|x_j - x'_j\| + b\left\|u - u'\right\|,$$

for all $x, x' \in E^{(\mathbb{Z}^d\setminus\{0\})}$ and $u, u' \in \Xi$.

We quote that assumption **(H2')** is more restrictive than **(H2)**

Proposition 1 *Assume* **(H1)** *and* **(H2')**.

1) *If the random field ξ is $\eta-$weakly dependent, with weak dependence coefficients $\eta_\xi(r)$, then X is $\eta-$weakly dependent with*

$$\eta(r) \le C \inf_{p\in\mathbb{N}^*}\left\{\sum_{\|j\|>p} a_j + \inf_{n\in\mathbb{N}^*}\left\{a^n + p^n \eta_\xi\left((r-2pn)\vee 0\right)\right\}\right\}$$

where $C > 0$ is a constant.

2) *If the random field ξ is $\lambda-$weakly dependent, with dependence coefficients denoted $\lambda_\xi(r)$, then X is $\lambda-$weakly dependent with*

$$\lambda(r) \le C \inf_{p\in\mathbb{N}^*}\left\{\sum_{\|j\|>p} a_j + \inf_{n\in\mathbb{N}^*}\left\{a^n + p^{2n} \lambda_\xi\left((r-2pn)\vee 0\right)\right\}\right\}$$

for some constant $C > 0$.

Remark. For models with finite interactions, *i.e.* $F(x;u) = f(x_{j_1}, \ldots, x_{j_k}; u)$ for $x = (x_j)_{j\neq 0}$, this simply writes

$$\begin{aligned}\eta(r) &\le c \inf_{n\in\mathbb{N}^*}\left\{a^n + k^n \eta_\xi\left((r-2\rho n)\vee 0\right)\right\},\\ \lambda(r) &\le c \inf_{n\in\mathbb{N}^*}\left\{a^n + k^{2n} \lambda_\xi\left((r-2\rho n)\vee 0\right)\right\},\end{aligned}$$

here $\rho = \max\{\|j_1\|, \ldots, \|j_k\|\}$. If $\eta_\xi(r)$ or $\lambda_\xi(r)$ have geometric or Riemannian decay the same holds for the output random field. More precisely set $a = e^{-\alpha}$ and $k = e^{\kappa}$ under η-dependence and $k^2 = e^{\kappa}$ under λ-dependence, then decay rates of the outputs (X_t) write

Geometric decays:	$e^{-\frac{\alpha\beta}{\alpha+2\rho\beta+\kappa}r}$,	for dependence decays of the inputs with order	$e^{-\beta r}$,
Riemannian decays:	$r^{-\frac{\alpha b}{\alpha+\kappa}}$,	for dependence decays of the inputs with order	r^{-b}.

2.3 Causality

For $d = 1$, the recurrence equation $X_t = \xi_t(a + bX_{t-1})$ is given with $F(x; u) = u(a + bx_1)$. There exist a stationary solution with ξ_t and X_{t-1} independent. Here **(H2)** implies that innovations are bounded, which seems unrealistic. In this example, instead of $H\left((\xi_t)_{t\in\mathbb{Z}}\right) \in \mathbb{L}^m(\mu)$, this is enough to exhibit solutions $H\left((\xi_t)_{t\geq 0}\right) \in \mathbb{L}^m(\mu)$ (which is independent of $(\xi_s)_{s<0}$). This allows to replace *suprema* by *integrals* in **(H2)** in order to derive a contraction principle. Causality of random fields has been considered in Helson and Lowdenslager (1959) [12]; we adapt this idea in order to relax the previous assumption.

Definition 2 (causality) *If $A \subset \mathbb{Z}^d \backslash \{0\}$, we denote $c(A)$ the convex cone of $\mathbb{R}^d$ generated by A,*

$$c(A) = \left\{ \sum_{i=1}^{k} r_i j_i \Big/ \ (j_1, \ldots, j_k) \in A^k, (r_1, \ldots, r_k) \in \mathbb{R}_+^k, \ k \geq 1 \right\}.$$

1) The set A is a causal subset of $\mathbb{Z}^d$ if $c(A) \cap \big(-c(A)\big) = \{0\}$.
2) If F is measurable with respect to the σ-algebra $\mathfrak{F}_A \otimes \mathcal{B}(E')$ for some causal set A, then the equation $X_t = F\left((X_{t-j})_{j\in I}; \xi_t\right)$ is A-causal.

For a causal set $A \subset \mathbb{Z}^d$, we denote by $\widetilde{A}$ the subset $c(A) \cap \mathbb{Z}^d$.

Examples. A singleton is causal, as well as $\{i, j\}$ if and only if $-j \notin i \cdot \mathbb{R}^+$. The half plane $\{(i, j) \in \mathbb{Z}^2 / i > 0\} \bigcup \{(0, j); j > 0\} \subset \mathbb{Z}^2$ is also causal.

One consequence of this notion is the elementary lemma:

Lemma 4 *If A is a causal subset of $\mathbb{Z}^d$, then $\forall (j, j') \in A \times \widetilde{A}$ we have $j + j' \neq 0$.*

For a linear basis $b = (b_1, \ldots, b_d)$ of $\mathbb{R}^d$, $(x_1, \ldots, x_d) \mapsto x_1 b_1 + \cdots + x_d b_d$, defines an isomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$. We denote by $\leq_b$ the total order relation on $\mathbb{R}^d$ defined by:

$$u \leq_b v \Leftrightarrow f^{-1}(u) \leq_{lex} f^{-1}(v)$$

with $\leq_{lex}$ the lexicographic order on $\mathbb{R}^d$.

Proposition 2 (characterization of causal sets) *If B is a convex cone of $\mathbb{R}^d$ such that $B \cap (-B) = \{0\}$ there exists a basis b of $\mathbb{R}^d$ such that $B \subset \{j \in \mathbb{R}^d / 0 \leq_b j\}$. Moreover if b is a basis of $\mathbb{R}^d$, $\{j \in \mathbb{Z}^d / 0 <_b j\}$ is a causal set of $\mathbb{Z}^d$ witch will be called maximal causal subset.*

Remarks.

- The maximal causal subsets of $\mathbb{Z}$ are $\{1, 2, 3, \ldots\}$ and $\{-1, -2, \ldots\}$. An example of maximal causal subset of $\mathbb{Z}^2$ is $\{(i, j) \in \mathbb{Z}^2 / i > 0 \text{ or } (i = 0, j > 0)\}$.

- Helson and Lowdenslager (1959) [12] define symmetric half planes as subsets $S \subset \mathbb{Z}^2$ such that S is stable by addition and $S \cup (-S) = \mathbb{Z}^2$, $S \cap (-S) = \{0\}$. A nice review of this causality condition is given in Loubaton (1989) [13], applications are essentially given in terms of linear random fields.

  Note that $S \setminus \{0\}$ is a maximal causal subset of $\mathbb{Z}^2$. This notion plays a prominent part in prediction theory of 2-D stationary process (see [13]).

If $D \subset \mathbb{Z}^d$, we denote by π_s (respectively π'_s) the coordinate applications in E^{Z^d} (resp. in $(E')^{\mathbb{Z}^d}$), $\mathfrak{F}_D = \sigma(\pi_s; s \in D)$ and $\mathfrak{F}'_D = \sigma(\pi'_s; s \in D)$. Hence we denote by $\mathbb{L}^m_D(\mu)$ the subspace of $\mathbb{L}^m(\mu)$ of functions μ-measurable with respect to $\mathfrak{F}'_D$. The following result takes this definition into account to relax the assumptions in theorem 1,

Theorem 3 *Let $X_t = F\left((X_{t-j})_{j\in\mathbb{Z}^d\setminus\{0\}}; \xi_t\right)$ be a A-causal equation with iid inputs ξ. Besides the assumption* **(H1)** *we assume the following condition:*
(H3) *there exist nonnegative constants with $\sum_{j\in A} a_j = e^{-\alpha} < 1$ and*

$$\|F(x;\xi_0) - F(x';\xi_0)\|_m \le \sum_{j\in A} a_j \|x_j - x'_j\|, \qquad \forall x, x' \in E^{(\mathbb{Z}^d\setminus\{0\})}.$$

Then there exists a unique strictly stationary solution X of this equation in $\mathbb{L}^m$ if for each $t \in \mathbb{Z}^d$, X_t is measurable wrt $\sigma\left(\xi_{t-j}/j \in \widetilde{A}\right)$.
This solution writes $X_t = H\left((\xi_{t-j})_{j\in\mathbb{Z}^d}\right)$ where $H \in \mathbb{L}^m_{\widetilde{A}}$. and it is $\eta-$weakly dependent; moreover relation (3) still holds for a constant $C > 0$.

Now the function F is extended as follows:

Lemma 5 *Suppose* **(H1)** *and* **(H3)**. *If ξ_0 is independent of $\sigma\big((X_j, X'_j)/j \in A\big)$ for two random fields X and X' in $\mathbb{L}^m$ then,*

1) $\lim_{p\to\infty} F\big((X_j \mathbf{1}_{0<\|j\|\le p})_{j\neq 0}; \xi_0\big)$ *exists in $\mathbb{L}^m$ and it is denoted $F\big((X_j)_{j\neq 0}; \xi_0\big)$.*

2) $\left\|F\big((X_j)_{j\neq 0}; \xi_0\big) - F\big((X'_j)_{j\neq 0}; \xi_0\big)\right\|_m \le \sum_{j\in A} a_j \left\|X_j - X'_j\right\|_m.$

2.4 Simulation of the model

Simulations of those models are deduced from the proof of the existence theorems based on the Picard fixed point theorem. Consider the shift operators $\theta_j : \left(E'\right)^{\mathbb{Z}^d} \to \left(E'\right)^{\mathbb{Z}^d}$ defined as $(x_k)_{k\in\mathbb{Z}^d} \mapsto (x_{k+j})_{k\in\mathbb{Z}^d}$. For $H \in \mathbb{L}^m(\mu)$ we note

$$\Phi_p(H) = F\left(\left((H\circ\theta_j)\mathbf{1}_{\|j\|\le p}\right)_j ; \pi_0\right)$$

It is shown in theorem 1's proof that the application $\Phi : \mathbb{L}^m(\mu) \to \mathbb{L}^m(\mu)$ given by

$$\Phi(H) = F((H\circ\theta_j)_{j\neq 0}; \pi_0).$$

is well defined and has a fixed point in $\mathbb{L}^m(\mu)$.
The proof of theorem 3 shows that it is also the case for a $A-$causal equation if we replace $\mathbb{L}^m(\mu)$ by $\mathbb{L}^m_{\widetilde{A}}(\mu)$.
For $n, p \in \mathbb{N}^*$, $t \in \mathbb{Z}^d$ we denote $X^n_t = \Phi^{(n)}(0)\left((\xi_{t-j})_{j\in\mathbb{Z}^d}\right)$ and $X^n_{p,t} = \Phi^{(n)}_p(0)\left((\xi_{t-j})_{j\in\mathbb{Z}^d}\right)$.

Lemma 6 *We assume that conditions in theorem 1 or in theorem 3 hold for some $m \ge 1$. Let $n \in \mathbb{N}$ then:*

1. *For every $t \in \mathbb{Z}^d$, $\|X_t - X^n_t\|_m \le a^n \|X_0\|_m$, hence $\lim_{n\to\infty} X^n_t = X_t$ a.s.*

2. *if* $p \in \mathbb{N}$ *we have,* $\left\|X_t - X^n_{p,t}\right\|_m \le \|X_0\|_m \left\{a^n + \frac{1}{1-a}\sum_{\|j\|>p} a_j\right\}$. *Thus if* $p = p_n$ *is chosen such that* $\sum_{n\ge1}\Big(\sum_{\|j\|>p_n} a_j\Big)^m < \infty$ *then*

$$\lim_{n\to\infty} X^n_{p_n,t} = X_t, \quad a.s. \tag{4}$$

Remarks.

- If the random field has finitely many interactions, then 1. provides a simulation scheme.
- For each finite p the operator Φ_p can be calculated thus relation (4) provides an explicit simulation scheme even for infinitely many interactions.
- *A.s.* convergence rates may also be evaluated in the previous lemma. They write $o_{a.s.}(a^n n^\epsilon)$ in the first point for each $\epsilon > 1/m$ and $o_{a.s.}(n^{-\epsilon})$ for $0 < \epsilon < \alpha - 1/m$ if $\sum_{\|j\|>p_n} a_j \le Cn^{-\alpha}$ for some $C > 0, \alpha > 1/m$ in the point 2.
- If $T \subset \mathbb{Z}^d$ is a finite set the random field X may be analogously simulated over T and $(X_t)_{t\in T}$ is estimated by $\left(X^n_{p_n,t}\right)_{t\in T}$.

2.4.1 Simulation scheme for finitely many interactions

Let $F(x;u) = f(x_{j_1},\dots,x_{j_k};u)$. The sequence of random fields X^n is defined from:

$$X^1_t = f(0;\xi_t), t\in\mathbb{Z}^d, \quad X^{n+1}_t = f\big(X^n_{t-j_1},\dots,X^n_{t-j_k};\xi_t\big), \qquad \text{for } n\ge 0$$

We now simulate samples $(X^{10}_t)_{1\le t_1,t_2\le 15}$ of LARCH models with $d=2$, $k=k'=1$ and $p=10$:

$$X_t = \xi_t\Big(1 + \sum_{0<\|j\|\le p} a_j X_{t-j}\Big)$$

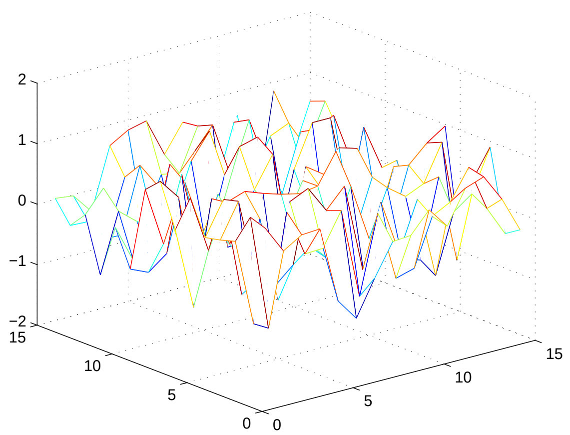

Figure 1: Non causal LARCH field

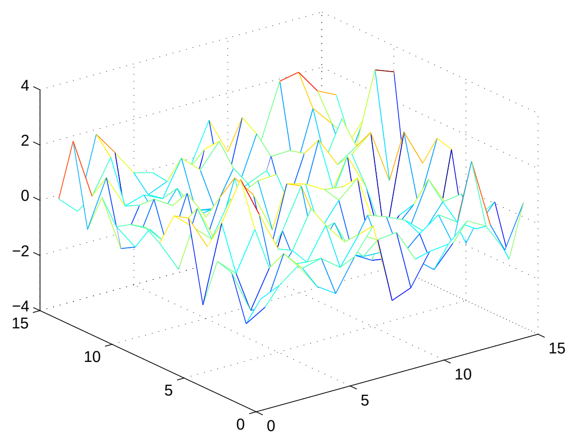

Figure 2: Causal LARCH field

1) In the figure 1, we represent the non causal case with $a_j = \dfrac{0.05}{j_1^2 + j_2^2}$ and ξ_0 is uniform on $[-1, 1]$.

2) Figure 2 deals with the causal case with $a_j = \dfrac{0.05}{j_1^2 + j_2^2}$ if $0 \le j_1, j_2 \le 10$ and $a_j = 0$ otherwise. In this case, ξ_0 is $N(0,1)$-distributed.

3 Examples

Theorems 2 and 3 are now applied to examples of random fields with infinite interactions. Causality will allow to weaken moment conditions. In fact, theorem 2 proves a contraction principle in $\mathbb{L}^m$ for each value of m while theorem 3 only works with one fixed value of m.

3.1 Finite interactions random fields

If $\xi_t = (\zeta_t, \gamma_t)$ with $\zeta_t \in \mathbb{R}^p$ and γ_t a $p \times q$ matrix, and functions $f(\cdot) \in \mathbb{R}^p$ and $g(\cdot) \in \mathbb{R}^q$

$$X_t = f(X_{t-\ell_1}, \ldots, X_{t-\ell_k}) + \gamma_t g(X_{t-\ell_1}, \ldots, X_{t-\ell_k}) + \zeta_t \tag{5}$$

with $\ell_1, \ldots, \ell_k \neq 0$.
E.g. non linear auto-regression corresponds to $\gamma_t \equiv 0$ and ARCH type models are obtained with $\zeta_t = 0$ (classically $p = q = 1$, f is linear and $g^2(x_1, \ldots, x_k)$ is an affine function of $x_1^2, \ldots, x_k^2$).
Theorems 1, 2, 3 imply the following lemma,

Corollary 1 *Suppose $\|\zeta_0\|_m < \infty$ and*

$$\left\{ \begin{array}{lcl} \|f(x_1, \ldots, x_k) - f(y_1, \ldots, y_k)\| & \le & \sum_{i=1}^k b_i \|x_i - y_i\|, \\ \|g(x_1, \ldots, x_k) - g(y_1, \ldots, y_k)\| & \le & \sum_{j=1}^k c_j \|x_j - y_j\| \end{array} \right. .$$

1. *If ξ is iid and $\displaystyle\sum_{i=1}^k \left(b_i + \|\gamma_0\|_\infty c_i\right) = e^{-\alpha} < 1$, then $\eta(r) \le C\left(e^{-\frac{\alpha}{2k}}\right)^r$ for the model (5).*

   If the equation (5) is causal and $\displaystyle\sum_{i=1}^k \left(b_i + \|\gamma_0\|_m c_i\right) = e^{-\alpha} < 1$ the same relation holds.

2. *If now ξ is η or λ-weakly dependent, g bounded and $\displaystyle\sum_{i=1}^k \left(b_i + \|\gamma_0\|_\infty c_i\right) = e^{-\alpha} < 1$, then X is η or λ-weakly dependent. Decay rates are given according to proposition 1.*

The remark following proposition 1 states precise decay rates under standard decay rates (*i.e.* Riemannian or geometric decays) of the weak dependence coefficients of the input process.
The volatility coefficients γ_t need to be bounded in the general case and they only have finite moments under causality.
Note that functions f and g may only depend on a strict subset of the indices $1, \ldots, k$.

3.2 Linear fields

Let X be a solution of the equation

$$X_t = \sum_{j\in A} \alpha_t^j X_{t-j} + \zeta_t, \tag{6}$$

innovations ζ_t are vectors of $E = \mathbb{R}^k$ and coefficients α_t^j are $k\times k$ matrices, $\|\cdot\|$ is a norm of algebra on this set of matrices and X will be an E valued random field. Let $A \subset \mathbb{Z}^d \setminus \{0\}$, we assume that the iid random field $\xi = \left((\alpha_t^j)_{i\in A}, \zeta_t\right)_{t\in\mathbb{Z}^d}$ takes now its values in $(M_{k\times k})^A \times E$; here $M_{k\times k}$ denotes the set of $k\times k$ matrices.

Proposition 3 *If $b = \sum_{j\in A} \|\alpha_0^j\|_\infty < 1$, then theorem 2 applies with $a_j = \|\alpha_0^j\|_\infty$.*
For a causal equation if $b = \sum_{j\in A} \|\alpha_0^j\|_m < 1$ theorem 3 applies with $a_j = \|\alpha_0^j\|_m$.
In both cases the solution of equation (6) writes a.s. and in $\mathbb{L}^m$,

$$X_t = \zeta_t + \sum_{j\in A} \alpha_t^j \xi_{t-j} + \sum_{i=2}^\infty \sum_{j_1,\ldots,j_i\in A} \alpha_t^{j_1}\alpha_{t-j_1}^{j_2}\cdots\alpha_{t-j_1-\cdots-j_{i-1}}^{j_i}\zeta_{t-(j_1+\cdots+j_i)}.$$

This means that the random coefficients are bounded in the general case and they need only to have have finite moments under causality.

Examples. If the sequence $(\alpha_t^j)_t$ is deterministic then those models extend on linear auto-regressive models.
If only a finite number of coefficients α_t^j do not vanish we obtain auto-regressive models with random coefficients, see [15].

3.3 LARCH(∞) random fields

Stationary innovations ξ_t are now $k\times k'$ matrices and $\|\cdot\|$ will denote a norm $k\times k'$ or $k'\times k$ matrices while $X_t \in \mathbb{R}^k$. For bounded innovations we first recall

Theorem 4 (Doukhan, Teyssière, Winant (2006)) *Let α_j be a $k'\times k$ matrix for $j \in \mathbb{Z}^d\setminus\{0\}$, note $A(x) = \sum_{\|j\|\ge x} \|\alpha_j\|$ and suppose that $\lambda = A(1)\|\xi_0\|_\infty < 1$, then*

$$X_t = \xi_t \left(a + \sum_{k=1}^\infty \sum_{j_1,\ldots,j_k\neq 0} \alpha_{j_1}\xi_{t-j_1}\cdots\alpha_{j_k}\xi_{t-j_1-\cdots-j_k}a \right) \tag{7}$$

is a solution of the equation

$$X_t = \xi_t \left(a + \sum_{j\neq 0} \alpha_j X_{t-j} \right), \quad t\in\mathbb{Z}^d \tag{8}$$

if moreover ξ is iid, then

$$\eta(r) \le \mathbb{E}\|\xi_0\| \left(\mathbb{E}\|\xi_0\| \sum_{k<r/2} \lambda^{k-1} A\left(\frac{r}{k}\right) + \frac{\lambda^{[r/2]}}{1-\lambda} \right) \|a\|.$$

If we use theorem 2 we also obtain that eqn. (8) admits a unique Bernoulli shift $\mathbb{L}^m$ solution. Note that this solution is bounded. Notice that for Riemannian decay the previous $A(u) \le Cu^{-c}$ relation yields $\eta(r) = \mathcal{O}(r^{-c})$ while theorem 3 only provides us this bound up to a log-loss; geometric decays yield the same result for both cases.

Bounded innovations ξ_t look unnatural hence we investigate below the causal case. Let A a causal subset of $\mathbb{Z}^d$ and

$$X_t = \xi_t \left(a + \sum_{s \in A} a_s X_{t-s} \right) \tag{9}$$

Proposition 4 *If $b\|\xi_0\|_m < 1$ with $b = \sum_{s\in A} \|a_s\|$, theorem 3 applies with $a_j = \|\xi_0\|_m \|\alpha_j\|$ to the solution (7) of eqn. (9) (we set $\alpha_j = 0$ for $j \notin A$).*

3.4 Non linear ARCH(∞) random fields

Models with

$$X_t = \xi_t \Big(a + \sum_{j \neq 0} g_j(X_{t-j}) \Big)$$

clearly extend on LARCH(∞) models; bounded functions g_j provide robust models.

Corollary 2 *If $\|g_j(x) - g_j(y)\| \le \alpha_j \|x - y\|$ and $\|\xi_0\|_\infty \sum_{j\neq 0} \alpha_j < 1$, theorem 2 holds with $a_i = \|\xi_0\|_\infty \alpha_i$ (innovations are bounded here).*
Assume now that $g_i \equiv 0$ for $i \notin A$, causal set then theorem 3 holds with $a_i = \|\xi_0\|_m \alpha_i$ (and now the innovations do not need anymore to be bounded).

This causality argument improves on [7] by only assuming finite moments for innovations instead of boundedness.

3.5 Mean field type model

Consider innovations in $\mathbb{R}^{k'}$ and $k \times k$ matrices α_i,

$$X_t = f\Big(\xi_t, \sum_{s \neq t} \alpha_{s-t} X_s \Big) \tag{10}$$

Corollary 3 *Assume that $f : \mathbb{R}^{k'} \times \mathbb{R}^k \to \mathbb{R}^k$ satisfies*

$$\sup_{u \in \mathbb{R}^{k'}} \|f(u,x) - f(u,y)\| \le b\|x-y\|, \ \forall x, y \in \mathbb{R}^k, \qquad b \sum_{i \neq 0} \|\alpha_i\| < 1.$$

then equation (10) admits a unique solution in $\mathbb{L}^m$ written as a Bernoulli shift and this solution is $\eta-$weakly dependent with $a_i = b\|\alpha_i\|_1$.
The same results hold if now $a_i = 0$ for $i \notin A$ with A is causal in $\mathbb{Z}^d$ and,

$$\big\| f\big(\xi_0, x)\big) - f\big(\xi_0, y\big) \big\|_m \le b\|x-y\|, \ \forall x, y \in \mathbb{R}^k, \qquad b \sum_{i \neq 0} \|\alpha_i\| < 1.$$

LARCH(∞) model is still special cases of this one.

4 Proofs

We begin with the proof of some lemmas which relate the assumptions to contraction conditions in the space of Bernoulli shifts. Then we give separated proofs for existence and weak dependence properties. Those proofs always follow two steps since we first consider models with a finite range. For shortness we write here $I = \mathbb{Z}^d \setminus \{0\}$.

4.1 Proof of lemma 1

For $p \in \mathbb{N}^*$, we set $Y_p = F\big((X_j \mathbf{1}_{0<\|t\|\le p})_j, \xi_0\big)$ and $Y'_p = F\big((X'_j \mathbf{1}_{0<\|t\|\le p})_j, \xi_0\big)$.

1. If $q \in \mathbb{N}^*$ from assumption (H2),

$$\|Y_p - Y_{p+q}\| \le \sum_{p<\|j\|\le p+q} a_j \|X_j\|, \qquad a.s.$$

   Since the serie $\sum_{i\in I} a_j \|X_j\|_m$ is convergent the serie $\sum_{i\in I} a_j \|X_j\|$ converges $a.s.$ Hence, we deduce that a.s $(Y_p)_{p\in\mathbb{N}^*}$ is a Cauchy sequence in E and then converges. We denote by $Y = F\big((X_j)_{j\neq 0}; \xi_0\big)$ this limit.
   Moreover, for $p \in \mathbb{N}^*$, we have:

$$\|Y_p - F(0;\xi_0)\|_m \le \sum_{0<\|j\|\le p} a_j \|X_j\|_m$$

   This proves that $Y_p \in \mathbb{L}^m$. Hence the convergence in $\mathbb{L}^m$ is a simply consequence of the Fatou lemma since:

$$\begin{aligned} \|Y - Y_p\|_m &\le \liminf_{q\to\infty} \|Y_q - Y_p\|_m \\ &\le \liminf_{q\to\infty} \sum_{p<\|j\|\le q} a_j \|X_0\|_m \\ &= \sum_{\|j\|>p} a_j \|X_0\|_m \end{aligned}$$

2. If $p \in \mathbb{N}^*$, we have using **(H2)**:

$$\|Y_p - Y'_p\|_m \le \sum_{j\neq 0} a_j \|X_j - X'_j\|_m$$

   Hence the result follows with $p \to \infty$. $\square$

4.2 Proof of the existence theorem 1

Assuming **(H1)** and **(H2)** we set $a = \sum_{j\neq 0} a_j$. We adopt the notations of paragraph 2.4. For $H \in \mathbb{L}^m(\mu)$ we note

$$\Phi_p(H) = F\left(\left((H\circ\theta_j)\mathbf{1}_{\|j\|\le p}\right)_j; \pi_0\right)$$

A direct consequence of lemma 1 is that $\lim_{p\to\infty}\Phi_p(H)$ exists in $\mathbb{L}^m(\mu)$. Denote this limit by $F((H\circ\theta_j)_{j\neq 0};\pi_0)$, the application $\Phi:\mathbb{L}^m(\mu)\to\mathbb{L}^m(\mu)$ is defined as

$$\Phi(H)=F\left((H\circ\theta_j)_{j\neq 0};\pi_0\right).$$

Let show that Φ is a contraction of $\mathbb{L}^m(\mu)$.If $H,\ H'\in\mathbb{L}^m(\mu)$, then applying the lemma 1 to the random fields X and X' defined as $X_j=H\circ\theta_j(\xi)$ and $X'_j=H'\circ\theta_j(\xi)$, we obtain:

$$\begin{aligned}\|\Phi(H)(\xi)-\Phi(H')(\xi)\|_m &\leq \sum_{j\neq 0}a_j\|H\circ\theta_j(\xi)-H'\circ\theta_j(\xi)\|_m\\ &\leq \sum_{j\neq 0}a_j\|H(\xi)-H'(\xi)\|_m\end{aligned}$$

Picard fixed point theorem applies since the space $\mathbb{L}^m(\mu)$ is complete. There exists a unique $H\in\mathbb{L}^m(\mu)$ with $\Phi(H)=H$ thus $H(\xi)=F\big((H\circ\theta_j(\xi))_{j\in\mathbb{Z}^d};\xi_0\big)$, *a.s.* Set $X_t=H\left((\xi_{t-i})_{i\in\mathbb{Z}^d}\right)$ then with stationarity of ξ and since $\mathbb{Z}^d$ is denumerable we get

$$X_t=F\left((X_{t-j})_{j\in\mathbb{Z}^d\setminus\{0\}};\xi_t\right),\quad \forall t\in\mathbb{Z}^d \qquad a.s.$$

Let Y be a stationary solution of this equation, we denote $u_t=\|X_t-Y_t\|_1$ for each $t\in\mathbb{Z}^d$. We obtain

$$u_t\leq\sum_{j\neq 0}a_ju_{t-j}$$

As $\sup_t u_t\leq\|X_0\|_1+\|Y_0\|_1<\infty$ we note that the previous relation implies $\sup_t u_t\leq a\sup_t u_t$. Hence $u_t=0$ for each t. Thus $X_t=Y_t$ *a.s* for each t. $\square$

4.3 Proof of theorem 2

For an independent copy $(\xi'_t)_{t\in\mathbb{Z}^d}$ of $\xi=(\xi_t)_{t\in\mathbb{Z}^d}$ and $s\in\mathbb{R}^+$, we set $\xi^{(s)}=(\xi_t^{(s)})_{t\in\mathbb{Z}^d}$ with $\xi_t^{(s)}=\xi_t$ if $\|t\|<s$ and $\xi_s^{(s)}=\xi'_s$ else. For a Bernoulli shift defined by H a straightforward extension of a result in [6] to random fields implies

$$\eta(r)\leq 2\delta_{r/2},\qquad \text{where}\qquad \delta_r=\left\|H\left(\xi\right)-H\big(\xi^{(r)}\big)\right\|_1 \tag{11}$$

4.3.1 Weak dependence under finite interactions

We first assume that F depends finitely many variables

$$X_t=F(X_{t-j_1},\ldots,X_{t-j_k};\xi_0)$$

Lipschitz coefficients of F in condition **(H2)** write $a_1,\ldots,a_k$ and we set $a=\sum_{i=1}^k a_i<1$. Let H be the element of $\mathbb{L}^m(\mu)$ with $X_t=H\left((\xi_{t-i})_{i\in\mathbb{Z}^d}\right)$ and $\delta_r=\mathbb{E}\left\|H(\xi)-H(\xi^{(r)})\right\|=\mathbb{E}\left\|X_0-X_0^{(r)}\right\|$ with $X_t^{(r)}=H((\xi_{t-i}^{(r)})_{i\in\ \mathbb{Z}^d})$.

Lemma 7 *Assume that* **(H1)** *and* **(H2)** *hold, then* $\delta_r\leq 2\|X_0\|_1a^{\frac{r}{\rho}}$ *hence* $\delta_r\to_{r\to\infty}0$.

Proof of lemma 7. Set $\rho = \max\{\|j_1\|, \ldots, \|j_k\|\}$ and $r > 0$. Since ξ and $\xi^{(r)}$ admit the same distribution, we have for each t:

$$\begin{aligned} X_t &= F(X_{t-j_1}, \ldots, X_{t-j_k}; \xi_t) \\ X_t^{(r)} &= F(X_{t-j_1}^{(r)}, \ldots, X_{t-j_k}^{(r)}; \xi_t^{(r)}) \end{aligned}$$

If $\|t\| < r$ then $\xi_t^{(r)} = \xi_t$ and using **(H2)**, we have :

$$\|X_t - X_t^{(r)}\|_1 \le \sum_{l=1}^{k} a_l \left\| X_{t-j_l} - X_{t-j_l}^{(r)} \right\|_1 \tag{12}$$

Set now $i = -[-\frac{r}{\rho}]$ if $r \ge \rho$, then if $u \le i-1$ and $l_1, \ldots, l_u \in \{1, \ldots, k\}$: $\|j_{l_1} + j_{l_2} + \cdots + j_{l_u}\| < r$. We use inequality (12) to derive recursively the bounds

$$\begin{aligned} \left\| X_0 - X_0^{(r)} \right\|_1 &\le \sum_{l_1=1}^{k} a_{l_1} \left\| X_{-j_{l_1}} - X_{-j_{l_1}}^{(r)} \right\|_1 \\ &\le \cdots \\ &\le \sum_{l_1=1}^{k} a_{l_1} \sum_{l_2=1}^{k} a_{l_2} \cdots \sum_{l_i=1}^{k} a_{l_i} \left\| X_{-(j_{l_1}+j_{l_2}+\cdots+j_{l_i})} - X_{-(j_{l_1}+j_{l_2}+\cdots+j_{l_i})}^{(r)} \right\|_1 \\ &\le 2\|X_0\|_1 a^i \end{aligned}$$

From $i \ge r/\rho$ we get $\|X_0 - X_0^{(r)}\|_1 \le 2\|X_0\|_1 \, a^{\frac{r}{\rho}}$ thus $\delta_r \le 2\|X_0\|_1 a^{\frac{r}{\rho}}$.
If now $r < \rho$,

$$\left\| X_0 - X_0^{(r)} \right\|_1 \le \sum_{l_1=1}^{k} a_{l_1} \left\| X_{-j_{l_1}} - X_{-j_{l_1}}^{(r)} \right\|_1$$

Thus $\delta_r \le 2\|X_0\|_1 a^i \le 2\|X_0\|_1 \, a^{\frac{r}{\rho}}$. The result follows with $a < 1$. □

We now set a useful result. $(X_t)_{t\in\mathbb{Z}^d}$ and $(X_{p,t})_{t\in\mathbb{Z}^d}$ will denote for $p \ge 0$ the previous unique solution of the equations (1) and $Z_t = F\left((Z_{t-j} 1_{\{0<\|j\|\le p\}})_{j\in\mathbb{Z}^d\setminus\{0\}}; \xi_t \right)$.

Lemma 8 *Assume that the conditions in theorem 1 hold. Then $X_{p,t} \to_{s\to\infty} X_t$ in $\mathbb{L}^m$, for each $t \in \mathbb{Z}^d$.*

Proof.

$$\begin{aligned} \|X_{p,0} - X_0\|_m &\le \sum_{0<\|j\|\le p} a_j \|X_{p,-j} - X_{-j}\| + \sum_{\|j\|>p} a_j \|X_{-j}\|_m \\ &\le a\|X_{p,0} - X_0\|_m + \|X_0\|_m \sum_{\|j\|>p} a_j \end{aligned}$$

thus $\|X_{p,0} - X_0\|_m \le (1-a)^{-1} \|X_0\|_m \sum_{\|j\|>p} a_j$ which entails the first result. We also quote that $\sup_p \|X_{p,0}\|_m < \infty$. □

4.3.2 Weak dependence

Lemma 9 *Assume that the conditions in theorem 1 hold. Then the random field $(X_t)_{t\in\mathbb{Z}^d}$ is η-weakly dependent.*

Proof. Recall that $\sup_p \|X_{p,0}\|_m < \infty$; if $m \geq 1$, weak dependence follows from

$$\begin{aligned}\mathbb{E}\|X_0^{(r)} - X_0\| &\leq \mathbb{E}\|X_0^{(r)} - X_{p,0}^{(r)}\| + \mathbb{E}\|X_{p,0}^{(r)} - X_{p,0}\| + \mathbb{E}\|X_{p,0} - X_0\| \\ &= 2\mathbb{E}\|X_{p,0} - X_0\| + \mathbb{E}\|X_{p,0}^{(r)} - X_{p,0}\|\end{aligned}$$

For $r \geq p$, from lemma 7 we derive

$$\mathbb{E}\|X_{p,0}^{(r)} - X_{p,0}\| \leq 2\|X_{p,0}\|_1 \Big(\sum_{\|j\|\leq p} a_j \Big)^{\frac{r}{p}}$$

Hence

$$\begin{aligned}\delta_r &= \mathbb{E}\|X_0^{(r)} - X_0\| \\ &\leq 2 \cdot \frac{\|X_0\|_1}{1-a} \sum_{\|j\|>p} a_j + 2\|X_{p,0}\|_1 \Big(\sum_{\|j\|\leq p} a_j \Big)^{\frac{r}{p}} \\ &\leq 2 \cdot \frac{\|X_0\|_1}{1-a} \sum_{\|j\|>p} a_j + 2\|X_{p,0}\|_1 \, a^{\frac{r}{p}}\end{aligned}$$

Using the fact that $\sup_p \|X_{p,0}\|_1 < \infty$ there exists a constant $C > 0$ with

$$\delta_r \leq C \cdot \inf_p \Big\{ \sum_{\|j\|>p} a_j + a^{\frac{r}{p}} \Big\}$$

For any $\varepsilon > 0$ we choose $p_0 > 0$ such that $\sum_{\|j\|>p_0} a_j \leq \varepsilon/(2C)$ and r_0 such that $r \geq r_0$ implies $a^{\frac{r}{p}} \leq \varepsilon/(2C)$. Thus $\delta_r \leq \varepsilon$ if $r \geq r_0$ and $\delta_r \to_{r\to\infty} 0$. Using (11) we prove that $(X_t)_t$ is η-weakly dependent and $\eta(r) \leq \delta_{r/2}$. $\square$

4.3.3 Decay rates

Using the representation of the solution as a Bernoulli shift and the inequality (11) this will be enough to bound the expression of δ_r. Set $b_p = \#\{i \in \mathbb{Z}^d / \ \|i\| \leq p\}$ and $s_p = \#\{i \in \mathbb{Z}^d / \ \|i\| = p\}$ for $\|i\| = \max\{|i_1|, \ldots, |i_d|\}$ we obtain $b_p = (2p+1)^d$ and $s_p = b_p - b_{p-1} \leq Kp^{d-1}$ for a constant $K > 0$.

Proof of lemma 2.

$$\sum_{\|j\|>p} a_j = \sum_{q>p} \sum_{\|j\|=q} e^{-\beta q} \leq Kp^{d-1} \sum_{q>p} e^{-\beta q} \leq K \sum_{q\geq p+1} q^{d-1} e^{-\beta q}.$$

The function $x \mapsto x^{d-1}e^{-\beta x}$ decreases on $(p, +\infty)$ for large enough p, thus

$$\begin{aligned}\sum_{q\geq p+1} q^{d-1}e^{-\beta q} &\leq \int_p^{+\infty} x^{d-1}e^{-\beta x} dx \\ &= \frac{p^{d-1}e^{-\beta p}}{\beta} + (d-1) \int_p^{+\infty} x^{d-2}e^{-\beta x} dx \\ &= \mathcal{O}\left(p^{d-1}e^{-\beta p}\right).\end{aligned}$$

We thus find a constant C_1 such that

$$\delta_r \leq C_1 \inf_p \left\{p^{d-1}e^{-\beta p} + e^{-\alpha\frac{r}{p}}\right\} = C_1 \inf_p \left\{e^{-\beta p+(d-1)\ln p} + e^{-\alpha\frac{r}{p}}\right\}.$$

Assume $\beta p-(d-1)\ln p \sim \alpha r/p$ $(r \to \infty)$ which implies $\beta p^2 \sim \alpha r$, then $p \sim \sqrt{\alpha r/\beta}$. Thus, there is a constant C_2 such that:

$$\delta_r \leq C_2 r^{\frac{d-1}{2}} e^{-\sqrt{\alpha\beta r}}. \qquad \square$$

Proof of lemma 3. As before,

$$\sum_{\|j\|>p} a_j \leq K \sum_{q>p} q^{d-\beta-1} \leq K \int_p^{+\infty} x^{d-\beta-1}dx = K\frac{p^{d-\beta}}{\beta-d}.$$

Hence $\delta_r \leq c\cdot \inf_p \left\{e^{-\alpha r/p} + \dfrac{p^{d-\beta}}{\beta-d}\right\}$. Choosing $(\beta-d)\ln p \sim \alpha r/p$ thus $(\beta-d)p\ln p \sim \alpha r$ we derive $\ln p \sim \ln r$, thus $p \sim \frac{\alpha r}{(\beta-d)\ln r}$. There exists some constant C_3 with

$$\delta_r \leq C_3 \left(\frac{r}{\ln r}\right)^{d-\beta}. \qquad \square$$

4.4 Proof of proposition 1

4.4.1 Models with finite interactions

We assume first that there exist $k \geq 1$ and $j_1, \ldots, j_k \in I$ such $F(x;u)$ only depends on $x_{j_1}, \ldots, x_{j_k}$ for each $x = (x_j)_{j\neq 0} \in E^I$.
Hence writing a_i instead of a_{j_i} for $1 \leq i \leq k$, we have inequality

$$\|F(x;u) - F(y;u')\| \leq \sum_{i=1}^k a_i\|x_{j_i} - y_{j_i}\| + b\|u-u'\|, \qquad a = \sum_{i=1}^k a_i < 1$$

Now $h : E^k \times E' \to E$is such that $F(x;u) = h(x_{j_1}, \ldots, x_{j_k}, u)$. We will denote $\rho = \max\{\|j_1\|, \ldots, \|j_k\|\}$.

Lemma 10 *1) If the random field ξ is $\eta-$weakly dependent (the weak dependence coefficients are denoted $\eta_\xi(r)$) then X is $\eta-$weakly dependent with*

$$\eta(r) \leq C \inf_{n\in\mathbb{N}^*} \left\{a^n + k^n\eta_\xi\left((r-2\rho n)\vee 0\right)\right\},$$

where C is a positive constant.
2) If the random field ξ is $\lambda-$weakly dependent (the weak dependence coefficients are denoted $\lambda_\xi(r)$) then X is $\lambda-$weakly dependent with

$$\lambda(r) \leq C \inf_{n\in\mathbb{N}^*} \left\{a^n + k^{2n}\lambda_\xi\left((r-2\rho n)\vee 0\right)\right\},$$

for some constant $C > 0$.

Proof of lemma 10. We will use the lemma 6 and the following useful lemma 11.

Lemma 11 *1. For every x and $y \in C^{(\mathbb{Z}^d)}$ we have $\|\Phi(0)(x) - \Phi(0)(y)\| \leq b\|x_0 - y_0\|$ and if $n \geq 2$,*

$$\|\Phi^{(n)}(0)(x) - \Phi^{(n)}(0)(y)\| \\ \leq \sum_{l=1}^{n-1} \sum_{i_1,\dots,i_l=1}^{k} a_{i_1} \cdots a_{i_l} b \|x_{j_{i_1}+\dots+j_{i_l}} - y_{j_{i_1}+\dots+j_{i_l}}\| + b\|x_0 - y_0\|$$

2. Fix $x \in C^{(Z^d)}$. Then $\Phi(0)(x)$ only depends on x_0 and $\Phi(0)$ defines a $b-$Lipschitz function on C. We set $K_1 = b$ and $p_1 = 1$.
For $n \geq 2$ we set $A_n = \bigcup_{l=1}^{n-1} \{j_{i_1} + \dots + j_{i_l} \,/\, 1 \leq i_1, \dots, i_l \leq k\} \cup \{0\}$, $p_n = card\ A_n$ and $K_n = b\frac{1-a^n}{1-a}$. Then $\Phi^{(n)}(0)(x)$ only depends on x_j for $j \in A_n$. Moreover $\Phi^{(n)}(0)$ defines a Lipschitz function on C^{p_n} and Lip $\left(\Phi^{(n)}(0)\right) \leq K_n$.

Proof of lemma 11.

- The first point is easy to check. For $n \geq 2$ we use induction. For $n = 2$

$$\begin{aligned}
\|\Phi^{(2)}(0)(x) - \Phi^{(2)}(0)(y)\| &= \|h\left(\Phi(0) \circ \theta_1(x), \dots, \Phi(0) \circ \theta_k(x), x_0\right) \\
&\quad - h\left(\Phi(0) \circ \theta_1(y), \dots, \Phi(0) \circ \theta_k(y), y_0\right)\| \\
&\leq \sum_{i=1}^{k} a_i \|F(0, x_i) - F(0, y_i)\| + b\|x_0 - y_0\| \\
&\leq \sum_{i=1}^{k} a_i b \|x_i - y_i\| + b\|x_0 - y_0\|
\end{aligned}$$

Assuming that the inequality holds for an integer $n \geq 2$, we estimate $\phi_{n,x,y} = \|\Phi^{(n+1)}(0)(x) - \Phi^{(n+1)}(0)(y)\|$:

$$\begin{aligned}
\phi_{n,x,y} &= \|h\left(\Phi^{(n)}(0)(\theta_{j_1}x), \dots, \Phi^{(n)}(0)(\theta_{j_k}x), x_0\right) \\
&\quad - h\left(\Phi^{(n)}(0)(\theta_{j_1}y), \dots, \Phi^{(n)}(0)(\theta_{j_k}y), y_0\right) \\
&\leq \sum_{i=1}^{k} a_i \|\Phi^{(n)}(0)(\theta_{j_i}x) - \Phi^{(n)}(0)(\theta_{j_i}y)\| + b\|x_0 - y_0\| \\
&\leq \sum_{i=1}^{k} a_i \Big(\sum_{l=1}^{n-1} \sum_{1 \leq i_1,\dots,i_l \leq k} a_{i_1} \cdots a_{i_l} b \|x_{j_i+j_{i_1}+\dots+j_{i_l}} - y_{j_i+j_{i_1}+\dots+j_{i_l}}\| \\
&\quad + b\|x_{j_i} - y_{j_i}\| \Big) + b\|x_0 - y_0\| \\
&= \sum_{l=1}^{n} \sum_{1 \leq i_1,\dots,i_l \leq k} a_{i_1} \cdots a_{i_l} b \|x_{j_{i_1}+\dots+j_{i_l}} - y_{j_{i_1}+\dots+j_{i_l}}\| + b\|x_0 - y_0\|
\end{aligned}$$

Hence inequality holds for $n + 1$.

- The case $n = 1$ is easy to check. For the first point we use induction. For $n = 2$ the result is a consequence of:

$$\Phi^{(2)}(x)(0) = h\left(h(0, x_{j_1}), \ldots, h(0, x_{j_k}), x_0\right)$$

Suppose now the result true for an integer $n \geq 2$. Then the identity

$$\Phi^{(n+1)}(0)(x) = h\left(\Phi^{(n)}(0)(\theta_{j_1}x), \ldots, \Phi^{n}(0)(\theta_{j_k}x), x_0\right)$$

shows that $\Phi^{(n+1)}(0)(x)$ only depends of coordinates $(x_{j_i+j})_{1\leq i\leq k, j\in A_n}$ and x_0 that is to say coordinates $(x_j)_{j\in A_{n+1}}$.
For the second point, we use inequality in 1. We have:

$$\begin{aligned}
\phi_{n,x,y} &\leq \sum_{l=1}^{n-1} \sum_{1\leq i_1,\ldots,i_l\leq k} a_{i_1}\cdots a_{i_l} b\|x_{j_{i_1}+\ldots+j_{i_l}} - y_{j_{i_1}+\ldots+j_{i_l}}\| + b\|x_0 - y_0\| \\
&\leq b\Big(\sum_{l=1}^{n-1} a^l + 1\Big) \sum_{j\in A_n} \|x_j - y_j\| \\
&= b\cdot \frac{1-a^n}{1-a} \sum_{j\in A_n} \|x_j - y_j\|
\end{aligned}$$

End of the proof of lemma 10. We recall the notation $X_t^n = \Phi^{(n)}(0)\big((\xi_{t-j})_j\big)$ for $n \in \mathbb{N}^*$ and $t \in \mathbb{Z}^d$. Set $f_1 = f(X_{s_1}, \ldots, X_{s_u})$, $g_1 = g(X_{t_1}, \ldots, X_{t_v})$ and

$$f_1' = f\left(X_{s_1}^n, \ldots, X_{s_u}^n\right), \qquad g_1' = g\left(X_{t_1}^n, \ldots, X_{t_v}^n\right)$$

For each $t \in \mathbb{Z}^d$, if $n \in \mathbb{N}\setminus\{0,1\}$ then $A_{t,n} = \{t\} - A_n$. If $\|s_i - t_l\| \geq r$ for $1 \leq i \leq u$ and $1 \leq l \leq v$ then $d(A_{s_i,n}, A_{t_l,n}) \geq (r - 2\rho n)\vee 0 = d_{r,n}$. Thus

$$\begin{aligned}
|\mathrm{Cov}(f_1, g_1)| &\leq |\mathrm{Cov}(f_1 - f_1', g_1)| + |\mathrm{Cov}(f_1', g_1 - g_1')| + |\mathrm{Cov}(f_1', g_1')| \\
&\leq 4\mathbb{E}|f_1 - f_1'| + 4\mathbb{E}|g_1 - g_1'| \\
&\quad + \psi(up_n, vp_n, K_n\mathrm{Lip}\ (f), K_n\mathrm{Lip}\ (g))\varepsilon_\xi(d_{r,n}) \\
&\leq 4\mathrm{Lip}\ (f)\sum_{i=1}^{u} \|X_{s_i} - X_{s_i}^n\|_1 + 4\mathrm{Lip}\ (g)\sum_{i=1}^{v} \|X_{t_i} - X_{t_i}^n\|_1 \\
&+ \psi(up_n, vp_n, K_n\mathrm{Lip}\ (f), K_n\mathrm{Lip}\ (g))\varepsilon_\xi(d_{r,n}) \\
&\leq (4\mathrm{Lip}\ (f)u + 4\mathrm{Lip}\ (g)v)a^n\|X_0\|_1 \\
&\quad + \psi(up_n, vp_n, K_n\mathrm{Lip}\ (f), K_n\mathrm{Lip}\ (g))\varepsilon_\xi(d_{r,n})
\end{aligned}$$

Note that this result is still true for $n = 1$.

1) Under $\eta-$weak dependence, $\psi(u, v, a, b) = au + bv$,

$$|\mathrm{Cov}(f_1, g_1)| \leq (u\mathrm{Lip}\ f + v\mathrm{Lip}\ g)(4a^n\|X_0\|_1 + K_n p_n \eta_\xi(d_{r,n})$$

Thus $|\mathrm{Cov}(f_1, g_1)| \leq (u\mathrm{Lip}\ f + v\mathrm{Lip}\ g)\eta(r)$ where

$$\eta(r) \leq \inf_{n\in\mathbb{N}^*}\{4a^n\|X_0\|_1 + K_n p_n \eta_\xi(d_{r,n})\}$$

2) With $\lambda-$weak dependence $\psi(u,v,a,b)=au+bv+abuv$,

$$|\text{Cov}(f_1,g_1)|\leq(u\text{Lip }f+v\text{Lip }g+uv\text{Lip }f\text{Lip }g)(4a^n\|X_0\|_1+K_np_n\lambda_\xi(d_{r,n})$$

Now $|\text{Cov}(f_1,g_1)|\leq(u\text{Lip }f+v\text{Lip }g+uv\text{Lip }f\text{Lip }g)\lambda(r)$ with

$$\lambda(r)\leq\inf_{n\in\mathbb{N}^*}\{4a^n\|X_0\|_1+K_np_n^2\lambda_\xi(d_{r,n})\}$$

As $(K_n)_n$ is bounded and $p_n\leq\sum_{l=1}^{n-1}k^l=\frac{k-k^n}{1-k}$ for $n\geq 2$, we obtain the proposed bounds.

We now prove that $\lim_{r\to\infty}\lambda(r)=0$. We suppose that the sequence $(\lambda_\xi(r))_r$ nonincreasing without loss of generality. We use the bound

$$\lambda(r)\leq C\inf_{N+2\rho n=r,n\in\mathbb{N}^*}\left\{a^n+k^{2n}\lambda_\xi(N)\right\}$$

If $N\in\mathbb{N}$, we choose $n_N=[log(\lambda_\xi(N))/(\log a-2\log k)]$. Note that $\lim_{N\to\infty}n_N=\infty$ and $\lim_{N\to\infty}\left(a^{n_N}+k^{2n_N}\lambda_\xi(N)\right)=0$. For $r\geq r_N=N+2\rho n_N$, we have $N+r-r_N+2\rho n_N=r$, hence:

$$\lambda(r)\leq a^{n_N}+k^{2n_N}\lambda_\xi(N+r-r_N)\leq a^{n_N}+k^{2n_N}\lambda_\xi(N)\to_{N\to\infty}0$$

Hence $\lim_{r\to\infty}\lambda(r)=0$. Analogously, $\lim_{r\to\infty}\eta(r)=0$. $\square$

4.5 General case

Recall that we have denoted $(X_{p,t})_{t\in\mathbb{Z}^d}$ for $s>0$ the unique solution of the equation $Z_t=F\left(\left(Z_{t-j}1_{0<\|j\|\leq p\}}\right)_{j\neq 0};\xi_t\right)$. Denote $f_1=f(X_{s_1},\ldots,X_{s_u})$, $g_1=g(X_{t_1},\ldots,X_{t_v})$, $f_1'=f(X_{p,s_1},\ldots,X_{p,s_u})$ and $g_1'=f(X_{p,t_1},\ldots,X_{p,t_v})$, then

$$\begin{aligned}|\text{Cov}(f_1,g_1)| &\leq |\text{Cov}(f_1-f_1',g_1)|+|\text{Cov}(f_1',g_1-g_1')|+|\text{Cov}(f_1',g_1')|\\ &\leq 4\,\text{Lip }f\sum_{i=1}^{u}\|X_{s_i}-X_{p,s_i}\|_1\\ &\quad+4\,\text{Lip }g\sum_{i=1}^{v}\|X_{t_i}-X_{p,t_i}\|_1+|\text{Cov}(f_1',g_1')|\\ &\leq 4\,\|X_0-X_{s,0}\|_1(u\text{Lip }f+v\text{Lip }g)+|\text{Cov}(f_1',g_1')|\end{aligned}$$

Recall that from the proof of lemma 8 we have:

$$\|X_{p,0}-X_0\|_1\leq\frac{\|X_0\|_1}{1-a}\sum_{\|j\|>p}a_j$$

Moreover, the field $X_{p,t}$ is $k-$dependent with $k=(2p)^d$.

- Suppose first that the random field ξ is $\eta-$weakly dependent. From proposition 10,

$$|\text{Cov}(f_1',g_1')|\leq(u\text{Lip }f+v\text{Lip }g)C\inf_{n\in\mathbb{N}^*}\left\{a^n+p^{dn}\eta_\xi\left((r-2pn)\vee 0\right)\right\}$$

for a suitable positive constant C.
Hence we bound $|\mathrm{Cov}(f_1, g_1)|$ by,

$$(u\mathrm{Lip}\ f + v\mathrm{Lip}\ g)C\left(\sum_{\|j\|>p} a_j + \inf_{n\in\mathbb{N}^*}\left\{a^n + p^{dn}\eta_\xi\left((r-2\rho n)\vee 0\right)\right\}\right)$$

for another positive constant denoted C. Then we obtain the proposed bound.

- Suppose that the random field ξ is $\lambda-$weakly dependent. From proposition 10, $|\mathrm{Cov}(f_1', g_1')|$ is bounded by

$$(u\mathrm{Lip}\ f + v\mathrm{Lip}\ g + uv\mathrm{Lip}\ f\mathrm{Lip}\ g)\inf_{n\in\mathbb{N}^*}\left\{a^n + p^{2dn}\lambda_\xi\left((r-2pn)\vee 0\right)\right\}$$

up a suitable positive constant C. Hence we bound $|\mathrm{Cov}(f_1, g_1)|$ by,

$$(u\mathrm{Lip}\ f + v\mathrm{Lip}\ g + uv\mathrm{Lip}\ f\mathrm{Lip}\ g) \times\Big(\sum_{\|j\|>p} a_j + \inf_{n\in\mathbb{N}^*}\left\{a^n + p^{2dn}\lambda_\xi\left((r-2pn)\vee 0\right)\right\}\Big)$$

up to another positive constant C. Then we obtain the proposed bound.

4.6 Results on causality

4.6.1 Proof of proposition 2

We will use here the Euclidean norm on $\mathbb{R}^d$. We proceed by induction on d.
For $d = 1$, if there exists $r_1, r_2 \in B$ such that $r_1 > 0$ and $r_2 < 0$ then $B \cap (-B) \neq \{0\}$. Then we can choose $b_1 = 1$ if $B \subset \mathbb{R}_+$ or $b_1 = -1$ if $B \subset \mathbb{R}_-$.
Suppose the result true for $d-1$. We first define b_1.
1) If B° is empty, since B is convex and contain 0 there exists $b_1 \in \mathbb{R}^d \setminus \{0\}$ such that $B \subset H = \{x \in \mathbb{R}^d / x.b_1 = 0\}$($\cdot$ denotes the scalar product in $\mathbb{R}^d$).
2) Now if B° is not empty, like $B \cap (-B) = \{0\}$ it is clear that $0 \notin B^\circ$. Moreover B° is still convex and by application of the Hahn-Banach theorem ([2], theorem 3.3, page 108), there exists $b_1 \in R^d \setminus \{0\}$ such that $B^\circ \subset \{x \in \mathbb{R}^d / x.b_1 \geq 0\}$. Like for a convex $\overline{B^\circ} = \overline{B}$, then the same inclusion holds for B. We set here $H = \{x \in \mathbb{R}^d / x.b_1 = 0\}$.
We consider now the convex cone $C = B \cap H$. If g denote an isomorphism between H and $\mathbb{R}^{d-1}$, then $g(C)$ is a convex cone of $\mathbb{R}^{d-1}$ such that $g(C) \cap \big(-g(C)\big) = \{0\}$. Hence there exists a basis $c = (c_2, \ldots, c_d)$ such that $g(C) \subset \{x \in \mathbb{R}^{d-1} / 0 \leq_c x\}$. For $i = 2, \ldots, d$ we set $b_i = g^{-1}(c_i)$. Then $b = (b_1, \ldots, b_d)$ is a basis of $\mathbb{R}^d$ and if $x = x_1 b_1 + \ldots + x_d b_d \in B$, we have by the preceding two points $x_1 \geq 0$. Suppose that $x_1 = 0$, then $x \in C$ and $g(x) \geq_c 0 \Rightarrow (x_2, \ldots, x_d) \geq_{lex} 0$ in $\mathbb{R}^{d-1}$. Hence $(x_1, \ldots, x_d) \geq_{lex} 0$ in $\mathbb{R}^d$, in other word $x \geq_b 0$.

4.6.2 Proof of lemma 5

Proof of lemma 5 Denote for $p \in \mathbb{N}^*$, $Y_p = F\big((X_j \mathbf{1}_{0<\|j\|\leq p})_j; \xi_0\big)$.
1) We first prove that for $p \in \mathbb{N}^*$, $Y_p \in \mathbb{L}^m$.

Recall that here F is measurable wrt $\mathfrak{F}_{\widetilde{A}} \otimes \mathcal{B}(E')$. Let $x \in E^I$, then using the independence between ξ_0 and $\sigma\big(X_j, j \in A\big)$ and the condition **(H3)**, we have:

$$\begin{aligned}
\mathbb{E}\big(\|Y_p - F(0;\xi_0)\|^m / X_j = x_j, j \in A\big) &= \mathbb{E}\left\|F((x_j \mathbf{1}_{0<\|j\|\le p})_j; \xi_0) - F(0;\xi_0)\right\|^m \\
&\le \left(\sum_{0<\|j\|\le p} a_j \|x_j\| \right)^m
\end{aligned}$$

Hence by integration:

$$\|Y_p - F(0;\xi_0)\|_m \le \left\| \sum_{0<\|j\|\le p} a_j \|X_j\| \right\|_m \le \|X_0\|_m$$

As $F(0;\xi_0) \in \mathbb{L}^m$, we obtain the result.
It is enough to prove that $(Y_p)_{p\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathbb{L}^m$. Using the same method as in 1), we obtain if $q > 0$:

$$\|Y_{p+q} - Y_p\|_m \le \|X_0\|_m \sum_{\|j\|>p} a_j$$

This inegality imply the result.
2) Using the same method as in 1), we have for $p \in \mathbb{N}^*$:

$$\left\|Y_p - Y'_p\right\|_m \le \sum_{0<\|j\|\le p} a_j \left\|Y_j - Y'_j\right\|_m$$

Hence the result follows with $p \to \infty$. $\square$

4.7 Proof of theorem 3

4.7.1 Existence

If $H \in \mathbb{L}^m_{\widetilde{A}}(\mu)$, we denote by Y the random field defined as $Y_j = H \circ \theta_j(\xi)$ for $j \in \mathbb{Z}^d$. If $j \in A$, then $H \circ \theta_j$ is measurable wrt $\sigma(\pi'_{j+j'}/j' \in \widetilde{A}) \subset \mathfrak{F}'_{\widetilde{A}}$. Hence if $p \in \mathbb{N}^*$, $\Phi_p(H) \in \mathbb{L}^m_{\widetilde{A}}(\mu)$ and by the lemma 4, ξ_0 is independant of $\sigma(Y_j/j \in A)$. By application of the lemma 5, $\Phi_p(H)$ converges to an element of $\mathbb{L}^m_{\widetilde{A}}(\mu)$ witch is $F\big((H \circ \theta_j)_{j\neq 0}; \pi_0\big)$.
Lets show that the application $\Phi : \mathbb{L}^m_{\widetilde{A}}(\mu) \mapsto \mathbb{L}^m_{\widetilde{A}}(\mu)$ defined as $\Phi(H) = F\big((H \circ \theta_j)_{j\neq 0}; \pi_0\big)$ is contraction in $\mathbb{L}^m_{\widetilde{A}}(\mu)$. If $H, H' \in \mathbb{L}^m_{\widetilde{A}}(\mu)$ then the two random fields Y and Y' defined as $Y_j = H \circ \theta_j(\xi)$ and $Y'_j = H' \circ \theta_j(\xi)$ for $j \in \mathbb{Z}^d$ verify the assumptions of lemma 5. Indeed $\sigma(Y_j, Y'_j/j \in A) \subset \sigma(\xi_{j+j'}/j \in A, j' \in \widetilde{A})$ and using the lemma 4 we deduce the independence between ξ_0 and $\sigma(Y_j, Y'_j/j \in A)$. Hence, we have:

$$\begin{aligned}
\left\|\Phi(H)(\xi) - \Phi(H')(\xi)\right\|_m &\le \sum_{j\in A} a_j \left\|H \circ \theta_j(\xi) - H' \circ \theta_j(\xi)\right\|_m \\
&= \sum_{j\in A} a_j \left\|H(\xi) - H'(\xi)\right\|_m
\end{aligned}$$

witch shows the result.
The construction of X_t comes from theorem 2. The variable $H(\xi)$ being measurable wrt $\sigma\left(\xi_j; j \in \widetilde{A}\right)$ measurability of X_t is simply deduced. Then unicity is a consequence of the application of the fixed point theorem. $\square$

4.7.2 Weak dependence

Weak dependence of the solution is as in § 4.3.1 where **(H2)** replaces **(H3')**. The case of finite range corresponds to k-Markov systems on a finite causal set. To prove lemma 7, we use **(H3')** and independence of rvs $\left(X_{t-j_1},\ldots,X_{t-j_k},X_{t-j_1}^{(r)},\ldots,X_{t-j_k}^{(r)}\right)$ and ξ_t to derive (12). In the general case we note $(X_{p,t})_t$ the solution of $Z_t = F\left((Z_{t-j}1_{\{j\in A_p\}})_j;\xi_t\right)$ with $A_p = \{t \in A/\|t\| \le p\}$ and we conclude as in lemma 9. □

4.8 Proof of lemma 6

1) The proposed bound is a consequence of the fixed point theorem. Thus we deduce that for each $\varepsilon > 0$:

$$\begin{aligned}\sum_{n\ge1}\mathbb{P}\left(\|X_t-\Phi^{(n)}(0)(\xi_{t-j},j\in\mathbb{Z}^d)\|\ge\varepsilon\right) &\le \frac{1}{\varepsilon}\sum_{n\ge1}\|X_t-\Phi^{(n)}(0)(\xi_{t-j},j\in\mathbb{Z}^d)\|_1\\ &< \infty\end{aligned}$$

Hence by the Borel-Cantelli lemma, we deduce $\lim_{n\to\infty}\Phi^{(n)}(0)\left(\xi_{t-j},j\in\mathbb{Z}^d\right)=X_t$ *a.s.*
2) We use induction. For $n=1$

$$\begin{aligned}\|X_t-\Phi_p(0)(\xi_{t-j},j\in\mathbb{Z}^d)\|_1 &= \|F\left((X_{t-j})_j,\xi_t\right)-F(0,\xi_t)\|_1\\ &\le a\|X_0\|_1\\ &\le a\|X_0\|_1+\|X_0\|_1\sum_{\|j\|>p}a_j\end{aligned}$$

Suppose the result true for an integer $n\ge1$, then

$$\begin{aligned}&\|X_t-X_{p,t}^{n+1}\|_1\\ &\quad=\left\|F\left((X_{t-k})_k,\xi_t\right)-F\left(\left(\Phi_p^{(n+1)}(0)(\xi_{t-j-k},j\in\mathbb{Z}^d)\mathbf{1}_{\|k\|\le p}\right)_k,\xi_t\right)\right\|_1\\ &\quad\le\sum_{\|k\|\le p}a_k\left\|X_{t-k}-\Phi_p^{(n)}(0)(\xi_{t-j-k},j\in\mathbb{Z}^d)\right\|_1+\|X_0\|_1\sum_{\|k\|>p}a_k\\ &\quad\le a\Big(a^n\|X_0\|_1+\frac{1-a^n}{1-a}\|X_0\|_1\sum_{\|k\|>p}a_k\Big)+\|X_0\|_1\sum_{\|k\|>p}a_k\\ &\quad=a^{n+1}\|X_0\|_1+\frac{1-a^{n+1}}{1-a}\|X_0\|_1\sum_{\|k\|>p}a_k\end{aligned}$$

4.9 Proofs for the section 3

4.9.1 Proof of corollary 1

Here $F(x;u) = f(x_{\ell_1},\dots,x_{\ell_k}) + h(u)g(x_{\ell_1},\dots,x_{\ell_k}) + u$. Condition **(H1)** is easy to check and *e.g.* in the first case,

$$\begin{aligned}\|F(z;\xi_0) - F(z';\xi_0)\|_m &\le \|f(z_{\ell_1},\dots,z_{\ell_k}) - f(z'_{\ell_1},\dots,z'_{\ell_k})\| \\ &\quad + \|\gamma_t\|_\infty \|g(z_{\ell_1},\dots,z_{\ell_k}) - g(z'_{\ell_1},\dots,z'_{\ell_k})\| \\ &\le \sum_{i=1}^k b_i\|z_{\ell_i} - z'_{\ell_i}\| + \|\gamma_0\|_\infty \sum_{i=1}^k c_i \|z_{\ell_i} - z'_{\ell_i}\|. \qquad \square\end{aligned}$$

For dependent inputs, we remark that $(z,u)\mapsto F(z;u)$ is a Lipschitz function in order to apply proposition 1.

4.9.2 Proof of proposition 3

Normal convergence in $\mathbb{L}^m$ will justify all the forthcoming manipulations of series. We only consider the more complicated causal case. In order to prove that $X_t \in \mathbb{L}^m$ we will prove the normal convergence of the series.

Set $S = \|\xi_t\| + \sum_{i=1}^{+\infty} \sum_{j_1,\dots,j_i\in A} \|\alpha_t^{j_1}\cdots\alpha^{j_i}_{t-j_1-\dots-j_{i-1}}\xi_{t-(j_1+\dots+j_i)}\|_m$, we notice from causality that indices t and $(t-(j_1+\dots+j_\ell))$ are distinct if $1\le \ell\le i$ hence the independence of inputs implies

$$\begin{aligned}&\|\alpha_t^{j_1}\cdots\alpha^{j_i}_{t-j_1-\dots-j_{i-1}}\xi_{t-(j_1+\dots+j_i)}\|_m \\ &\qquad \le \Big\| \|\alpha_t^{j_1}\|\cdots\|\alpha^{j_i}_{t-j_1-\dots-j_{i-1}}\|\|\xi_{t-(j_1+\dots+j_i)}\|\Big\|_m \\ &\qquad\qquad = \|\alpha_t^{j_1}\|_m\cdots\|\alpha^{j_i}_{t-j_1-\dots-j_{i-1}}\|_m\|\xi_{t-(j_1+\dots+j_i)}\|_m\end{aligned}$$

$$\begin{aligned}S &\le \|\zeta_t\|_m + \sum_{j\in A}\sum_{j_1,\dots,j_i\in A}\|\alpha_t^{j_1}\|_m\cdots\|\alpha^{j_i}_{t-j_1-\dots-j_{i-1}}\|_m\|\zeta_{t-(j_1+\dots+j_i)}\|_m \\ &= \|\zeta_0\|_m\Big(1+\sum_{j_i n A}\sum_{j_1,\dots,j_i\in A}\|\alpha_0^{j_1}\|_m\cdots\|\alpha_0^{j_i}\|_m\Big) \\ &= \|\zeta_0\|_m\Big(1+\sum_{i=1}^{+\infty}\Big(\sum_{j\in A}\|a_0^j\|_m\Big)^i\Big) \\ &= \|\zeta_0\|_m\Big(1+\sum_{i=1}^{+\infty} b^i\Big) \\ &= \|\zeta_0\|_m\left(1+\frac{b}{1-b}\right) \\ &< \infty.\end{aligned}$$

In order to prove that X_t is solution of the equation, we expand it:

$$\begin{aligned} X_t &= \zeta_t + \sum_{j_1 \in A} \alpha_t^{j_1}\ \zeta_{t-j_1} + \sum_{i=2}^{\infty} \sum_{j_1,\ldots,j_i \in A} \alpha_t^{j_1} \cdots \alpha_{t-j_1-\cdots-j_{i-1}}^{j_i} \zeta_{t-(j_1+\cdots+j_i)} \\ &= \zeta_t + \sum_{j_1 \in A} \alpha_t^{j_1} \zeta_{t-j_1} \\ &\quad + \sum_{j_1 \in A} \alpha_t^{j_1} \sum_{i=2}^{\infty} \sum_{j_2,\ldots,j_i \in A} \alpha_{t-j_1}^{j_2} \cdots \alpha_{t-j_1-\cdots-j_{i-1}}^{j_i} \zeta_{t-j_1-(j_2+\cdots+j_i)} \\ &= \zeta_t + \sum_{j_1 \in A} \alpha_t^{j_1} X_{t-j_1} \end{aligned}$$

Here $F(x;(u,v)) = \sum_{j \in A} u_j x_j + v$ and we use notations in **(H3)**. As ξ is iid, the variables $(Z(\xi), Z'(\xi))$ are $(\alpha_0^j)_{j\in A}$ are independent and

$$\|F\left(z;\zeta_0\right) - F\left(z';\zeta_0\right)\|_m \le \sum_{j \in A} \|\alpha_0^j\|_m \|z_j - z'_j\|$$

Since $b = \sum_{j\in A} \|a_0^j\|_m < 1$, **(H3)** holds.
In the first non-causal case the above inequalities are only changed by using the bound

$$\|\alpha_t^{j_1} \cdots \alpha_{t-j_1-\cdots-j_{i-1}}^{j_i} \xi_{t-(j_1+\cdots+j_i)}\|_m \le \|\alpha_t^{j_1}\|_\infty \cdots \|\alpha_{t-j_1-\cdots-j_{i-1}}^{j_i}\|_\infty \|\xi_{t-(j_1+\cdots+j_i)}\|_m. \ \square$$

4.9.3 Proof of proposition 4

Here **(H1)** holds and with the notation in **(H3)**:

$$\|F(z,\xi_0) - F(z',\xi_0)\|_m \le \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \|\alpha_j\| \|\xi_0\|_m \|z_j - z'_j\|.$$

The proposed solution is in $\mathbb{L}^m$ from normal convergence of series

$$\begin{aligned} \|X_t\|_m &\le \|\xi_t\|_m \Big(\|a\| + \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k \in A} \|\alpha_{j_1}\| \|\xi_{t-j_1}\|_m \cdots \|\alpha_{j_k}\| \|\xi_{t-j_1-\cdots-j_k}\|_m \|a\| \Big) \\ &= \|\xi_0\|_m \|a\| \Big(1 + \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k \in A} \|\alpha_{j_1}\| \|\xi_{t-j_1}\|_m \cdots \|\alpha_{j_k}\| \|\xi_0\|_m \Big) \\ &= \|\xi_0\|_m \|a\| \Big(1 + \sum_{k=1}^{\infty} b^i \|\xi_0\|_m^i \Big) \\ &= \|\xi_0\|_m \|a\| \Big(1 + \frac{b\|\xi_0\|_m}{1 - b\|\xi_0\|_m} \Big) < \infty. \end{aligned}$$

Substitutions prove that this process is a solution of the equation.

$$\begin{aligned} X_t &= \xi_t\Big(a + \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \in A} \alpha_{j_1}\xi_{t-j_1}\cdots\alpha_{j_k}\xi_{t-j_1-\dots-j_k}a\Big) \\ &= \xi_t\Big(a + \sum_{j_1\in A} \alpha_{j_1}\xi_{t-j_1}\Big(a + \sum_{k=2}^{+\infty} \alpha_{j_2}\xi_{t-j_1-j_2}\dots\alpha_{j_k}\xi_{t-j_1-j_2-\dots-j_k}\Big)\Big) \\ &= \xi_t\Big(a + \sum_{j_1\in A} \alpha_{j_1}X_{t-j_1}\Big). \qquad \square \end{aligned}$$

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