# Nonradial blow-up solutions of sublinear elliptic equations with gradient term

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**Abstract**. Let $f$ be a continuous and non-decreasing function such that $f > 0$ on $(0, \infty)$, $f(0) = 0$, $\sup_{s\geq 1} f(s)/s < \infty$ and let $p$ be a non-negative continuous function. We study the existence and nonexistence of explosive solutions to the equation $\Delta u + |\nabla u| = p(x)f(u)$ in $\Omega$, where $\Omega$ is either a smooth bounded domain or $\Omega = \mathbb{R}^N$. If $\Omega$ is bounded we prove that the above problem has never a blow-up boundary solution. Since $f$ does not satisfy the Keller-Osserman growth condition at infinity, we supply in the case $\Omega = \mathbb{R}^N$ a necessary and sufficient condition for the existence of a positive solution that blows up at infinity.

**Key words**: explosive solution, elliptic equation, maximum principle, sublinear growth condition.

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## 1 Introduction and the main results

Explosive solutions of semilinear elliptic equations have been studied intensively in the last few decades. Most of such studies have been concerned with equations of the type

$$\Delta u = g(x, u),$$

in which the function $g$ takes various forms (see [2, 3, 4, 5, 6, 7, 16] and their references).

In this paper we study an elliptic problem involving a sublinear nonlinearity. Due to the lack of the Keller-Osserman condition [12, 17], we find a necessary and sufficient condition satisfied by the potential so that our problem admits a nonradial solution blowing up at infinity. More precisely, we consider the equation

$$\begin{cases} \Delta u + |\nabla u| = p(x)f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is either a smooth bounded domain or the whole space.

The presence of the gradient term can have significant influence on the existence of a solution, as well as on its asymptotic behavior. Problems of this type appear in stochastic control theory and have been first studied by Lasry and Lions [14]. The corresponding parabolic equation was considered in Quittner [18]. We also refer to Bandle and Giarrusso [1, 10] who established existence results and the asymptotic behavior of solutions for semilinear elliptic equations in bounded domains containing gradient term (see also [13] for another class of nonlinear elliptic problems involving gradient term).

Throughout this paper we assume that $p$ is a non-negative function such that $p \in C^{0,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ if $\Omega$ is bounded, and $p \in C^{0,\alpha}_{\rm loc}(\mathbb{R}^N)$, otherwise. The non-decreasing non-linearity $f$ fulfills

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$(f1)\qquad f \in C^{0,\alpha}_{\rm loc}[0,\infty),\ \ f(0)=0 \ \text{ and } \ f>0 \ \text{ on } \ (0,\infty).$

We also assume that $f$ is sublinear at infinity, in the sense that

$(f2)\qquad \Lambda \equiv \sup_{s\geq 1} \dfrac{f(s)}{s} < \infty.$

Cf. Véron [19], the non-decreasing non-linearity $f$ is called an absorption term.

A solution $u$ of the problem (1) with $u(x)\to\infty$ as $\operatorname{dist}(x,\partial\Omega)\to 0$ (if $\Omega$ is bounded) is called a *large (explosive, blow-up)* solution. If $\Omega=\mathbb{R}^N$, this condition can be rewritten as $u(x)\to\infty$ as $|x|\to\infty$. In this latter case such solution is called an *entire large (explosive)* solution. In terms of the dynamic programming approach, an explosive solution of (1) corresponds to a value function (or Bellman function) associated to an infinite exit cost (see [14]).

We note that in [9] it is studied the existence and nonexistence of large solutions for the corresponding system to (1) where the coefficients are radial functions.

If $\Omega$ is bounded we prove the following non-existence result.

**Theorem 1.** *Suppose $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain. Then (1) has no positive large solution in $\Omega$.*

Following Bandle and Giarrusso [1], in the presence of the Keller-Ossermann condition on $f$, equation (1) may have positive large solutions.

Next, we consider problem (1) when $\Omega=\mathbb{R}^N$. For all $r\geq 0$ we set

$$\varphi(r)=\max_{|x|=r} p(x), \qquad \psi(r)=\min_{|x|=r} p(x), \qquad \text{and} \qquad h(r)=\varphi(r)-\psi(r).$$

We suppose that

$$\int_0^\infty r h(r)\Psi(r)dr<\infty, \tag{2}$$

where

$$\Psi(r)=\exp\left(\Lambda_N\int_0^r s\psi(s)ds\right), \qquad \Lambda_N=\frac{\Lambda}{N-2}.$$

Obviously, if $p$ is radial then $h\equiv 0$ and (2) occurs. Assumption (2) shows that the variable potential $p(x)$ has a slow variation. An example of nonradial potential for which (2) holds is $p(x)=\dfrac{1+|x_1|^2}{(1+|x_1|^2)(1+|x|^2)+1}$. In this case $\varphi(r)=\dfrac{r^2+1}{(r^2+1)^2+1}$ and $\psi(r)=\dfrac{1}{r^2+2}$. If $\Lambda_N=1$, by direct computation we get $rh(r)\Psi(r)=O\left(r^{-2}\right)$ as $r\to\infty$ and so (2) holds.

Our analysis will be developed under the basic assumption (2).

**Theorem 2.** *Assume $\Omega=\mathbb{R}^N$ and $p$ satisfies (2). Then (1) has positive entire large solution if and only if*

$$\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt=\infty. \tag{3}$$

**Remark 1.** Since $\int_0^r e^r t^k dt = k! e^r \left( \sum_{s=1}^{k} (-1)^{k-s} \frac{t^s}{s!} \right) - (-1)^k k!$, for all integers $k \geq 1$, we can give some examples of potentials $p$ that verify both conditions (2) and (3). In the case where $\Lambda_N = 1$ such functions are

(i) $p(x) = 1 + |x|^m + |x_1| e^{-|x|^{m+2}}$, $m > 0$.

(ii) $p(x) = \frac{1 + |x_1| g(|x|) e^{-|x|}}{1 + |x|}$, $g \in C^{0,\alpha}_{\mathrm{loc}}[0,\infty) \cap L^1[0,\infty)$, $g \geq 0$.

**Remark 2.** We point out that a solution of (1) may exist even if condition (2) fails, as shown in what follows. Define

$$p(x) = 2|x|^2 + 6x_1^2 + \sqrt{|x|^2 + 3x_1^2} + N + 1,\ x \in \mathbb{R}^N.$$

and $f(t) = 2t$. For this choice of $p$ and $f$, the equation (1) has the nonradial entire large solution $u(x) = e^{|x|^2 + x_1^2}$. In this case $h(r) = 6r^2 + r$, so (2) fails to hold.

The above results also apply to problems on Riemannian manifolds if $\Delta$ is replaced by the Laplace–Beltrami operator

$$\Delta_B = \frac{1}{\sqrt{c}} \frac{\partial}{\partial x_i} \left( \sqrt{c}\, a_{ij}(x) \frac{\partial}{\partial x_i} \right), \qquad c := \det\left(a_{ij}\right),$$

with respect to the metric $ds^2 = c_{ij}\, dx_i dx_j$, where $(c_{ij})$ is the inverse of $(a_{ij})$. In this case our results apply to concrete problems arising in Riemannian geometry. For instance, (cf. Loewner-Nirenberg [15]) if $\Omega$ is replaced by the standard $N$–sphere $(S^N, g_0)$, $\Delta$ is the Laplace-Beltrami operator $\Delta_{g_0}$, $a = N(N-2)/4$, and $f(u) = (N-2)/[4(N-1)]\, u^{(N+2)/(N-2)}$, we find the prescribing scalar curvature equation with gradient term.

# 2 Proofs

## 2.1 Proof of Theorem 1

Suppose by contradiction that (1) has a positive large solution $u$ and define $v(x) = \ln(1 + u(x))$, $x \in \Omega$. It follows that $v$ is positive and $v(x) \to \infty$ as $\mathrm{dist}\,(x, \partial\Omega) \to 0$. We have

$$\Delta v = \frac{1}{1+u} \Delta u - \frac{1}{(1+u)^2} |\nabla u|^2 \qquad \text{in } \Omega$$

and so

$$\Delta v \leq p(x) \frac{f(u)}{1+u} \leq \|p\|_\infty \frac{f(u)}{1+u} \leq A \qquad \text{in } \Omega,$$

for some constant $A > 0$. Therefore

$$\Delta(v(x) - A|x|^2) < 0, \qquad \text{for all } x \in \Omega.$$

Let $w(x) = v(x) - A|x|^2$, $x \in \Omega$. Then $\Delta w < 0$ in $\Omega$. Moreover, since $\Omega$ is bounded, it follows that $w(x) \to \infty$ as $\mathrm{dist}(x, \partial\Omega) \to 0$.

Let $M > 0$ be arbitrary. We claim that $w \geq M$ in $\Omega$. For all $\delta > 0$, we set

$$\Omega_\delta = \{x \in \Omega\,;\ \mathrm{dist}(x, \partial\Omega) > \delta\}.$$

Since $w(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, we can choose $\delta > 0$ such that

$$w(x) \geq M \qquad \text{for all } \; x \in \Omega \setminus \Omega_\delta. \tag{4}$$

On the other hand,

$$\begin{aligned} -\Delta(w(x) - M) > 0 \qquad & \text{in } \Omega_\delta, \\ w(x) - M \geq 0 \qquad & \text{on } \partial\Omega_\delta. \end{aligned}$$

By the maximum principle we get $w(x) - M \geq 0$ in $\Omega_\delta$. So, by (4), $w \geq M$ in $\Omega$. Since $M > 0$ is arbitrary, it follows that $w \geq n$ in $\Omega$, for all $n \geq 1$. Obviously, this is a contradiction and the proof is now complete. ■

## 2.2 Proof of Theorem 2

Several times in the proof of Theorem 2 we shall apply the following inequality

$$\int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} g(s)\,ds\,dt \leq \frac{1}{N-2} \int_0^r t g(t)\,dt, \qquad \forall\, r > 0, \tag{5}$$

for any continuous function $g : [0, \infty) \to [0, \infty)$. Indeed, using an integration by parts in the left hand side we obtain

$$\begin{aligned} \int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} g(s)\,ds\,dt \quad & \leq \int_0^r t^{1-N} \int_0^t s^{N-1} g(s)\,ds\,dt \\ & = \frac{1}{2-N} \int_0^r \left(t^{2-N}\right)' \int_0^t s^{N-1} g(s)\,ds\,dt \\ & = \frac{1}{2-N} r^{2-N} \int_0^r t^{N-1} g(t)\,dt + \frac{1}{N-2} \int_0^r t g(t)\,dt \\ & \leq \frac{1}{N-2} \int_0^r t g(t)\,dt, \end{aligned}$$

so (5) follows.

Necessary condition. Suppose that (2) fails and the equation (1) has a positive entire large solution $u$. We claim that

$$\int_1^\infty e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \varphi(s)\,ds\,dt < \infty. \tag{6}$$

We first recall that $\varphi = h + \psi$. Thus

$$\begin{aligned} \int_1^\infty e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \varphi(s)\,ds\,dt \quad & = \int_1^\infty e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s)\,ds\,dt \\ & + \int_1^\infty e^{-t} t^{1-N} \int_0^t e^s s^{N-1} h(s)\,ds\,dt. \end{aligned}$$

By virtue of (5) we find

$$\begin{aligned}\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\varphi(s)dsdt &\le \int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt + \frac{1}{N-2}\int_0^\infty th(t)dt\\ &\le \int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt\\ &\quad+\frac{1}{N-2}\int_0^\infty th(t)\Psi(t)dt.\end{aligned}$$

Since $\int_1^\infty e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt < \infty$, by (2) we deduce that (6) follows.

Now, let $\bar{u}$ be the spherical average of $u$, i.e.,

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}}\int_{|x|=r} u(x)d\sigma_x, \qquad r \ge 0,$$

where $\omega_N$ is the surface area of the unit sphere in $\mathbb{R}^N$. Since $u$ is a positive entire large solution of (1) it follows that $\bar{u}$ is positive and $\bar{u}(r) \to \infty$ as $r \to \infty$. With the change of variable $x \to ry$, we have

$$\bar{u}(r) = \frac{1}{\omega_N}\int_{|y|=1} u(ry)\, d\sigma_y, \qquad r \ge 0$$

and

$$\bar{u}'(r) = \frac{1}{\omega_N}\int_{|y|=1} \nabla u(ry)\cdot y\, d\sigma_y, \qquad r \ge 0. \tag{7}$$

Hence

$$\bar{u}'(r) = \frac{1}{\omega_N}\int_{|y|=1} \frac{\partial u}{\partial r}(ry)\, d\sigma_y = \frac{1}{\omega_N r^{N-1}}\int_{|x|=r} \frac{\partial u}{\partial r}(x)\, d\sigma_x,$$

that is

$$\bar{u}'(r) = \frac{1}{\omega_N r^{N-1}}\int_{B(0,R)} \Delta u(x)\, dx, \qquad \text{for all } r \ge 0. \tag{8}$$

Due to the gradient term $|\nabla u|$ in (1), we cannot infer that $\Delta u \ge 0$ in $\mathbb{R}^N$ and so we cannot expect that $\bar{u}' \ge 0$ in $[0, \infty)$. We define the auxiliary function

$$U(r) = \max_{0\le t\le r} \bar{u}(t), \qquad r \ge 0. \tag{9}$$

Then $U$ is positive and non-decreasing. Moreover, $U \ge \bar{u}$ and $U(r) \to \infty$ as $r \to \infty$.

The assumptions $(f1)$ and $(f2)$ yield $f(t) \leq \Lambda(1+t)$, for all $t \geq 0$. So, by (7) and (8),

$$\begin{aligned}
\bar{u}'' + \frac{N-1}{r}\bar{u}' + \bar{u}' &\leq \frac{1}{\omega_N r^{N-1}} \int\limits_{|x|=r} [\Delta u(x) + |\nabla u|(x)]\, d\sigma_x \\
&= \frac{1}{\omega_N r^{N-1}} \int\limits_{|x|=r} p(r) f(u(x)) d\sigma_x \\
&\leq \Lambda\varphi(r) \frac{1}{\omega_N r^{N-1}} \int\limits_{|x|=r} (1 + u(x))\, d\sigma_x \\
&= \Lambda\varphi(r)\,(1 + \bar{u}(r)) \\
&\leq \Lambda\varphi(r)\,(1 + U(r))\,,
\end{aligned}$$

for all $r \geq 0$. It follows that

$$\left(r^{N-1} e^r \bar{u}'\right)' \leq \Lambda e^r r^{N-1} \varphi(r)\,(1 + U(r))\,, \qquad \text{for all } r \geq 0.$$

So, for all $r \geq r_0 > 0$,

$$\bar{u}(r) \leq \bar{u}(r_0) + \Lambda \int_{r_0}^{r} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \varphi(s)(1 + U(s)) ds dt.$$

The monotonicity of $U$ implies

$$\bar{u}(r) \leq \bar{u}(r_0) + \Lambda(1 + U(r)) \int_{r_0}^{r} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \varphi(s) ds dt, \tag{10}$$

for all $r \geq r_0 \geq 0$. By (6) we can choose $r_0 \geq 1$ such that

$$\int_{r_0}^{\infty} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \varphi(s) ds dt < \frac{1}{2\Lambda}. \tag{11}$$

Thus (10) and (11) yield

$$\bar{u}(r) \leq \bar{u}(r_0) + \frac{1}{2}(1 + U(r)), \qquad \text{for all } r \geq r_0. \tag{12}$$

By the definition of $U$ and $\lim_{r \to \infty} \bar{u}(r) = \infty$, we find $r_1 \geq r_0$ such that

$$U(r) = \max_{r_0 \leq t \leq r} \bar{u}(r), \qquad \text{for all } r \geq r_1. \tag{13}$$

Considering now (12) and (13) we obtain

$$U(r) \leq \bar{u}(r_0) + \frac{1}{2}(1 + U(r)), \qquad \text{for all } r \geq r_1.$$

Hence

$$U(r) \leq 2\bar{u}(r_0) + 1, \qquad \text{for all } r \geq r_1.$$

This means that $U$ is bounded, so $u$ is also bounded, a contradiction. It follows that (1) has no positive entire large solutions.

Sufficient condition. We need the following auxiliary comparison result.

**Lemma 1.** *Assume that (2) and (3) hold. Then the equations*

$$\Delta v + |\nabla v| = \varphi(|x|)f(v) \qquad \Delta w + |\nabla w| = \psi(|x|)f(w) \tag{14}$$

*have positive entire large solution such that*

$$v \le w \qquad \text{in } \mathbb{R}^N. \tag{15}$$

**Proof**. Radial solutions of (14) satisfy

$$v'' + \frac{N-1}{r}v' + |v'| = \varphi(r)f(v)$$

and

$$w'' + \frac{N-1}{r}w' + |w'| = \psi(r)f(w).$$

Assuming that $v'$ and $w'$ are non-negative, we deduce

$$\left(e^r r^{N-1}v'\right)' = e^r r^{N-1}\varphi(r)f(v)$$

and

$$\left(e^r r^{N-1}w'\right)' = e^r r^{N-1}\psi(r)f(w).$$

Thus any positive solutions $v$ and $w$ of the integral equations

$$v(r) = 1 + \int_0^r e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\varphi(s)f(v(s))dsdt, \qquad r \ge 0, \tag{16}$$

$$w(r) = b + \int_0^r e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)f(w(s))dsdt, \qquad r \ge 0, \tag{17}$$

provide a solution of (14), for any $b > 0$. Since $w \ge b$, it follows that $f(w) \ge f(b) > 0$ which yields

$$w(r) \ge b + f(b)\int_0^r e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)dsdt, \qquad r \ge 0.$$

By (3), the right hand side of this inequality goes to $+\infty$ as $r \to \infty$. Thus $w(r) \to \infty$ as $r \to \infty$. With a similar argument we find $v(r) \to \infty$ as $r \to \infty$.

Let $b > 1$ be fixed. We first show that (17) has a positive solution. Similarly, (16) has a positive solution.

Let $\{w_k\}$ be the sequence defined by $w_1 = b$ and

$$w_{k+1}(r) = b + \int_0^r e^{-t}t^{1-N}\int_0^t e^s s^{N-1}\psi(s)f(w_k(s))dsdt, \qquad k \ge 1. \tag{18}$$

We remark that $\{w_k\}$ is a non-decreasing sequence. To get the convergence of $\{w_k\}$ we will show that $\{w_k\}$ is bounded from above on bounded subsets. To this aim, we fix $R > 0$ and we prove that

$$w_k(r) \le be^{Mr}, \qquad \text{for any } 0 \le r \le R, \text{ and for all } k \ge 1, \tag{19}$$

where $M \equiv \Lambda_N \max_{t\in[0,R]} t\psi(t)$.

We achieve (19) by induction. We first notice that (19) is true for $k = 1$. Furthermore, the assumption $(f2)$ and the fact that $w_k \geq 1$ lead us to $f(w_k) \leq \Lambda w_k$, for all $k \geq 1$. So, by (18),

$$w_{k+1}(r) \leq b + \Lambda \int_0^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) w_k(s) ds dt, \qquad r \geq 0.$$

Using now (5) (for $g(t) = \psi(t)w_k(t)$) we deduce

$$w_{k+1}(r) \leq b + \Lambda_N \int_0^r t\psi(t)w_k(t)dt, \qquad \forall\, r \in [0, R].$$

The induction hypothesis yields

$$w_{k+1}(r) \leq b + bM \int_0^r e^{Mt} dt = be^{Mr}, \qquad \forall\, r \in [0, R].$$

Hence, by induction, the sequence $\{w_k\}$ is bounded in $[0, R]$, for any $R > 0$. It follows that $w(r) = \lim_{k\to\infty} w_k(r)$ is a positive solution of (17). In a similar way we conclude that (16) has a positive solution on $[0, \infty)$.

The next step is to show that the constant $b$ may be chosen sufficiently large so that (15) holds. More exactly, if

$$b > 1 + K\Lambda_N \int_0^\infty sh(s)\Psi(s)ds, \tag{20}$$

where $K = \exp\left(\Lambda_N \int_0^\infty th(t)dt\right)$, then (15) occurs.

We first prove that the solution $v$ of (16) satisfies

$$v(r) \leq K\Psi(r), \qquad \forall\, r \geq 0. \tag{21}$$

Since $v \geq 1$, from $(f2)$ we have $f(v) \leq \Lambda v$. We use this fact in (16) and then we apply the estimate (5) for $g = \varphi$. It follows that

$$v(r) \leq 1 + \Lambda_N \int_0^r s\varphi(s)v(s)ds, \qquad \forall\, r \geq 0. \tag{22}$$

By Gronwall's inequality we obtain

$$v(r) \leq \exp\left(\Lambda_N \int_0^r s\varphi(s)ds\right), \qquad \forall\, r \geq 0,$$

and, by (22),

$$v(r) \leq 1 + \Lambda_N \int_0^r s\varphi(s) \exp\left(\Lambda_N \int_0^s t\varphi(t)dt\right) ds, \qquad \forall\, r \geq 0.$$

Hence

$$v(r) \le 1 + \int_0^r \left( \exp\left( \Lambda_N \int_0^s t\varphi(t)dt \right) \right)' ds, \qquad \forall\, r \ge 0,$$

that is

$$v(r) \le \exp\left( \Lambda_N \int_0^r t\varphi(t)dt \right), \qquad \forall\, r \ge 0. \tag{23}$$

Inserting $\varphi = h + \psi$ in (23) we have

$$v(r) \le e^{\Lambda_N \int_0^r th(t)dt} \Psi(r) \le K\Psi(r), \qquad \forall\, r \ge 0,$$

so (21) follows.

Since $b > 1$ it follows that $v(0) < w(0)$. Then there exists $R > 0$ such that $v(r) < w(r)$, for any $0 \le r \le R$. Set

$$R_\infty = \sup\{\ R > 0 \,|\, v(r) < w(r),\ \ \forall\, r \in [0, R]\,\}.$$

In order to conclude our proof, it remains to show that $R_\infty = \infty$. Suppose the contrary. Since $v \le w$ on $[0, R_\infty]$ and $\varphi = h + \psi$, from (16) we deduce

$$\begin{aligned} v(R_\infty) \quad &= 1 + \int_0^{R_\infty} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} h(s) f(v(s)) ds dt \\ &+ \int_0^{R_\infty} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(v(s)) ds dt. \end{aligned}$$

So, by (5),

$$v(R_\infty) \le 1 + \frac{1}{N-2} \int_0^{R_\infty} th(t) f(v(t)) dt + \int_0^{R_\infty} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w(s)) ds dt.$$

Taking into account that $v \ge 1$ and the assumption $(f2)$, it follows that

$$v(R_\infty) \le 1 + K\Lambda_N \int_0^{R_\infty} th(t)\Psi(t) dt + \int_0^{R_\infty} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w(s)) ds dt.$$

Now, using (20) we obtain

$$v(R_\infty) < b + \int_0^{R_\infty} e^{-t} t^{1-N} \int_0^t e^s s^{N-1} \psi(s) f(w(s)) ds dt = w(R_\infty).$$

Hence $v(R_\infty) < w(R_\infty)$. Therefore, there exists $R > R_\infty$ such that $v < w$ on $[0, R]$, which contradicts the maximality of $R_\infty$. This contradiction shows that inequality (15) holds and the proof of Lemma 1 is now complete. ■

**Proof of Theorem 2 completed.** Suppose that (3) holds. For all $k \geq 1$ we consider the problem

$$\begin{cases} \Delta u_k + |\nabla u_k| = p(x)f(u_k) & \text{in } B(0,k), \\ u_k(x) = w(k) & \text{on } \partial B(0,k). \end{cases} \tag{24}$$

Then $v$ and $w$ defined by (16) and (17) are positive sub and super–solutions of (24). So this problem has at least a positive solution $u_k$ and

$$v(|x|) \leq u_k(x) \leq w(|x|) \qquad \text{in } B(0,k), \text{ for all } k \geq 1.$$

By Theorem 14.3 in [11], the sequence $\{\nabla u_k\}$ is bounded on every compact set in $\mathbb{R}^N$. Hence the sequence $\{u_k\}$ is bounded and equicontinuous on compact subsets of $\mathbb{R}^N$. So, by the Arzela-Ascoli Theorem, the sequence $\{u_k\}$ has a uniform convergent subsequence, $\{u_k^1\}$ on the ball $B(0,1)$. Let $u^1 = \lim_{k\to\infty} u_k^1$. Then $\{f(u_k^1)\}$ converges uniformly to $f(u^1)$ on $B(0,1)$ and, by (24), the sequence $\{\Delta u_k^1 + |\nabla u_k^1|\}$ converges uniformly to $pf(u^1)$. Since the sum of Laplacian and Gradient is a closed operator, we deduce that $u^1$ satisfies (1) on $B(0,1)$.

Now, the sequence $\{u_k^1\}$ is bounded and equicontinuous on the ball $B(0,2)$, so it has a convergent subsequence $\{u_k^2\}$. Let $u^2 = \lim\limits_{k\to\infty} u_k^2$ on $B(0,2)$ and $u^2$ satisfies (1) on $B(0,2)$. Proceeding in the same way, we construct a sequence $\{u^n\}$ so that $u^n$ satisfies (1) on $B(0,n)$ and $u^{n+1} = u^n$ on $B(0,n)$ for all $n$. Moreover, the sequence $\{u^n\}$ converges in $L^\infty_{\text{loc}}(\mathbb{R}^N)$ to the function $u$ defined by

$$u(x) = u^m(x), \qquad \text{for } x \in B(0,m).$$

Since $v \leq u^n \leq w$ on $B(0,n)$ it follows that $v \leq u \leq w$ on $\mathbb{R}^N$, and $u$ satisfies (1). From $v \leq u$ we deduce that $u$ is a positive entire large solution of (1). This completes the proof. ∎

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