LIMIT LAWS FOR EMBEDDED TREES. APPLICATIONS TO THE INTEGRATED SUPERBROWNIAN EXCURSION

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Abstract. We study three families of labelled plane trees. In all these trees, the root is labelled 0, and the labels of two adjacent nodes differ by $0, 1$ or -1.

One part of the paper is devoted to enumerative results. For each family, and for all $j \in \mathbb{N}$, we obtain closed form expressions for the following three generating functions: the generating function of trees having no label larger than j; the (bivariate) generating function of trees, counted by the number of edges and the number of nodes labelled j; and finally the (bivariate) generating function of trees, counted by the number of edges and the number of nodes labelled *at least* j. Strangely enough, all these series turn out to be algebraic, but we have no combinatorial intuition for this algebraicity.

The other part of the paper is devoted to deriving limit laws from these enumerative results. In each of our families of trees, we endow the trees of size n with the uniform distribution, and study the following random variables: M_n, the largest label occurring in a (random) tree; $X_n(j)$, the number of nodes labelled j; and $X_n^+(j)$, the number of nodes labelled j or more. We obtain limit laws for scaled versions of these random variables.

Finally, we translate the above limit results into statements dealing with the integrated superBrownian excursion (ISE). In particular, we describe the law of the supremum of its support (thus recovering some earlier results obtained by Delmas), and the law of its distribution function at a given point. We also conjecture the law of its density (at a given point).

1. **Introduction**

We study in this paper three families of labelled plane trees. In all these trees, the root is labelled 0, and the labels of two adjacent nodes differ by $0, 1$ or -1.

More precisely, the first family we consider is the set of plane trees, and the increments of the labels along edges are constrained to be ± 1. In the closely related second family, these increments can be $0, \pm 1$. The third family is a bit different. It is simply the set of (incomplete) binary trees, in which the nodes are labelled in a deterministic way: the label of a node is the difference between the number of right steps and the number of left steps occurring in the path that yields from the root to the node under consideration. See Figure 1 for an illustration. We call this labelling the *natural labelling* of the binary tree. Note that the label of each node is simply its abscissa, if we draw the tree in the plane in such a way the right (resp. left) son of a node lies one unit to the right (resp. left) of its father. For this reason, we will sometimes call these labelled binary trees *naturally embedded binary trees.* More generally, for *any* plane labelled tree, we may consider that the label of each node tells where to embed it in $\mathbb{Z}$; hence the title of the paper.

In each of these three families, we endow the set of trees having a given size (say, n edges) with the uniform distribution. We address (via generating functions) the following three questions:

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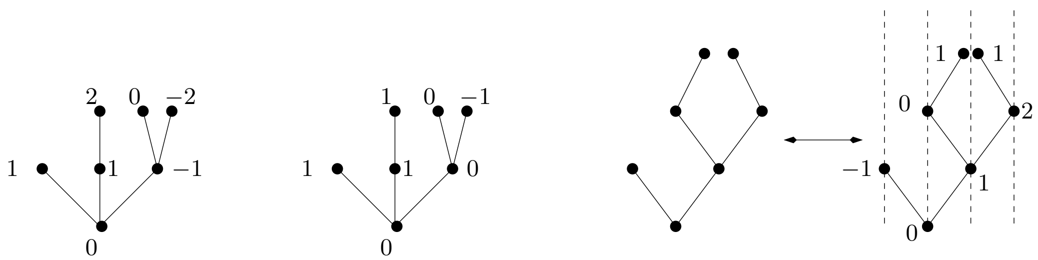

FIGURE 1. A labelled plane tree with increments ± 1. — A labelled tree with increments $0, \pm 1$. — A naturally embedded binary tree.

(1) What is the maximal label that occurs in the tree? This label is in fact a random variable M_n. We prove that $M_n/n^{1/4}$ converges in distribution to a random variable N having a density. We give this density explicitly. We also compute the moments of N and prove the convergence of the moments of $M_n/n^{1/4}$ to those of N.

(2) How many nodes of the tree have label j? Let $X_n(j)$ denote the corresponding random variable. If j is fixed, and n goes to infinity, then the answer to this question is independent of j. We prove that for any $j \in \mathbb{Z}$, the variable $X_n(j)/n^{3/4}$ converges in distribution to $cT^{-1/2}$, where c is a constant depending on which family of trees we consider, and T follows a unilateral stable law of parameter $2/3$.

Given that the maximal label grows like $n^{1/4}$, we get a better insight on the label distribution by asking how many nodes in a tree of size n have label $\lfloor \lambda n^{1/4} \rfloor$. We prove that, for any $\lambda \in \mathbb{R}$, the random variable $X_n(\lfloor \lambda n^{1/4} \rfloor)/n^{3/4}$ converges in distribution to a limit variable $Y(\lambda)$. This variable admits a Laplace transform, which we give explicitly. The convergence of the Laplace transform, and of the moments, hold as well. We say we have obtained a *local limit law* for embedded trees, because we look at *one* value of the labels only.

(3) Finally, we also obtain a *global limit law* by studying the variable $X_n^+(j)$ that gives the number of nodes having label j at least. Remarkably, we prove that $X_n^+(0)/n$, the (normalized) number of nodes having a non-negative label, converges to the uniform distribution on $[0, 1]$. More generally, for $\lambda \in \mathbb{R}$, the variable $X_n^+(\lambda n^{1/4})/n$ converges in distribution to a variable $Y^+(\lambda)$. This variable admits a Laplace transform, which we give explicitly. Once again, the convergence of the Laplace transform, and of the moments, hold as well.

The laws of N, $Y(\lambda)$ and $Y^+(\lambda)$ naturally depend on which family of trees we consider, but only by a simple normalization factor.

1.1. **Embedded trees and the integrated superBrownian excursion**

Why should one study such labelled trees?

The first two classes of trees we consider have a close connection with certain families of planar maps [6, 8, 11]. In particular, the diameter of a random quadrangulation having n faces is distributed like the largest label in *non-negative* random trees of our second family. Moreover, once scaled by $n^{1/4}$, this diameter has the same limit law as $(M_n - m_n)n^{-1/4}$, where M_n (resp. m_n) is the largest (resp. smallest) label occurring in a random tree of our second family [8].

The third class we study is the good old family of binary trees, and this may suffice to motivate its study! More seriously, the three questions addressed above have, for binary trees, a natural geometric formulation. The random variable M_n (the maximum label) tells us about the "true width" of a binary tree (as opposed to the maximal number of nodes lying at the same level, which is known to grow like $\sqrt{n}$). More generally, the variables $X_n(j)$ tell

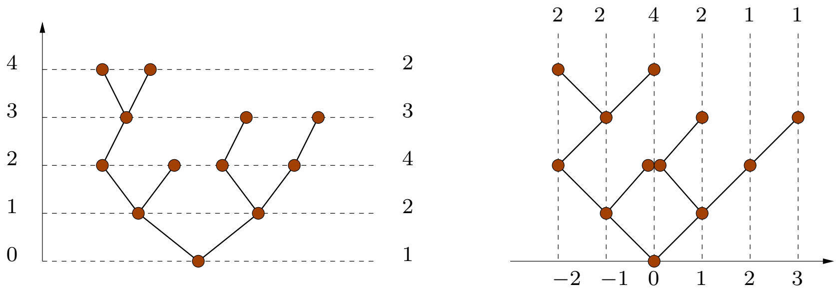

FIGURE 2. An (incomplete) binary tree having horizontal profile $[1, 2, 4, 3, 2]$ and vertical profile $[2, 2, 4; 2, 1, 1]$.

us about the *vertical profile* of the tree (as opposed to the *horizontal profile* which describes the repartition of nodes by level [13]). See Figure 2.

We may also invoke an *a posteriori* justification to the study of these trees: the form of the generating functions we obtain is remarkable, whatever family of trees we consider, and suggests that there must be some beautiful hidden combinatorics in these problems, which should be explored further.

However, the main motivation for this work is the connection between embedded trees and the *integrated superBrownian excursion* (ISE). Choose one of the three families of trees, and consider the following *random* probability distribution on $\mathbb{R}$:

$$\mu_n = \frac{1}{n+1} \sum_{j \in \mathbb{Z}} X_n(j) \delta_{cjn^{-1/4}}, \tag{1}$$

where $X_n(j)$ is the (random) number of nodes labelled j, δ_x denotes the Dirac measure at x, and the constant c equals $\sqrt{2}$ for the first family, $\sqrt{3}$ for the second one and 1 for the family of binary trees. Then μ_n is known to converge weakly to a limiting random probability distribution called the ISE [1, 23, 22, 20]. See Figure 3 for simulations of μ_n.

Our limit results provide some information about the law of the ISE. For instance, we prove that $cM_n n^{-1/4}$, the largest point having a positive weight under μ_n, converges in law

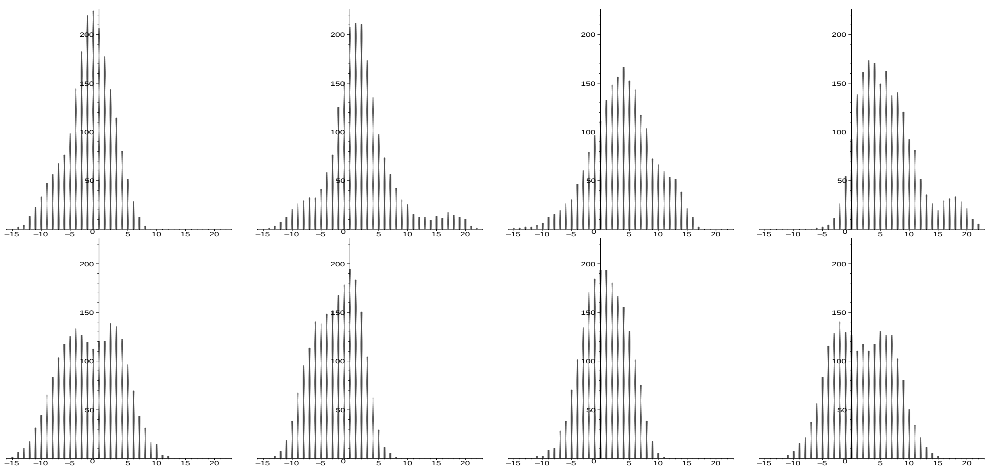

FIGURE 3. The plot of $X_n(j)$ vs. j for random binary trees with $n = 1000$ nodes.

to N_{ISE}, the supremum of the support of the ISE. We denote this by

$$cM_n n^{-1/4} \xrightarrow{\text{d}} N_{\text{ISE}}.$$

The results we obtain for the limit law of $M_n n^{-1/4}$ thus translate into expressions of the moments, distribution function and density of the supremum of the ISE. Note that the moments were already obtained by Delmas [12]. Our second limit result deals with the random variables $X_n(\lfloor \lambda n^{1/4} \rfloor)$. Observe that

$$\mu_n(c\lambda - cn^{-1/4}, c\lambda] = \frac{1}{n+1} X_n(\lfloor \lambda n^{1/4} \rfloor). \tag{2}$$

This leads us to *conjecture* that the random variable $Y(\lambda)$ involved in our local limit law satisfies

$$Y(\lambda) \stackrel{\text{d}}{=} cf_{\text{ISE}}(c\lambda) \tag{3}$$

where f_{ISE} is the (random) density of the ISE. Similarly,

$$\mu_n[c\lambda, +\infty) = \frac{1}{n+1} X_n^+(\lceil \lambda n^{1/4} \rceil),$$

and we *prove* that the random variable $Y^+(\lambda)$ involved in our global limit law satisfies

$$Y^+(\lambda) \stackrel{\text{d}}{=} g_{\text{ISE}}(c\lambda)$$

where g_{ISE} is the (random) tail distribution function of the ISE. The results we obtain about the laws of $Y(\lambda)$ and $Y^+(\lambda)$ thus translate into formulas for the Laplace transforms of $f_{\text{ISE}}(\lambda)$ and $g_{\text{ISE}}(\lambda)$ (the formula for $f_{\text{ISE}}(\lambda)$ being conjectural).

Our conjecture on f_{ISE} is naturally supported by the fact that the law of $Y(\lambda/c)/c$ is independent of the tree family we start from. This is one of the reasons why we consider as many as three families of trees. The other reasons involve the connections with planar maps, the remarkable form of the generating functions we obtain, and our unshakeable interest in binary trees. The details of the calculations are only given for the first of the three families (Sections 2 to 5), while the results are merely stated for the other two families (Section 6).

Let us finally mention that the moments of the *center of mass* of the ISE have recently been determined by two different approaches [7, 19]. In our discrete setting, this boils down to studying the convergence of the variable

$$\frac{1}{n^{5/4}} \sum_{j \in \mathbb{Z}} j X_n(j).$$

1.2. **Overview of the paper**

The starting point of our approach is a series of exact enumerative results dealing with our first class of trees: plane trees in which the labels of adjacent nodes differ by ± 1. These results are gathered in the next section. We obtain for instance an explicit expression for the bivariate generating function of labelled trees, counted by the number of edges and the number of nodes labelled j (for j fixed). This section includes, and owes a lot to, some results recently obtained by Bouttier, Di Francesco and Guitter [5, 6] on the enumeration of trees having no label greater than j. This part of our work raises a number of challenging combinatorial questions — why are these expressions so simple? — which are not addressed in this paper.

The limit behaviours of the random variables M_n, $X_n(\lfloor \lambda n^{1/4} \rfloor)$ and $X_n^+(\lambda n^{1/4})$ are respectively established in the next three sections (Sections 3 to 5). The main technique that we use is the "analysis of singularities" of Flajolet and Odlyzko [17]. It permits to extract the asymptotic behaviour of the coefficients of a generating function. This technique has already proved useful in numerous occasions, in particular for proving limit theorems that are similar in flavour to the ones obtained in this paper: these theorems deal with the height of simply generated trees and their profile, which are known to be related to the height of the Brownian excursion and its local time [16, 13]. This technique is carefully exemplified in Section 3 (which is devoted to the maximal label) before the more difficult questions of the local and global limit laws are attacked (Sections 4 and 5).

Finally, two other families of trees are briefly studied in Section 6: trees with increments $0, \pm 1$ and naturally embedded binary trees. The emphasis is put on their enumerative properties, which turn out to be as remarkable and surprising as those of our first family of trees. The limit laws we obtain are (up to a scalar) the same as for the first family.

Let us conclude with some notation and a few definitions on formal power series and generating functions. Let $\mathbb{K}$ be a field. We denote by $\mathbb{K}[t]$ the ring of polynomials in t with coefficients in $\mathbb{K}$, and by $\mathbb{K}(t)$ the field of rational functions in t with coefficients in $\mathbb{K}$. We denote by $\mathbb{K}[[t]]$ the ring of formal power series in t with coefficients in $\mathbb{K}$. If $A(t) \in \mathbb{K}[[t]]$ and $n \in \mathbb{N}$, the notation $[t^n]A(t)$ stands for the coefficient of t^n in $A(t)$. The series $A(t)$ is said to be *algebraic over* $\mathbb{K}(t)$ if it satisfies a non-trivial polynomial equation of the form $P(t, A(t)) = 0$, where P is a bivariate polynomial with coefficients in $\mathbb{K}$. In this case, the *degree* of $A(t)$ is the smallest possible degree of P (in its second variable).

Let $\mathcal{A}$ be a set of discrete objects, equipped with a *size* that takes nonnegative integer values. Assume that for all $n \in \mathbb{N}$, the number of objects of $\mathcal{A}$ of size n is finite, and denote this number by a_n. The *generating function of the objects of $\mathcal{A}$, counted by their size,* is the formal power series

$$A(t) = \sum_{n \geq 0} a_n t^n.$$

The above notions generalize in a straightforward way to multivariate power series. Such series arise naturally when enumerating objects according to several parameters.

2. Enumerative results

We consider in this section (and in the three following ones) our first family of labelled plane trees: the root is labelled 0, and the labels of two adjacent nodes differ by ± 1.

2.1. Trees with small labels

The first enumerative problem we address has already been studied by Bouttier, Di Francesco and Guitter [5, 6]. It deals with the largest label occurring in a tree. For $j \in \mathbb{N}$, let $T_j \equiv T_j(t)$ be the generating function of labelled trees in which all labels are less than or equal to j. The indeterminate t keeps track of the number of edges. Let $T \equiv T(t)$ be the generating function of all labelled trees. Clearly, T_j converges to T (in the space of formal power series in t) as j goes to infinity. It is very easy to describe an infinite set of equations that completely defines the collection of series T_j.

Lemma 1. *The series T satisfies*

$$T = 1 + 2tT^2. \tag{4}$$

More generally, for $j \geq 0$,

$$T_j = 1 + t(T_{j-1} + T_{j+1})T_j$$

while $T_j = 0$ for $j < 0$.

Proof. The two ingredients of the proof will be useful for the other enumerative problems we address below. Firstly, replacing each label k by $j - k$ shows that T_j is also the generating function of trees *rooted at* j and having only non-negative labels (we say that a tree is rooted at j if its root has label j). Secondly, consider such a tree and assume it is not reduced to a single node. The root has a leftmost child, which is the root of a labelled subtree, rooted at $j \pm 1$ and having only non-negative labels. Deleting this subtree leaves a smaller tree rooted at j, having only non-negative labels (see Figure 4). The result follows.

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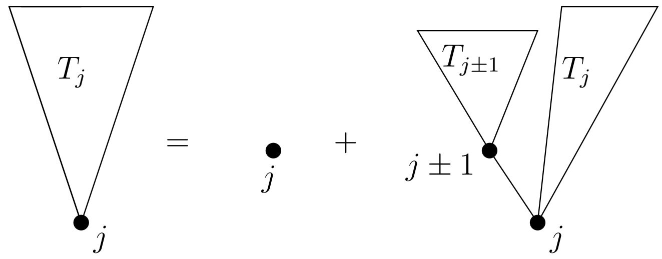

FIGURE 4. The decomposition of plane labelled trees.

The above lemma shows that the series T, counting labelled trees by edges, is algebraic, and the short proof we have given provides a simple combinatorial explanation for this property. What is far less clear — but nevertheless true — is that each of the series T_j is algebraic too, as stated in the proposition below, which we borrow from [5, 6]. These series will be expressed in terms of the series $T \equiv T(t)$ and of the unique formal power series $Z \equiv Z(t)$, with constant term 0, satisfying

$$Z = t\,\frac{(1+Z)^4}{1+Z^2}. \tag{5}$$

Observe that T and Z are related by:

$$T = \frac{(1+Z)^2}{1+Z^2}. \tag{6}$$

Proposition 2 (**Trees with small labels** [5, 6]**).** *Let $T_j \equiv T_j(t)$ be the generating function of trees having no label greater than j. Then T_j is algebraic of degree (at most) 2. In particular,*

$$T_0 = 1 - 11\,t - t^2 + 4\,t\,(3+2\,t)\,T_0 - 16\,t^2 {T_0}^2.$$

Moreover, for all $j \geq -1$,

$$T_j = T\,\frac{(1-Z^{j+1})(1-Z^{j+5})}{(1-Z^{j+2})(1-Z^{j+4})}, \tag{7}$$

where $Z \equiv Z(t)$ is given by (5)*.*

Proof. It is very easy to check, using (5–6), that the above values of T_j satisfy the recurrence relation of Lemma 1 and the initial condition $T_{-1} = 0$. How to *discover* such a formula is another story, which is told in [5]. The remarkable product form of T_j still awaits a combinatorial explanation.

The equation satisfied by T_0 is obtained by eliminating T and Z from the case $j = 0$ of (7). Then an induction of j, based on Lemma 1, implies that each T_j is quadratic (at most) over $\mathbb{Q}(t)$.

∎

Remarks

1. The product form (7), combined with the facts that T is quadratic over $\mathbb{Q}(t)$ and Z is quadratic over $\mathbb{Q}(T)$, shows that T_j belongs to an extension of $\mathbb{Q}(t)$ of degree 4. This is true, but not optimal, since T_j is actually quadratic over $\mathbb{Q}(t)$. Hence this product form does not give the best possible information on the degree of T_j.

2. The trees counted by T_0 (equivalently, the trees having only non-negative labels) are known to be in bijection with certain planar maps called *Eulerian triangulations* [6]. Through this bijection, the number of edges of the tree is sent to the number of black faces of the triangulation. These triangulations are nothing but the dual maps of the *bicubic* (that is, bipartite and trivalent) maps, which were first enumerated by Tutte [26]. In particular, the coefficients of $T_0(t)$ are remarkably simple:

$$T_0(t) = \frac{(1-8t)^{3/2} - 1 + 12t + 8t^2}{32t^2} = 1 + \sum_{n\geq 1} \frac{3.2^{n-1}}{(n+1)(n+2)} \binom{2n}{n} t^n.$$

2.2. **The number of nodes labelled j**

Let us now turn our attention to a bivariate counting problem. For $j \in \mathbb{Z}$, let $S_j \equiv S_j(t,u)$ be the generating function of labelled trees, counted by the number of edges (variable t) and the number of nodes labelled j (variable u). Clearly, $S_j(t,1) = T(t)$ for all j. Moreover, an obvious symmetry entails that $S_j = S_{-j}$.

Lemma 3. *For $j \neq 0$,*

$$S_j = 1 + t(S_{j-1} + S_{j+1})S_j \tag{8}$$

while for $j = 0$,

$$S_0 = u + t(S_{-1} + S_1)S_0 = u + 2tS_1S_0. \tag{9}$$

Proof. Observe that $S_j \equiv S_j(t,u)$ is also the generating function of labelled trees rooted at j, counted by the number of edges and the number of nodes labelled 0. The decomposition of trees illustrated in Figure 4 then provides the lemma. The only difference between the cases $j = 0$ and $j \neq 0$ lies in the generating function of the tree reduced to a single node.

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Again, the series $S_j(t,u)$ turn out to be algebraic, for reasons that currently remain mysterious (from the combinatorics viewpoint). They can be expressed in terms of the series T and Z given by (5–6).

Proposition 4 (**The number of nodes labelled** j)**.** *For any $j \in \mathbb{Z}$, the generating function $S_j \equiv S_j(t,u)$ that counts labelled trees by the number of edges and the number of nodes labelled j is algebraic over $\mathbb{Q}(T,u)$ of degree at most 3 (and hence has degree at most 6 over $\mathbb{Q}(t,u)$). More precisely,*

$$\frac{(T-S_0)^2}{(u-1)^2} = 1 - \frac{2(1-T^2)}{2+S_0-S_0T}, \tag{10}$$

and all the S_j belong to $\mathbb{Q}(t,u,S_0)$. Moreover, for all $j \geq 0$,

$$S_j = T\,\frac{(1+\mu Z^j)(1+\mu Z^{j+4})}{(1+\mu Z^{j+1})(1+\mu Z^{j+3})}, \tag{11}$$

where $Z \equiv Z(t)$ is given by (5) and $\mu \equiv \mu(t,u)$ is the unique formal power series in t satisfying

$$\mu = (u-1)\frac{(1+Z^2)(1+\mu Z)(1+\mu Z^2)(1+\mu Z^3)}{(1+Z)(1+Z+Z^2)(1-Z)^3(1-\mu Z^2)}. \tag{12}$$

The series $\mu(t,u)$ has polynomial coefficients in u, and satisfies $\mu(t,1) = 0$. It has degree 3 over $\mathbb{Q}(Z,u)$ and 12 over $\mathbb{Q}(t,u)$.

At some point, we will need a closed form expression for μ in terms of Z. Here is one.

Proposition 5. *Write*

$$v = \frac{(u-1)Z(1+Z^2)}{(1+Z)(1+Z+Z^2)(1-Z)^3}.$$

Then the algebraic series μ involved in the expression (11) of S_j, and defined by (12), is

$$\mu(t,u) = \frac{1}{Z^2}\left(\frac{2}{1+v(1-Z)^2/3+2/3\sqrt{3+v^2(1-Z)^4}\cos(\phi/3)} - 1\right)$$

where

$$\phi = \arccos\left(\frac{-9v(1+4Z+Z^2)+v^3(1-Z)^6}{(3+v^2(1-Z)^4)^{3/2}}\right).$$

Proof of Propositions 4 and 5.[1] First, observe that the family of series $S_0, S_1, S_2, \ldots$ is completely determined by (8) (taken for $j > 0$) and the second part of (9). The fact that for any series $\mu \in \mathbb{Q}(u)[[t]]$, the expression (11) satisfies (8) for all $j > 0$ is a straighforward verification, once t and T have been expressed in terms of Z (see (5) and (6)). The form of (11) is borrowed from [5]. In order for (11) to be the correct expression of S_j, it remains to satisfy the second part of (9). This last condition provides a polynomial equation relating μ, T, Z, t and u. In this equation, replace t and T by their expressions in terms of Z (given by (5–6)). This gives exactly (12). It can be easily checked that μ has degree 6 over $\mathbb{Q}(T, u)$ and degree 12 over $\mathbb{Q}(t, u)$.

The equation (10) satisfied by S_0 is obtained by eliminating μ and Z (using (12) and (6)) from the expression (11) of S_0. This equation gives an equation of degree 6 over $\mathbb{Q}(t, u)$ if one eliminates T thanks to (4).

Now the equations (9), (8) and (4), combined with an induction on j, imply that for $j \geq 1$, the series S_j belongs to the field $\mathbb{Q}(T, u, S_0)$, which has just been proved to be an extension of $\mathbb{Q}(T, u)$ of degree 3. This concludes the proof of Proposition 4.

Let us finally prove Proposition 5. The equation (12) that defines μ can be rewritten

$$\mu = \frac{v}{Z}\frac{(1+\mu Z)(1+\mu Z^2)(1+\mu Z^3)}{1-\mu Z^2}.$$

Hence μ is the unique formal power series in v (with rational coefficients in Z) that satisfies the above equation and equals 0 when v is 0. It is not hard to check that the closed form expression we give satisfies these two conditions.

■

Remarks

1. The product form (11) of Proposition 4 refines the product form (7) that deals with trees with small labels. Indeed, when $u = 0$, Eq. (12) gives $\mu = -1$, and the expression of $S_j(t, 0)$ coincides, as it should, with the expression of $T_{j-1}(t)$ given by Proposition 2.

2. There exists an alternative way to derive an equation for S_0 from the system of Lemma 3. As was observed in [6, p. 645] for the problem of counting trees with bounded labels, Eq. (8) implies that for $j \geq 1$,

$$I(S_{j-1}, S_j) = I(S_j, S_{j+1})$$

where the "invariant" function I is given by

$$I(x, y) = xy(1-tx)(1-ty) + txy - x - y.$$

But S_j converges to T as j goes to infinity, in the set of formal power series in t. This implies

$$I(S_0, S_1) = I(T, T).$$

Eliminating S_1 between the above equation and (9) gives an equation between S_0, T and t.

2.3. The number of nodes labelled j or more

Let us finally study our third and last enumeration problem. For $j \in \mathbb{Z}$, let $R_j \equiv R_j(t, u)$ be the generating function of labelled trees, counted by the number of edges (variable t) and the number of nodes labelled j at least (variable u).

[1] All the calculations in this paper have been done using Maple. We do not recommend the reader to check them by hand.

Lemma 6. *The set of series $R_0, R_1, R_2, \dots$ is completely determined by the following equations: for $j \geq 1$,*

$$R_j = 1 + tR_j(R_{j-1} + R_{j+1}) \tag{13}$$

and

$$R_0(t,u) = uR_1(tu, 1/u). \tag{14}$$

More generally, for all $j \in \mathbb{Z}$, one has:

$$R_{-j}(t,u) = uR_{j+1}(tu, 1/u). \tag{15}$$

Proof. For all $j \in \mathbb{Z}$, the series $R_j \equiv R_j(t,u)$ is also the generating function of trees rooted at j, counted by their number of edges and the number of nodes having a non-positive label. The equation satisfied by j, for $j \geq 1$, follows once again from the decomposition of trees illustrated in Figure 4. It remains to prove the symmetry relation (15). For any tree τ, let $n_{\leq 0}(\tau)$ denote the number of nodes of τ having a non-positive label. We use similar notations for the number of nodes having label at most j, etc. Let $\mathcal{T}_{j,n}$ denote the set of trees rooted at j and having n edges. As observed above,

$$R_{-j}(t,u) = \sum_{n\geq 0} t^n \sum_{\tau \in \mathcal{T}_{-j,n}} u^{n_{\leq 0}(\tau)} = \sum_{n \geq 0} t^n \sum_{\tau \in \mathcal{T}_{-j,n}} u^{n+1-n_{>0}(\tau)},$$

because a tree with n edges has a total of $n+1$ nodes. A translation of all labels by -1 gives

$$R_{-j}(t,u) = u \sum_{n \geq 0} (tu)^n \sum_{\tau \in \mathcal{T}_{-j-1,n}} u^{-n_{\geq 0}(\tau)},$$

while replacing each label k by $-k$ finally gives

$$R_{-j}(t,u) = u \sum_{n\geq 0} (tu)^n \sum_{\tau \in \mathcal{T}_{j+1,n}} u^{-n_{\leq 0}(\tau)} = uR_{j+1}(tu, 1/u).$$

∎

Again, the series R_j are algebraic, and admit a closed form expression in terms of T and Z.

Proposition 7 (The number of nodes labelled j or more). *Let $j \in \mathbb{Z}$. The generating function $R_j(t,u) \equiv R_j$ that counts labelled trees by the number of edges and the number of nodes labelled j or more is algebraic of degree at most 2 over $\mathbb{Q}(T(t), T(tu))$. Hence it has degree at most 8 over $\mathbb{Q}(t,u)$. More precisely, it belongs to the extension of $\mathbb{Q}(T(t), T(tu))$ generated by*

$$\sqrt{(T+\tilde{T})^2 - 4T\tilde{T}(T-1)(\tilde{T}-1)}$$

where $T \equiv T(t)$ and $\tilde{T} \equiv T(tu)$.
Moreover, for all $j \geq 0$,

$$R_j = T\,\frac{(1+\nu Z^j)(1+\nu Z^{j+4})}{(1+\nu Z^{j+1})(1+\nu Z^{j+3})}, \tag{16}$$

where $Z \equiv Z(t)$ is given by (5) and $\nu \equiv \nu(t,u)$ is a formal power series in t, with polynomial coefficients in u, which is algebraic of degree 4 over $\mathbb{Q}(u, Z)$, and of degree 16 over $\mathbb{Q}(t,u)$. This series satisfies $\nu(t,1) = 0$. The first terms in its expansion are:

$$\nu(t,u) = (u-1)\Big(1 + 2\,ut + \left(7\,u + 6\,u^2\right)t^2 + \left(32\,u + 36\,u^2 + 23\,u^3\right)t^3 + O(t^4)\Big).$$

Before we prove this proposition, let us give something like a closed form for ν. Since ν has degree 4 over $\mathbb{Q}(u,Z)$, and Z has degree 4 over $\mathbb{Q}(t)$, the series ν is in theory expressible in terms of radicals... It turns that this expression is less terrible than one could fear.

Proposition 8. *Define the following four formal power series in t with polynomial coefficients in u:*

$$\delta \equiv \delta(t,u) = 1 - 8(u-1)\frac{Z(1+Z^2)}{(1-Z)^4} = \frac{1-8tu}{1-8t},$$

$$V \equiv V(t,u) = \frac{1-\sqrt{\delta}}{4} = \frac{1-\sqrt{\frac{1-8tu}{1-8t}}}{4},$$

$$\Delta \equiv \Delta(t,u) = (1-V)^2 - \frac{4ZV^2}{(1+Z)^2},$$

and

$$P = (1+Z)\frac{1-V-\sqrt{\Delta}}{2VZ}.$$

Then P has degree 16 over $\mathbb{Q}(t,u)$, degree 2 over $\mathbb{Q}(V,Z)$, and satisfies the following "Lagrangian" equation:

$$P = \frac{V}{1+Z}(1+P)(1+ZP).$$

Moreover, the algebraic series ν involved in the expression (16) of R_j is

$$\nu = \frac{P}{Z}\frac{1-P(1+Z)-P^2(1+Z+Z^2)}{1+Z+Z^2+PZ(1+Z)-P^2Z^2}.$$

Proof of Proposition 7. We have already checked, in the proof of Proposition 4, that for any formal power series ν in t, the series defined by (16) for $j \geq 0$ satisfy the recurrence relation (13) for $j \geq 1$. It remains to prove that one can choose ν so as to satisfy (14). For any formal power series A in t having rational coefficients in u, we denote by $\tilde{A}$ the series $\tilde{A}(t,u) = A(tu, 1/u)$. Observe that $\tilde{\tilde{A}} = A$. With this notation, if R_j is of the generic form (16), the relation (14) holds if and only if

$$1+\nu = u\,\frac{\tilde{T}}{T}\,\frac{(1+\nu Z)(1+\nu Z^3)(1+\tilde{\nu}\tilde{Z})(1+\tilde{\nu}\tilde{Z}^5)}{(1+\nu Z^4)(1+\tilde{\nu}\tilde{Z}^2)(1+\tilde{\nu}\tilde{Z}^4)}. \tag{17}$$

Let $\mathbb{R}_m[u]$ denote the space of polynomials in u, with real coefficients, of degree at most m. Let $\mathbb{R}_n[u][[t]]$ denote the set of formal power series in t with polynomial coefficients in u such that for all $m \leq n$, the coefficient of t^m has degree at most m. Observe that this set of series in stable under the usual operations on series: sum, product, and quasi-inverse. Write $\nu = \sum_{n\geq 0} \nu_n(u)t^n$. We are going to prove, by induction on n, that (17) determines uniquely each coefficient $\nu_n(u)$, and that this coefficient belongs to $\mathbb{R}_{n+1}[u]$.

First, observe that for any formal power series ν, the right-hand side of (17) is $u + O(t)$. This implies $\nu_0(u) = u - 1$. Now assume that our induction hypothesis holds for all $m < n$. Recall that Z is a multiple of t: this implies that νZ belongs to $\mathbb{R}_n[u][[t]]$. The induction hypothesis also implies that the coefficient of t^m in $u\tilde{\nu}$ belongs to $\mathbb{R}_{m+1}[u]$, for all $m < n$. Note that $\tilde{Z} = Z(tu) = tu + O(t^2)$ is a multiple of t and u and also belongs to $\mathbb{R}_n[u][[t]]$. This implies that $\tilde{\nu}\tilde{Z}$ belongs to $\mathbb{R}_n[u][[t]]$ too. The same is true for all the other series occurring in the right-hand side of (17), namely $T, \tilde{T}, Z, \tilde{Z}$. Given the closure properties of the set $\mathbb{R}_n[u][[t]]$, we conclude that the right-hand side of (17), divided by u, belongs to this set. Moreover, the fact that Z and $\tilde{Z}$ are multiples of t guarantees that the coefficient of t^n in this series only involves the $\nu_i(u)$ for $i < n$. By extracting the coefficient of t^n in (17), we conclude that $\nu_n(u)$ is uniquely determined and belongs to $u\mathbb{R}_n[u] \subset \mathbb{R}_{n+1}[u]$.

This completes the proof of the existence and uniqueness of the series ν satisfying (17). Also, setting $u = 1$ (that is, $\tilde{T} = T$ and $\tilde{Z} = Z$) in this equation shows that $\nu(t,1) = 0$.

Let us now replace t by tu and u by $1/u$ in (17). This gives:

$$1+\tilde{\nu} = \frac{1}{u}\,\frac{T}{\tilde{T}}\,\frac{(1+\tilde{\nu}\tilde{Z})(1+\tilde{\nu}\tilde{Z}^3)(1+\nu Z)(1+\nu Z^5)}{(1+\tilde{\nu}\tilde{Z}^4)(1+\nu Z^2)(1+\nu Z^4)}. \tag{18}$$

In the above two equations, replace T by its expression (6) in terms of Z. Similarly, replace $\tilde{T}$ by its expression in terms of $\tilde{Z}$. Finally, it follows from (5) and from the fact that $\tilde{Z} = Z(tu)$ that

$$u = \frac{\tilde{Z}}{Z} \frac{(1+Z)^4(1+\tilde{Z}^2)}{(1+\tilde{Z})^4(1+Z^2)}. \tag{19}$$

Replace u by this expression in (17) and (18). Eliminate $\tilde{\nu}$ between the resulting two equations: this gives a polynomial equation that relates ν, Z and $\tilde{Z}$, of degree 2 in ν. The elimination of $\tilde{Z}$ between this quadratic equation and (19) provides an equation of degree 4 in ν that relates ν to Z and u. Finally, the elimination of Z shows that ν is algebraic of degree 16 over $\mathbb{Q}(t,u)$.

Let us now focus on the first part of the proposition. From the form (16), and the fact that ν has degree 4 over $\mathbb{Q}(u,Z)$ and Z has degree 4 over $\mathbb{Q}(t)$, we conclude that the degree of R_j over $\mathbb{Q}(t,u)$ is a divisor of 16. Let us prove that is is actually a divisor of 8. The proof goes as follows:

(1) Using the generic form (16), and the equations satisfied by T, Z and ν, we obtain a polynomial equation of degree 8 over $\mathbb{Q}(t,u)$ for R_0.

(2) Using (4) to express t in terms of T, and

$$u = \frac{T^2}{\tilde{T}^2} \frac{1-\tilde{T}}{1-T},$$

(which also follows from (4)), we convert the equation satisfied by R_0 into a polynomial equation (still of degree 8 in R_0) relating R_0 to T and $\tilde{T}$. This equation factors into four quadratic polynomials in R_0. The factor that actually vanishes is identified by setting $u = 1$ (in which case $\tilde{T} = T = R_0$).

(3) From this equation, we conclude that R_0 belongs to the extension of $\mathbb{Q}(T,\tilde{T})$ generated by

$$\sqrt{\Delta_1} = \sqrt{(T+\tilde{T})^2 - 4T\tilde{T}(T-1)(\tilde{T}-1)}.$$

Observe that this extension of $\mathbb{Q}(t,u)$ is left invariant by the transformation $A \mapsto \tilde{A}$.

(4) From the fact that $R_1 = u\tilde{R}_0$ (see (15)), we conclude that R_1 also belongs to $\mathbb{Q}(T,\tilde{T},\sqrt{\Delta_1})$.

(5) The recurrence relation (13) on the R_j allows us to extends this to all R_j, for $j \geq 0$.

(6) Finally, (15) shows that our algebraicity result actually holds for all R_j, for $j \in \mathbb{Z}$.

∎

Proof of Proposition 8. In the course of the proof of Proposition 7, we have obtained a polynomial equation $P(\nu, Z, u) = 0$, of degree 4 in ν, relating the series $\nu(t,u), Z(t)$, and the variable u. This equation is not written in the paper (it is a bit too big), but it follows from (17) and (18). In this equation, replace u by its expression in terms of δ and Z. Then replace δ by its expression in terms of V: the resulting equation factors into two terms! Each of them is quadratic in ν. In order to decide which of these factors cancels, one uses the fact that when $u = 1$ (that is, $V = 0$), the series ν must be 0. It remains to solve a quadratic equation in ν. Its discriminant is found to be Δ, and one may find convenient to introduce the series P which is Lagrangian in V.

∎

Remark. Again, the product form (16) of Proposition 7 includes as a special case the enumeration of trees with labels at most $j-1$, obtained when $u = 0$. Indeed, (17) shows that $\nu = -1$ when $u = 0$, and (16) then reduces to (7).

3. The largest label, and the support of the ISE

Let $\mathcal{T}_0$ denote the set of labelled trees (rooted at 0), and let $\mathcal{T}_{0,n}$ denote the subset of $\mathcal{T}_0$ formed by trees having n edges. We endow $\mathcal{T}_{0,n}$ with the uniform distribution. In other words, any of its elements occurs with probability

$$\frac{1}{2^n C_n}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, and is well-known to be the number of (unlabelled) plane trees with n edges.

Let M_n denote the random variable equal to the largest label occurring in a random tree of $\mathcal{T}_{0,n}$. The law of M_n is related to the series T_j studied in Proposition 2:

$$\mathbb{P}\left(M_n \le j\right) = \frac{[t^n]T_j}{2^n C_n}.$$

Let us define a normalized version of M_n by

$$N_n = \frac{M_n}{n^{1/4}}.$$

The aim of this section is to prove the convergence of N_n in distribution[2].

Theorem 9. *As n goes to infinity, the random variable N_n converges in distribution to a non-negative random variable N. The tail distribution function of N, defined by $G(\lambda) = \mathbb{P}(N > \lambda)$, satisfies*

$$G(\lambda) = \frac{12}{i\sqrt{\pi}} \int_\Gamma \frac{v^5 e^{v^4}}{\sinh^2(\lambda v)} dv = \frac{6}{\sqrt{\pi}\lambda^6} \int_0^\infty \frac{1-\cos u \cosh u}{(\cosh u - \cos u)^2} u^5 e^{-u^4/(4\lambda^4)} du$$

where the contour Γ is formed of two half-lines:

$$\Gamma = \{1 - te^{-i\pi/4}, t \in (\infty, 0]\} \cup \{1 + te^{-i\pi/4}, t \in [0, \infty)\}.$$

Equivalently, the variable N has density

$$f(\lambda) = \frac{24}{i\sqrt{\pi}} \int_\Gamma \frac{\cosh(\lambda v) v^6 e^{v^4}}{\sinh^3(\lambda v)} dv = \frac{6}{\sqrt{\pi}\lambda^{11}} \int_0^\infty \frac{1-\cos u \cosh u}{(\cosh u - \cos u)^2} u^5 (6\lambda^4 - u^4) e^{-u^4/(4\lambda^4)} du$$

with respect to the Lebesgue measure on $\mathbb{R}_+$. The moments of N are finite, and admit simple expressions:

$$\mathbb{E}(N) = \frac{3\sqrt{\pi}}{2\Gamma(3/4)}, \quad \mathbb{E}(N^2) = 3\sqrt{\pi},$$

and for $k \ge 3$,

$$\mathbb{E}(N^k) = \frac{24\sqrt{\pi} k! \zeta(k-1)}{2^k \Gamma((k-2)/4)}.$$

Finally, the moments of $N_n = M_n/n^{1/4}$ converge to the moments of N.

The functions G and f are plotted in Figure 5.
The proof of this theorem will be split into four subsections (Sections 3.1 to 3.4). In view of the following proposition, this theorem gives the density, distribution function and moments of the supremum of the support of the ISE.

[2]The above convention will be used throughout the paper: if a random variable depending on n is denoted by some letter of the alphabet, then its suitably normalized version is denoted by *the next letter* of the alphabet.

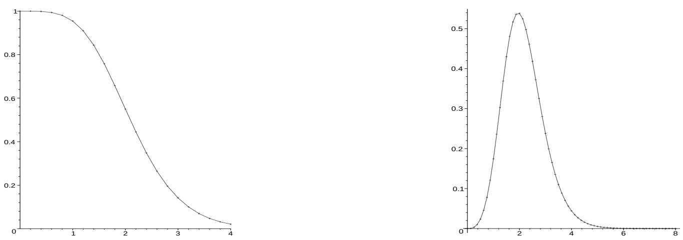

FIGURE 5. The tail distribution function G and the density f of the limit distribution N.

Proposition 10 (The supremum of the support of the ISE). *Let N_{ISE} denote the supremum of the support of the ISE*

$$N_{\text{ISE}} = \sup\{y : \mu_{\text{ISE}}(y, \infty) > 0\}.$$

Then N_{ISE} has the same law as the random variable $\sqrt{2}N$ described in Theorem 9.

Remark. The moments of N_{ISE} are thus

$$\mathbb{E}(N_{\text{ISE}}) = \frac{3\sqrt{\pi}}{\sqrt{2}\Gamma(3/4)}, \quad \mathbb{E}(N_{\text{ISE}}^2) = 6\sqrt{\pi},$$

and for $k \geq 3$,

$$\mathbb{E}(N_{\text{ISE}}^k) = \frac{24\sqrt{\pi}k!\zeta(k-1)}{\sqrt{2}^k\Gamma((k-2)/4)}.$$

They were already obtained by Delmas [12] using a completely different (and continuous) approach. The expressions he gives actually differ from ours by a factor $2^{k/4}$, due to a different choice of normalization. Note that the zeta function also appears in the moments of the maximum of the Brownian excursion, which follows a theta law [10]. This law is known to describe the limiting normalized height of simple trees [16]. Finally, let us mention that another, more complicated expression of the density of the limiting variable N was obtained in [5] (maybe in a slightly less rigorous way). Proposition 10 is proved in Section 3.5.

3.1. Convergence of the distribution function

We prove in this section that the tail distribution function of N_n converges pointwise. Let $\lambda \geq 0$ and $j = \lfloor \lambda n^{1/4} \rfloor$. The probability we are interested in is

$$\mathbb{P}(N_n > \lambda) = \mathbb{P}(M_n > \lambda n^{1/4}) = \mathbb{P}(M_n > j) = \frac{[t^n]U_j(t)}{2^n C_n}, \tag{20}$$

where

$$U_j(t) \equiv U_j = T - T_j = \frac{(1+Z)^2 Z^{j+1} \left(1 + Z + Z^2\right)(1-Z)^2}{(1+Z^2)(1-Z^{j+2})(1-Z^{j+4})} \tag{21}$$

is the generating function of trees having at least one label greater than j. This algebraic series has a positive radius of convergence[3], and by Cauchy's formula,

$$\begin{aligned} [t^n]U_j &= \frac{1}{2i\pi}\int_{\mathcal{C}} U_j(t)\frac{dt}{t^{n+1}} \\ &= \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{(1+Z)^2 Z^{j+1}\left(1+Z+Z^2\right)(1-Z)^2}{(1+Z^2)(1-Z^{j+2})(1-Z^{j+4})}\frac{dt}{t^{n+1}}, \end{aligned} \tag{22}$$

for any contour $\mathcal{C}$ included in the analyticity domain of U_j and enclosing positively the origin.

This leads us to study the singularities of U_j, and therefore those of Z. We gather in the following lemma a few properties of this series.

[3]So do all algebraic power series

Lemma 11 (**Analytic properties of** Z). *Let $Z \equiv Z(t)$ be the unique formal power series in t with constant term 0 satisfying (5). This series has non-negative integer coefficients. It has radius of convergence $1/8$, and can be continued analytically on the domain $\mathcal{D} = \mathbb{C} \setminus [1/8, +\infty)$. In the neighborhood of $t = 1/8$, one has*

$$Z(t) = 1 - 2(1-8t)^{1/4} + O(\sqrt{1-8t}). \tag{23}$$

Moreover, $|Z(t)| < 1$ on the domain $\mathcal{D}$. More precisely, the only roots of unity that are accumulation points of the set $Z(\mathcal{D})$ are 1 and -1, and they are only approached by $Z(t)$ when t tends to $1/8$ and when $|t|$ tends to ∞, respectively.

Proof. In order to establish the first statement, we observe that

$$Z = W(1+Z)^2$$

where $W \equiv W(t)$ is the only formal power series in t with constant term zero satisfying

$$W = t + 2W^2. \tag{24}$$

These equations imply that both W and Z have non-negative integer coefficients.

The general approach for studying the singularities of algebraic series (see for instance [18]) gives the second part of the lemma (up to (23)). The polynomial equation defining $Z(t)$ has leading coefficient t and discriminant $4(1-8t)^3$, so that the only possible singularity of Z is $1/8$. Alternatively, one can exploit the following closed form expression:

$$Z(t) = \frac{\sqrt{1-4t+\sqrt{1-8t}}\left(\sqrt{1-4t+\sqrt{1-8t}} - \sqrt{2}(1-8t)^{1/4}\right)}{4t}. \tag{25}$$

Let us now come to the third part of the lemma, and prove that $|Z(t)|$ never reaches 1 on the domain $\mathcal{D}$. Assume $Z(t) = e^{i\theta}$, with $\theta \in [-\pi, \pi]$. From (5), one has

$$t = t_\theta \quad \text{where} \quad t_\theta = \frac{\cos\theta}{8\cos^4(\theta/2)} \quad \text{and} \quad \theta \in (-\pi, \pi).$$

This shows that t is real, and belongs to $(-\infty, 1/8)$. But the expression (25) of $Z(t)$ shows that $Z(t)$ is real, which contradicts the hypothesis $Z(t) = e^{i\theta}$, unless $\theta = 0$. But then $t = 1/8$ and does not belong to the domain $\mathcal{D}$. Hence the modulus of Z never reaches 1 on $\mathcal{D}$. One can actually prove that, for $\theta \in (-\pi, 0)$,

$$Z(t_\theta) = \frac{1+\sin\theta}{\cos\theta},$$

but we do not need so much precision here.

Finally, if a sequence t_n of $\mathcal{D}$ is such that $Z(t_n) \to e^{i\theta}$ as $n \to \infty$, with $\theta \in (-\pi, \pi]$, then either $\theta = \pi$ and, by (5), the sequence $|t_n|$ tends to ∞, or $\theta \in (-\pi, \pi)$ and t_n converges to t_θ. But then by continuity, $Z(t_n)$ actually converges to $Z(t_\theta)$, which, as argued above, only coincides with $e^{i\theta}$ when $\theta = 0$, that is, $t_\theta = 1/8$. In this case, $Z(t_n) \to 1$.

■

Let us now go back to the evaluation of the tail distribution function of N_n via the integral (22). We choose a contour $\mathcal{C} = \mathcal{C}_n$ that depends on n and consists of two parts $\mathcal{C}_n^{(1)}$ and $\mathcal{C}_n^{(2)}$ (see Figure 6):

- $\mathcal{C}_n^{(1)}$ is an arc of radius $r_n/8 = (1 + \log^2 n/n)/8$, centered at the origin; note that its radius tends to $1/8$ as n goes to infinity,
- $\mathcal{C}_n^{(2)}$ is a Hankel contour around $1/8$, at distance $1/(8n)$ of the real axis, which meets $\mathcal{C}_n^{(1)}$ at both ends; this contour shrinks around $1/8$ as n goes to infinity; more precisely, as t runs along $\mathcal{C}_n^{(2)}$, the variable z defined by

$$t = \frac{1}{8}\left(1 + \frac{z}{n}\right)$$

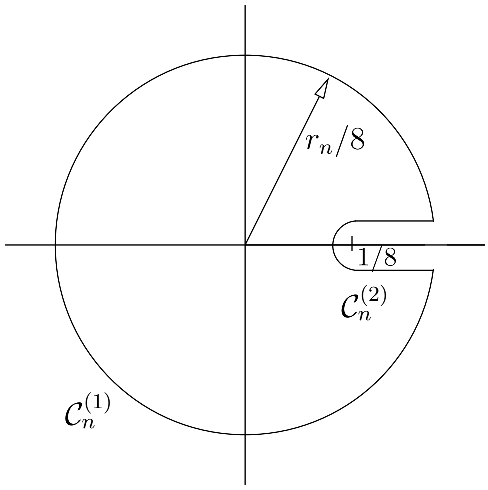

FIGURE 6. The integration contour $\mathcal{C}_n$.

runs over the truncated Hankel contour $\mathcal{H}_n$ shown on the right of Figure 7:

$$\mathcal{H}_n = \{x - i,\ x \in [0, x_n]\} \cup \left\{-e^{i\theta},\ \theta \in [-\pi/2, \pi/2]\right\} \cup \{x + i,\ x \in [0, x_n]\}$$

where $(1 + x_n/n)^2 + 1/n^2 = r_n^2$, so that $x_n \leq \log^2 n$ and $x_n = \log^2 n + O(1/n)$.

We denote by $z_n = x_n + i$ the top right end of $\mathcal{H}_n$. This point tends to infinity as n does.

The integral (22) on $\mathcal{C} = \mathcal{C}_n$ is the sum of the contributions of the contours $\mathcal{C}_n^{(1)}$ and $\mathcal{C}_n^{(2)}$. We shall see that the dominant contribution is that of $\mathcal{C}_n^{(2)}$, because of the vicinity of the singularity at $t = 1/8$.

Let us first bound carefully $Z(t)$ for $t \in \mathcal{C}_n$. Let $t_n \in \mathcal{C}_n$ be such that

$$|Z(t_n)| = \max_{t \in \mathcal{C}_n} |Z(t)|.$$

By Lemma 11, $|Z(t_n)|$ tends to 1 as n grows. Moreover, every accumulation point a of the sequence t_n satisfies $|a| \leq 1/8$ and $|Z(a)| = 1$. This forces $a = 1/8$, and we conclude that $t_n \to 1/8$. Write $t_n = (1 - u_n)/8$. Then $u_n \to 0$, but $|u_n| \geq 1/n$. By (23),

$$Z(t_n) = 1 - 2u_n^{1/4}\left(1 + o(1)\right).$$

Let us write, for short, $v_n = 1 - Z(t_n)$. Then $v_n \to 0$ but

$$|v_n| = 2|u_n|^{1/4}\left(1 + o(1)\right) \geq n^{-1/4} \tag{26}$$

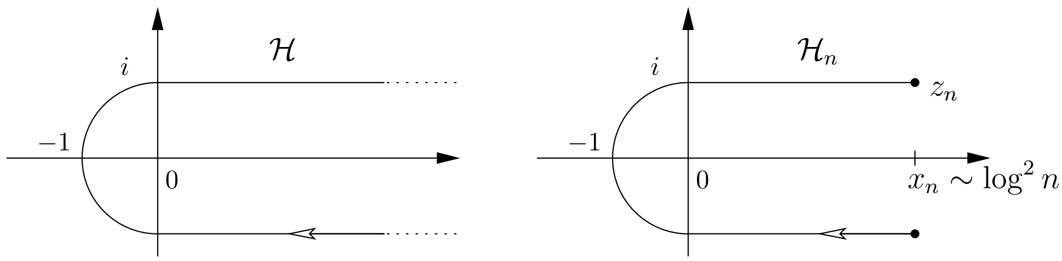

FIGURE 7. The Hankel contour $\mathcal{H}$ and its truncated version $\mathcal{H}_n$.

for n large enough. Moreover,

$$|\arg v_n| = \frac{1}{4}|\arg(u_n)| + o(1) \leq \frac{\pi}{4} + o(1),$$

so that

$$\cos(\arg v_n) \geq \frac{1}{\sqrt{2}} + o(1).$$

Finally,

$$|Z(t_n)|^2 = |1 - v_n|^2 = 1 - 2|v_n|\cos(\arg v_n) + |v_n|^2 \leq 1 - \sqrt{2}|v_n|\,(1 + o(1)),$$

that is,

$$|Z(t_n)| \leq 1 - \frac{1}{\sqrt{2}}|v_n|\,(1 + o(1)) \leq 1 - \frac{1}{2}n^{-1/4}.$$

The latter inequality follows from (26), and holds for n large enough. Finally, for $t \in \mathcal{C}_n$,

$$1 - |Z(t)| \geq \frac{1}{2}n^{-1/4}. \tag{27}$$

Let us now consider the integral on the contour $\mathcal{C}_n^{(1)}$. By Lemma 11, the quantity

$$\frac{(1+Z)^2\, Z^{j+1}\left(1 + Z + Z^2\right)(1 - Z)^2}{1 + Z^2}$$

is uniformly bounded on this contour by some constant c, independant of n and t. Moreover,

$$|1 - Z^{j+2}| \geq 1 - |Z|^{j+2} \geq 1 - |Z| \geq \frac{1}{2}n^{-1/4}$$

by (27). The same bound holds for the term $1 - Z^{j+4}$. Therefore the modulus of the contribution of $\mathcal{C}_n^{(1)}$ in the integral (22) is bounded by

$$4c\, 8^n n^{1/2}\, r_n^{-n} = O(8^n n^{1/2 - \log n}) = o(8^n/n^m) \tag{28}$$

for any $m > 0$.

Let us now study the contribution of the contour $\mathcal{C}_n^{(2)}$. As t varies along $\mathcal{C}_n^{(2)}$, the variable z defined by $t = (1 + z/n)/8$ varies along the contour $\mathcal{H}_n$. As n goes to infinity, this contour converges to the contour $\mathcal{H}$ shown on the left side of Figure 7. Let $z \in \mathcal{H}$. Then $z \in \mathcal{H}_n$ for n large enough, $|z| \leq |z_n| \sim \log^2 n$, and, as n goes to infinity, the following approximations hold with error terms independent of z:

$$\left\{\begin{array}{rcl} Z(t) &=& 1 - 2(-z)^{1/4}n^{-1/4} + O\left(n^{-1/2}\log n\right) \\ 1 - Z(t) &=& 2(-z)^{1/4}n^{-1/4}\left(1 + O(n^{-1/4}\sqrt{\log n})\right) \\ Z(t)^j &=& \exp(-2\lambda(-z)^{1/4})\left(1 + O(n^{-1/4}\log n)\right) \quad (\text{recall } j = \lfloor \lambda n^{1/4} \rfloor) \\ t^{-n-1} &=& 8^{n+1}e^{-z}\left(1 + O(\log^4 n/n)\right). \end{array}\right. \tag{29}$$

Observe that, for $z \in \mathcal{H}$, the real part of $(-z)^{1/4}$ is bounded from below by a positive constant α. Hence

$$|\exp(-2\lambda(-z)^{1/4})| = \exp(-2\lambda\Re(-z)^{1/4}) \leq \exp(-2\lambda\alpha),$$

so that $\exp(-2\lambda(-z)^{1/4})$ does not approach 1. This allows us to write

$$\frac{1}{1 - Z^{j+2}} = \frac{1}{1 - \exp(-2\lambda(-z)^{1/4})}\left(1 + O(n^{-1/4}\log n)\right).$$

Hence, uniformly in $t \in \mathcal{C}_n^{(2)}$, we have

$$\begin{aligned} U_j(t)t^{-n-1} &= \frac{(1+Z)^2 Z^{j+1}\left(1+Z+Z^2\right)(1-Z)^2}{(1+Z^2)(1-Z^{j+2})(1-Z^{j+4})} t^{-n-1} \\ &= \frac{6.8^{n+1}}{n^{1/2}} \frac{\sqrt{-z}e^{-z}}{\sinh^2(\lambda(-z)^{1/4})}(1+O(n^{-1/4}\log n)) \end{aligned}$$

with $8t = 1 + z/n$. Let us now integrate this over $\mathcal{C}_n^{(2)}$:

$$\begin{aligned} \int_{\mathcal{C}_n^{(2)}} U_j(t)\frac{dt}{t^{n+1}} &= \frac{6.8^n}{n^{3/2}} \int_{\mathcal{H}_n} \frac{\sqrt{-z}e^{-z}(1+O(n^{-1/4}\log n))}{\sinh^2(\lambda(-z)^{1/4})} dz \\ &= \frac{6.8^n}{n^{3/2}} \left(\int_{\mathcal{H}} \frac{\sqrt{-z}e^{-z}}{\sinh^2(\lambda(-z)^{1/4})} dz + o(1) \right). \end{aligned}$$

We now put together our estimates of the integrals on $\mathcal{C}_n^{(1)}$ (Eq. (28)) and $\mathcal{C}_n^{(2)}$ and obtain

$$[t^n]U_j(t) = \frac{6.8^n n^{-3/2}}{2i\pi} \left(\int_{\mathcal{H}} \frac{\sqrt{-z}e^{-z}}{\sinh^2(\lambda(-z)^{1/4})} dz + o(1) \right).$$

Using (20) and the estimation $C_n \sim 4^n n^{-3/2}/\sqrt{\pi}$, this gives

$$\mathbb{P}(N_n > \lambda) \to \frac{3}{i\sqrt{\pi}} \int_{\mathcal{H}} \frac{\sqrt{-z}e^{-z}}{\sinh^2(\lambda(-z)^{1/4})} dz.$$

The next step in our proof of Theorem 9 is to set $v = (-z)^{1/4}$ in the above integral. As z runs on $\mathcal{H}$, the variable v runs on the contour $\mathcal{J}$ of Figure 8, and the corresponding integral is easily seen to coincide with the integral on the contour Γ defined in the statement of the theorem. This gives the first expression of $G(\lambda)$.

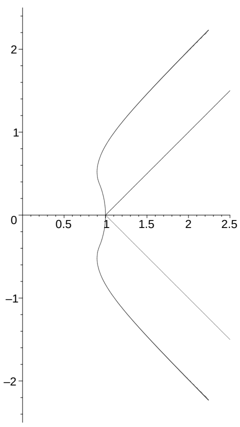

FIGURE 8. The contours Γ (two half lines) and $\mathcal{J}$.

We now want to express $G(\lambda)$ as a real integral. We first observe that the integration contour Γ can be replaced by its translated version

$$\Gamma_0 = \{-re^{-i\pi/4}, r \in (\infty, 0]\} \cup \{re^{i\pi/4}, r \in [0, \infty)\}.$$

This parametrization of Γ_0 by r splits the integral into two real integrals, and one finds:

$$\begin{aligned} G(\lambda) &= -\frac{12}{\sqrt{\pi}} \int_0^\infty \left(\frac{1}{\sinh^2(\lambda r e^{i\pi/4})} + \frac{1}{\sinh^2(\lambda r e^{-i\pi/4})} \right) r^5 e^{-r^4} dr \\ &= \frac{48}{\sqrt{\pi}} \int_0^\infty \frac{1-\cos(\sqrt{2}\lambda r)\cosh(\sqrt{2}\lambda r)}{(\cosh(\sqrt{2}\lambda r)-\cos(\sqrt{2}\lambda r))^2} r^5 e^{-r^4} dr. \end{aligned}$$

The expected expression of $G(\lambda)$ follows, upon setting $u=\sqrt{2}\lambda r$.

3.2. **The limit law and its density**

We now want to prove that $G(\lambda)$ is the tail distribution function of a random variable. Since it is the limit of non-increasing functions, it is non-increasing. Its integral expressions show that it is a continuous, and even a differentiable function of λ on $(0,+\infty)$. In order to conclude, we still need to prove that [3, Thm. 14.1]

$$\lim_{\lambda\to\infty} G(\lambda)=0 \text{ and } \lim_{\lambda\to 0} G(\lambda)=1.$$

In order to prove the first statement, we use the second expression of $G(\lambda)$ given in the theorem. We note that the function

$$u\mapsto \frac{1-\cos u\cosh u}{(\cosh u-\cos u)^2}$$

is well-defined, bounded and continuous on $[0,+\infty)$. Moreover, as u goes to infinity,

$$\left|\frac{1-\cos u\cosh u}{(\cosh u-\cos u)^2}\right| = O(e^{-u}),$$

so that the integral

$$\int_0^\infty \left|\frac{1-\cos u\cosh u}{(\cosh u-\cos u)^2}\right| u^5du$$

is convergent. The term $1/\lambda^6$ in the expression of $G(\lambda)$ then implies the convergence of $G(\lambda)$ to 0 as $\lambda\to\infty$.

In order to study the limit of $G(\lambda)$ as $\lambda\to 0^+$, we consider instead the first expression of $G(\lambda)$. Since $x^2/\sinh^2(x)$ is analytic in the disk of radius π, with expansion $1-x^2/3+O(x^4)$, there exists a constant c such that for $|v|\le \pi/(2\lambda)$,

$$\left|\frac{1}{\sinh^2(\lambda v)}-\frac{1}{\lambda^2v^2}+\frac{1}{3}\right| \le c\lambda^2|v|^2. \tag{30}$$

Let us write

$$\int_\Gamma \frac{v^5e^{v^4}}{\sinh^2(\lambda v)}dv = \int_\Gamma \left(\frac{1}{\sinh^2(\lambda v)}-\frac{1}{\lambda^2v^2}+\frac{1}{3}\right)v^5e^{v^4}dv + \int_\Gamma\left(\frac{v^3}{\lambda^2}-\frac{v^5}{3}\right)e^{v^4}dv.$$

Recall the Hankel expression of the reciprocal of the Gamma function, valid for any $s\in\mathbb{C}$:

$$\frac{1}{\Gamma(s)}=\frac{1}{2i\pi}\int_{\mathcal{H}}(-z)^{-s}e^{-z}dz=\frac{2}{i\pi}\int_\Gamma v^{3-4s}e^{v^4}dv. \tag{31}$$

Consequently,

$$\int_\Gamma v^3e^{v^4}dv=\frac{i\pi}{2\Gamma(0)}=0, \qquad \int_\Gamma v^5e^{v^4}dv=\frac{i\pi}{2\Gamma(-1/2)}=-\frac{i\sqrt{\pi}}{4},$$

and we can rewrite

$$G(\lambda)=\frac{12}{i\sqrt{\pi}}\int_\Gamma \frac{v^5e^{v^4}}{\sinh^2(\lambda v)}dv = 1+\frac{12}{i\sqrt{\pi}}\int_\Gamma\left(\frac{1}{\sinh^2(\lambda v)}-\frac{1}{\lambda^2v^2}+\frac{1}{3}\right)v^5e^{v^4}dv.$$

Let us cut the above integral into two parts, $|v|\le\pi/(2\lambda)$ and $|v|>\pi/(2\lambda)$. The first part is easily seen to tend to 0 as λ does, thanks to (30). For the second part, we observe that for $\lambda|v|>\pi/2$,

$$\left|\frac{1}{\sinh^2(\lambda v)}-\frac{1}{\lambda^2v^2}+\frac{1}{3}\right|$$

is bounded (by a constant independent of λ and v), that the integral of $v^5e^{v^4}$ on Γ is absolutely convergent, and that the contour $\{v\in\Gamma : |v|>\pi/(2\lambda)\}$ "shrinks to ∞" as $\lambda\to 0$. We finally conclude that $G(\lambda)$ tends to 1 as $\lambda\to 0$.

Consequently, there exists a random variable N having distribution function $1 - G(\lambda)$, and N_n converges in law to N. Since G is differentiable, N has a density with respect to the Lebesgue measure on $\mathbb{R}_+$, which is $f(\lambda) = -G'(\lambda)$. The two expressions of G given in the theorem provide the two expressions of f.

3.3. **The moments of N**

Let us first prove that for all $k \geq 0$, the tail distribution function of N satisfies

$$G(\lambda) = o(\lambda^{-k}) \text{ as } \lambda \to \infty. \tag{32}$$

This is easily seen to imply the existence of moments of N of all orders. In order to prove the above bound, we write

$$G(\lambda) = \frac{24}{i\lambda\sqrt{\pi}} \int_\Gamma \frac{v^4(5+4v^4)e^{v^4}}{e^{2\lambda v} - 1} dv.$$

This is obtained from the first expression of $G(\lambda)$ using an integration by parts. Now, for $\lambda > 0$ and $v \in \Gamma$,

$$|e^{2\lambda v} - 1| \geq |e^{2\lambda v}| - 1 = e^{2\lambda \Re(v)} - 1 \geq e^{2\lambda} - 1.$$

From this, and from the term e^{v^4} in the integral, we conclude that there exists a constant c such that

$$G(\lambda) \leq \frac{c}{e^{2\lambda} - 1}.$$

The bound (32) follows. This bounds also guarantees that for $k \geq 1$,

$$\mathbb{E}(N^k) = k \int_0^\infty \lambda^{k-1} G(\lambda) d\lambda. \tag{33}$$

The generic case: $k \geq 3$. Recall the following integral representations of the Riemann zeta function: for $\Re(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{w^{s-1}}{e^w - 1} dw = \frac{1}{4\Gamma(s+1)} \int_0^\infty \frac{w^s}{\sinh^2(w/2)} dw = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^\infty \frac{y^s}{\sinh^2(y)} dy.$$

The second expression follows from the first one after an integration by parts.

Let us now combine (33) with the first expression of $G(\lambda)$:

$$\mathbb{E}(N^k) = \frac{12k}{i\sqrt{\pi}} \int_0^\infty \lambda^{k-1} d\lambda \int_\Gamma \frac{v^5 e^{v^4}}{\sinh^2(\lambda v)} dv. \tag{34}$$

Assume for the moment that we can exchange the order of integration (this will be justified later). Exchange the integrals, and replace the variable λ by y/v, where y is a new variable:

$$\mathbb{E}(N^k) = \frac{12k}{i\sqrt{\pi}} \int_\Gamma v^{5-k} e^{v^4} dv \int_{v\mathbb{R}_+} \frac{y^{k-1}}{\sinh^2(y)} dy.$$

For $k \geq 3$, the function $y \mapsto y^{k-1}/\sinh^2(y)$ is meromorphic on $\mathbb{C}$, with poles at $ik\pi$ for $k \in \mathbb{Z}$ and $k \neq 0$. From this, and from the strong decay of this function as $\Re(y) \to \infty$, it follows that the integral on y is actually independent of the choice of $v \in \Gamma$. In particular, it is equal to its value at $v = 1$, which is

$$\int_0^\infty \frac{y^{k-1}}{\sinh^2(y)} dy = \frac{4\Gamma(k)\zeta(k-1)}{2^k},$$

as recalled above. The integral on v is then evaluated in terms of the Gamma function using (31), and the expected expression of $\mathbb{E}(N^k)$ follows.

It remains to justify the exchange of integrals in (34). Observe that

$$|\sinh(y)| = |e^y - e^{-y}|/2 \geq \left(|e^y| - |e^{-y}|\right)/2 = \sinh(\Re(y)),$$

so that for $v \in \Gamma$,

$$\frac{1}{|\sinh^2(\lambda v)|} \le \frac{1}{\sinh^2(\lambda)}.$$

Moreover, the integral of $v^5 e^{v^4}$ along Γ is absolutely convergent, and so is the integral of $\lambda^{k-1}/\sinh^2(\lambda)$ over R_+. It follows that the integral (34), once converted into two real integrals, is absolutely convergent, so that the integrals can be exchanged.

The case $k = 1$. We cannot apply exactly the same procedure as above, because the integral of $1/\sinh^2(\lambda)$ over $\mathbb{R}_+$ is divergent. However, in view of (31), we can write

$$G(\lambda) = \frac{12}{i\sqrt{\pi}} \int_\Gamma v^5 e^{v^4} \left(\frac{1}{\sinh^2(\lambda v)} - \frac{1}{\lambda^2 v^2} \right).$$

Also, replacing Γ by Γ_0 in the latter integral does not change its value. The technique is then the same as above:

$$\begin{aligned} \mathbb{E}(N) &= \frac{12}{i\sqrt{\pi}} \int_0^\infty d\lambda \int_{\Gamma_0} v^5 e^{v^4} \left(\frac{1}{\sinh^2(\lambda v)} - \frac{1}{\lambda^2 v^2} \right) dv \\ &= \frac{12}{i\sqrt{\pi}} \int_{\Gamma_0} v^4 e^{v^4} dv \int_{v\mathbb{R}_+} \left(\frac{1}{\sinh^2(y)} - \frac{1}{y^2} \right) dy \end{aligned} \tag{35}$$

(assuming we can change the order of integration). Again, the integral on y is independent of v, and equal to

$$\int_0^\infty \left(\frac{1}{\sinh^2(y)} - \frac{1}{y^2} \right) dy = \left[\frac{1}{y} - \frac{2}{e^{2y} - 1} \right]_0^\infty = -1.$$

Using again (31) to evaluate the integral on v, one finds

$$\mathbb{E}(N) = -\frac{6\sqrt{\pi}}{\Gamma(-1/4)} = \frac{3\sqrt{\pi}}{2\Gamma(3/4)}.$$

In order to justify the exchange of integrals in (35), we wish to prove that (35) is absolutely convergent. In order to do so, we split the integral over Γ_0 into two real integrals, corresponding respectively to $v = re^{i\pi/4}$ and $v = re^{-i\pi/4}$. We are thus led to prove that

$$\int_0^\infty d\lambda \int_0^\infty r^5 e^{-r^4} \left| \frac{1}{\sinh^2(\lambda r e^{i\pi/4})} - \frac{1}{i\lambda^2 r^2} \right| dr$$

is absolutely convergent (and a similar result when i is replaced by $-i$). But we can exchange the order of integration in this integral of *positive* functions. Doing so, and setting $\lambda = y/r$ as above, proves that this integral is finite.

The case $k = 2$. Let us start from another expression of $G(\lambda)$, obtained by writing $v = w/\lambda$:

$$G(\lambda) = \frac{12}{i\sqrt{\pi}\lambda^6} \int_{\lambda\Gamma} \frac{w^5}{\sinh^2(w)} e^{w^4/\lambda^4} dw = \frac{12}{i\sqrt{\pi}\lambda^6} \int_\Gamma \frac{w^5}{\sinh^2(w)} e^{w^4/\lambda^4} dw.$$

The second expression follows from the analyticity properties of the integrand. Now, take $\epsilon > 0$, and let us evaluate

$$\begin{aligned} \int_\epsilon^\infty \lambda G(\lambda) d\lambda &= \frac{12}{i\sqrt{\pi}} \int_\epsilon^\infty \frac{1}{\lambda^5} d\lambda \int_\Gamma \frac{w^5}{\sinh^2(w)} e^{w^4/\lambda^4} dw \\ &= \frac{12}{i\sqrt{\pi}} \int_\Gamma \frac{w^5}{\sinh^2(w)} dw \int_\epsilon^\infty \frac{e^{w^4/\lambda^4}}{\lambda^5} d\lambda \\ &= \frac{12}{i\sqrt{\pi}} \int_\Gamma \frac{w^5}{\sinh^2(w)} \left[-\frac{e^{w^4/\lambda^4}}{4w^4} \right]_\epsilon^\infty dw \\ &= \frac{3}{i\sqrt{\pi}} \int_\Gamma \frac{w}{\sinh^2(w)} \left(e^{w^4/\epsilon^4} - 1 \right) dw. \end{aligned} \tag{36}$$

The absolute convergence of integrals that legitimates the exchange of integrals in (36) is, this time, obvious (thanks to the fact that $\lambda > \epsilon$). Now, the analyticity of the function $w \mapsto w/\sinh^2(w)$ for $\Re(w) > 0$, and its strong decay as $\Re(w) \to \infty$, imply that

$$\int_\Gamma \frac{w}{\sinh^2(w)} dw = 0.$$

Hence

$$\begin{aligned} \int_\epsilon^\infty \lambda G(\lambda) d\lambda &= \frac{3}{i\sqrt{\pi}} \int_\Gamma \frac{w e^{w^4/\epsilon^4}}{\sinh^2(w)} dw &= \frac{3\epsilon^2}{i\sqrt{\pi}} \int_\Gamma \frac{v e^{v^4}}{\sinh^2(\epsilon v)} dv \\ &= \frac{3}{i\sqrt{\pi}} \int_\Gamma \frac{e^{v^4}}{v} dv + o(1) &= \frac{3\sqrt{\pi}}{2} \qquad \text{(by (31))}. \end{aligned}$$

Now, observe that

$$2\int_\epsilon^\infty \lambda G(\lambda) d\lambda = \mathbb{E}(N^2 \mathbb{1}_{N>\epsilon}) - \epsilon^2 G(\epsilon).$$

The announced expression of the second moment of N follows.

3.4. **Convergence of the moments of** N_n

In this section, we prove that the moments of $N_n = M_n/n^{1/4}$ converge to the corresponding moments of N. In order to do so, we first express $\mathbb{E}(M_n^k)$ as the coefficient of t^n in a certain series. Then, we apply the general consequences of the analysis of singularities: if this series is regular enough (with a precise meaning of regular), one can derive the asymptotic behaviour of its coefficients from the singular behaviour of the series near its dominant singularities [17].

Recall that the series U_j, given by (21), counts the trees that contain at least one label larger than j. Hence $U_{j-1} - U_j$ counts the trees having maximal label j. Also, note that

$$U_j = V(Z^j) - V(Z^{j+2}), \tag{37}$$

where

$$V(x) = \frac{xZ(1+Z)(1-Z^3)}{(1+Z^2)(1-xZ^2)}.$$

Consequently, for $k \geq 1$,

$$\mathbb{E}(M_n^k) = \frac{1}{2^n C_n} \sum_{j\geq 1} j^k [t^n](U_{j-1} - U_j) = \frac{1}{2^n C_n} [t^n] \sum_{j \geq 0} \left((j+1)^k - j^k\right) U_j. \tag{38}$$

For $k = 1$, this gives

$$2^n C_n \mathbb{E}(M_n) = [t^n] \sum_{j\geq 0} \left(V(Z^j) - V(Z^{j+2})\right) = [t^n]\left(V(1) + V(Z)\right) = [t^n] \frac{Z(1+2Z+2Z^2)}{1+Z^2}.$$

By Lemma 11, the latter series is analytic in $\mathbb{C} \setminus [1/8, \infty)$. The generic consequences of the analysis of singularities apply: one can derive the asymptotic behaviour of the coefficients from the singular behaviour of the series [17]. Given that, when $t \to 1/8$,

$$\frac{Z(1+2Z+2Z^2)}{1+Z^2} = \frac{5}{6} - 6(1-8t)^{1/4} + O(\sqrt{1-8t}),$$

the behaviour of the nth coefficient of this series is

$$[t^n] \frac{Z(1+2Z+2Z^2)}{1+Z^2} = -6 \frac{8^n n^{-5/4}}{\Gamma(-1/4)} (1+o(1)) = \frac{3}{2} \frac{8^n n^{-5/4}}{\Gamma(3/4)} (1+o(1)).$$

It remains to divide by $2^n C_n \sim 8^n n^{-3/2}/\sqrt{\pi}$ to conclude that

$$\mathbb{E}(M_n n^{-1/4}) \to \frac{3\sqrt{\pi}}{2\Gamma(3/4)},$$

which is also the first moment of N.

Now, by combining the expression (37) of U_j and (38), one obtains, for $k \geq 2$,

$$2^n C_n \mathbb{E}(M_n^k) = [t^n] \left(V(1) + (2^k - 1)V(Z) + \sum_{j\geq 2} \left((j+1)^k - j^k - (j-1)^k + (j-2)^k\right) V(Z^j) \right). \tag{39}$$

Observe that $(j+1)^k - j^k - (j-1)^k + (j-2)^k$ is a polynomial in j of degree $k-2$ and leading coefficient $2k(k-1)$. Let

$$A_\ell(t) = \sum_{j\geq -1} (j+2)^\ell V(Z^j).$$

We are going to prove that, for $\ell \in \mathbb{N}$,

$$a_n(\ell) := [t^n] A_\ell(t) = \begin{cases} \dfrac{3}{4}\dfrac{8^n}{n} & \text{if } \ell = 0, \\ \dfrac{3.8^n \ell! \zeta(\ell+1) n^{\ell/4-1}}{2^\ell \Gamma(\ell/4)} & \text{if } \ell \geq 1. \end{cases} \tag{40}$$

Assume for the moment this is proved, and let us conclude about the limiting moments of $N_n = M_n n^{-1/4}$. First, we observe that for $j \geq 0$, $V(Z^j)$ has (only) a fourth root singularity, so that the coefficient of t^n in $V(Z^j)$ grows like $8^n n^{-5/4}$, up to a multiplicative constant. This observation, combined with (39) and the above asymptotics of $a_n(\ell)$, implies that the dominant term in the asymptotic behaviour of $2^n C_n \mathbb{E}(M_n^k)$ is that of $2k(k-1)a_n(k-2)$. After normalizing by $2^n C_n n^{k/4}$, this gives

$$\mathbb{E}(M_n^k n^{-k/4}) \to \begin{cases} 3\sqrt{\pi} & \text{if } k = 2, \\ \dfrac{24\sqrt{\pi} k! \zeta(k-1)}{2^k \Gamma((k-2)/4)} & \text{if } k \geq 3. \end{cases}$$

These limiting moments are exactly those of N.

It remains to study the asymptotic behaviour of the numbers $a_n(\ell)$ (for ℓ fixed, and n going to infinity). We have:

$$\begin{aligned} A_\ell(t) &= \frac{(1+Z)(1-Z^3)}{Z(1+Z^2)} \sum_{j\geq 1} j^\ell \frac{Z^j}{1-Z^j} \\ &= \frac{(1+Z)(1-Z^3)}{Z(1+Z^2)} \sum_{j\geq 1, m\geq 1} j^\ell Z^{jm} \\ &= \frac{(1+Z)(1-Z^3)}{Z(1+Z^2)} \sum_{N\geq 1} Z^N \sigma_\ell(N) \end{aligned}$$

where

$$\sigma_\ell(N) = \sum_{j|N} j^\ell.$$

The function

$$D_\ell(z) = \sum_{N\geq 1} z^N \sigma_\ell(N)$$

is easily seen to have radius of convergence 1. Moreover, as z tends to 1 in such a way $|\arg(1-z)| < \phi < \pi/2$,

$$D_\ell(z) \sim \begin{cases} \dfrac{1}{1-z} \log\left(\dfrac{1}{1-z}\right) & \text{if } \ell = 0, \\ \dfrac{\ell! \zeta(\ell+1)}{(1-z)^{\ell+1}} & \text{if } \ell \geq 1 \end{cases}$$

(this can be obtained using a Mellin transform [16, 15]). The above expression of $A_\ell(t)$, combined with Lemma 11 and these properties of $D_\ell(z)$, shows that $A_\ell(t)$ is analytic in the domain $\mathcal{D} = \mathbb{C} \setminus [1/8, \infty)$. Moreover, since $|\arg(1-Z)| \le \pi/4 + o(1)$ as $t \to 1/8$ in $\mathcal{D}$, we can use the above estimates of $D_\ell(z)$. This gives

$$A_\ell(t) \sim \begin{cases} -3\log 2 + \dfrac{3}{4}\log\left(\dfrac{1}{1-8t}\right) & \text{if } \ell = 0, \\ \dfrac{3\ell!\zeta(\ell+1)}{2^\ell(1-8t)^{\ell/4}} & \text{if } \ell \ge 1. \end{cases}$$

The generic results derived from the analysis of singularities apply, and give the asymptotic behaviour (40) of the numbers $a_n(\ell)$. This concludes the proof of Theorem 9.

3.5. **The supremum of the support of the ise**

Let us finally prove Proposition 10. The following argument requires a detour via discrete snakes and Brownian snakes. We refer to [21, 23, 20] for definitions and notation[4]. In particular, we use the following integral representation of the random measure μ_{ISE}: for any continuous bounded function g on $\mathbb{R}$,

$$\int_{\mathbb{R}} g(y) d\mu_{\mathrm{ISE}}(y) = \int_0^1 g(r(t))dt \tag{41}$$

where $r(.)$ is a random process, continuous on $[0,1]$, called the head of the Brownian snake. In other words, μ_{ISE} is the *occupation measure* of the process r. (Again, the definition of r varies from one paper to the other. The above formula fixes our normalization of r.)

The random variable $N_n = M_n n^{-1/4}$ coincides with $\max(r_n)$, where r_n is the (normalized) head of the discrete snake associated with our tree family. The random process $\sqrt{2}r_n$ converges weakly to r, the head of the Brownian snake [23]. Since max is a continuous functional on $\mathcal{C}[0,1]$, this implies that $\sqrt{2}N_n = \sqrt{2}\max(r_n)$ converges in distribution to $\max(r)$. Thus $\max(r)$ has density $f(\lambda/\sqrt{2})/\sqrt{2}$, where f is defined in Theorem 9.

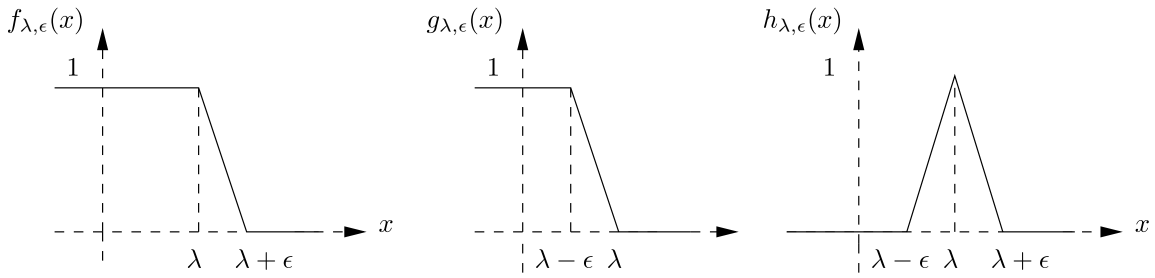

FIGURE 9. The functions $f_{\lambda,\epsilon}$, $g_{\lambda,\epsilon}$ and $h_{\lambda,\epsilon}$.

It remains to prove that $\max(r)$ is equal (in distribution) to N_{ISE}, the supremum of the support of the ISE. Let $\lambda \in \mathbb{R}$ and $\epsilon > 0$. Let $f_{\lambda,\epsilon}$ be the function plotted on the left-hand side of Figure 9. We have

$$N_{\mathrm{ISE}} \le \lambda \iff \mu_{\mathrm{ISE}}(-\infty, \lambda] = 1 \iff \int_{\mathbb{R}} f_{\lambda,1}(y) d\mu_{\mathrm{ISE}}(y) = 1.$$

Thanks to (41), this gives

$$N_{\mathrm{ISE}} \le \lambda \iff \int_0^1 f_{\lambda,1}(r(t))dt = 1$$

[4]We warn the reader that normalizations change from one paper to another.

Taking probabilities yields to

$$\mathbb{P}(N_{\text{ISE}} \le \lambda) = \mathbb{P}\left(\int_0^1 f_{\lambda,1}(r(t))dt = 1\right) = \mathbb{P}(\max(r) \le \lambda),$$

since r is almost surely continuous.

4. **A local limit law**

For $j \in \mathbb{Z}$, let $X_n(j)$ denote the random variable equal to the number of nodes having label j in a random tree of $\mathcal{T}_{0,n}$. This quantity is related to the series $S_j(t,u)$ studied in Proposition 4. In particular,

$$\mathbb{E}\left(e^{aX_n(j)}\right) = \frac{[t^n]S_j(t,e^a)}{2^n C_n}.$$

Also, observe that

$$X_n(j) = 0 \Longleftrightarrow M_n < j,$$

where M_n is the largest label, studied in the previous section. Let us define a normalized version of $X_n(j)$ by

$$Y_n(j) = \frac{X_n(j)}{n^{3/4}}.$$

Let $\lambda \in \mathbb{R}$. The aim of this section is to prove that $Y_n(\lfloor \lambda n^{1/4} \rfloor)$ converges in distribution, as n goes to infinity, to a random variable $Y(\lambda)$ that we describe by its Laplace transform. This is achieved in Theorem 14 below, but we first want to present two consequences of this theorem, which have a simpler formulation. The first consequence deals with the case $\lambda = 0$. Recall that, up to a normalization by $n^{3/4}$, the random variable $Y_n(0)$ gives the number of nodes labelled 0 in a tree rooted at 0.

Proposition 12 (**The number of nodes labelled** 0)**.** *As n goes to infinity, the random variable $3Y_n(0)/\sqrt{2}$ converges in distribution to $T^{-1/2}$, where T follows a unilateral stable law of parameter $2/3$. The convergence of the moments holds as well: for $k \ge 0$,*

$$\mathbb{E}\left(Y_n(0)^k\right) \to \left(\frac{\sqrt{2}}{3}\right)^k \frac{\Gamma(1+3k/4)}{\Gamma(1+k/2)} = \left(\frac{\sqrt{2}}{3}\right)^k \mathbb{E}(T^{-k/2}).$$

This proposition will be proved in Section 4.2. I am indebted to Alain Rouault, who recognized that the above moments were related to T. Recall that T is given by its Laplace transform:

$$\mathbb{E}(e^{-aT}) = e^{-a^{2/3}} \text{ for } a \ge 0.$$

The second consequence of Theorem 14 is an explicit expansion in λ of the limiting first moment of $Y_n(j)$.

Proposition 13 (**The first moment**)**.** *Let $\lambda \in \mathbb{R}$. Denote $j = \lfloor \lambda n^{1/4} \rfloor$. Then, as n goes to infinity,*

$$\mathbb{E}\left(Y_n(j)\right) \to \frac{1}{\sqrt{\pi}} \sum_{m \ge 0} \frac{(-2|\lambda|)^m}{m!} \cos\frac{(m+1)\pi}{4} \Gamma\left(\frac{m+3}{4}\right).$$

This function of λ is plotted on Figure 10.

Similar, but more and more complicated expressions may be written for the next moments of $Y(\lambda)$. This proposition will be proved in Section 4.3. Let us, finally, state our main theorem, from which the two above propositions derive.

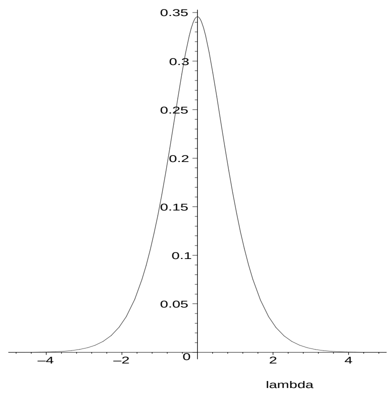

FIGURE 10. The average number of nodes labelled $\lfloor \lambda\sqrt{n} \rfloor$ in a tree of size n, when $n \to \infty$.

Theorem 14 (**A local limit law).** *Let $\lambda \geq 0$. The sequence $Y_n(\lfloor \lambda n^{1/4} \rfloor)$ converges in distribution to a non-negative random variable $Y(\lambda)$ whose Laplace transform is given, for $|a| < 4/\sqrt{3}$, by*

$$\mathbb{E}\left(e^{aY(\lambda)}\right) = L(\lambda, a)$$

where

$$L(\lambda, a) = 1 + \frac{48}{i\sqrt{\pi}} \int_\Gamma \frac{A(a/v^3)e^{-2\lambda v}}{(1 + A(a/v^3)e^{-2\lambda v})^2} v^5 e^{v^4} dv,$$

$A(x) \equiv A$ is the unique solution of

$$A = \frac{x}{24} \frac{(1 + A)^3}{1 - A} \tag{42}$$

satisfying $A(0) = 0$, and the integral is taken over

$$\Gamma = \{1 - te^{-i\pi/4}, t \in (\infty, 0]\} \cup \{1 + te^{-i\pi/4}, t \in [0, \infty)\}.$$

More precisely, the Laplace transform of $Y_n(\lfloor \lambda n^{1/4} \rfloor)$ converges pointwise to $L(\lambda, \cdot)$ on the interval $(-4/\sqrt{3}, 4/\sqrt{3})$. The convergence of moments holds as well.

It is believed (or known?) that the random measure μ_{ISE} is almost surely absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Eq. (2) leads us to the following conjecture.

Conjecture 15 (**The density of the ISE).** *There exists a random continuous process $f_{\mathrm{ISE}}(\lambda)$, defined for $\lambda \in \mathbb{R}$, such that $\mu_{\mathrm{ISE}} = f_{\mathrm{ISE}}Leb$, where Leb denotes the Lebesgue measure on $\mathbb{R}$. Moreover, $f_{\mathrm{ISE}}(\lambda)$ satisfies*

$$f_{\mathrm{ISE}}(\lambda) \stackrel{\mathrm{d}}{=} \frac{1}{\sqrt{2}} Y\left(\frac{|\lambda|}{\sqrt{2}}\right),$$

where the law of $Y(\lambda)$ is given in Theorem 14.

Comments

1. The limit random variable $Y(\lambda)$ equals 0 with a *positive probability* as soon as $\lambda > 0$. Indeed, by the portmanteau Theorem [14, Thm. 11.1.1],

$$\mathbb{P}(Y(\lambda) = 0) \geq \limsup \mathbb{P}(Y_n(\lfloor \lambda n^{1/4} \rfloor) = 0) = \limsup \mathbb{P}(M_n < \lfloor \lambda n^{1/4} \rfloor).$$

But, by Theorem 9,

$$\mathbb{P}(M_n < \lfloor \lambda n^{1/4} \rfloor) \to 1 - G(\lambda) > 0.$$

2. Let us add a few words on the series $A(x)$ defined by (42), in order to convince ourselves that the integral giving $L(\lambda, a)$ is well-defined. Clearly, the expansion of $A(x)$ at $x = 0$ has non-negative coefficients. Looking at the discriminant of the equation that defines A shows that A has radius of convergence at least $4/\sqrt{3}$. Moreover, it is easy to prove that $A(4/\sqrt{3}) = 2 - \sqrt{3} = 0.26\ldots$ Consequently, $|A(x)|$ is bounded by $2 - \sqrt{3}$ for $|x| \le 4/\sqrt{3}$. Since $|v| \ge 1$ for $v \in \Gamma$, the modulus of $A(a/v^3)$ is bounded from above by $2-\sqrt{3}$. Moreover, $\Re(v) \ge 1$, so that $|e^{-2\lambda v}| \le e^{-2\lambda} < 1$. Hence

$$\frac{A(a/v^3)e^{-2\lambda v}}{(1 + A(a/v^3)e^{-2\lambda v})^2}$$

is uniformly bounded on Γ, and $L(\lambda, a)$ is well-defined.

Note that the series $A(x)$ admits the following closed form expression:

$$A(x) = \frac{2}{1 + \frac{2}{\sqrt{3}} \cos(\frac{\arccos(-x\sqrt{3}/4)}{3})} - 1. \tag{43}$$

This can be checked by proving that this expression satisfies (42) and the initial condition $A(0) = 0$.

4.1. **Proof of Theorem 14**

Let $\lambda \ge 0$ and $j = \lfloor \lambda n^{1/4} \rfloor$. Let us first express the Laplace transform of $Y_n(j)$ in terms of the generating functions $S_j(t, u)$ of Proposition 4:

$$\mathbb{E}\left(e^{aY_n(j)}\right) = \mathbb{E}\left(e^{an^{-3/4}X_n(j)}\right) = \frac{[t^n]S_j(t, e^{an^{-3/4}})}{2^n C_n}. \tag{44}$$

Again, we will evaluate this Laplace transform thanks to the analysis of singularities [17]. We wish to use again the integration contour $\mathcal{C}_n$ of Figure 6. This requires to prove that $S_j(t, u)$ is analytic in a neigborhood of this contour (for n large and $u = e^{an^{-3/4}}$). This is guaranteed by the following lemma. This lemma naturally includes some properties of the series μ involved in the product form (11) of S_j. We denote by $\mathcal{I}_n$ the part of the complex plane enclosed by $\mathcal{C}_n$ (including $\mathcal{C}_n$ itself).

Lemma 16 (**Analytic properties of** μ **and** S_j)**.** *Let a be a real number such that $|a| < 4/\sqrt{3}$. Then there exists $\epsilon > 0$ such that for n large enough, the series $\mu(t, u_n)$, with $u_n = e^{an^{-3/4}}$, is analytic in the domain*

$$\mathcal{E}_n = \{t : |t - 1/8| > 1/((8 + \epsilon)n)\} \setminus [1/8, +\infty).$$

In particular, $\mu(t, u_n)$ is analytic in a neighborhood of $\mathcal{I}_n$. Its modulus in $\mathcal{I}_n$ is smaller than α, for some $\alpha < 1$ independent of a and n. The series $S_j(t, u_n)$ is also analytic in a neighborhood of $\mathcal{I}_n$.

Proof. The lemma is clear if $a = 0$: in this case, $u_n = 1$, the series $\mu(t, u_n)$ vanishes, and the series S_j reduces to the size generating function of labelled trees, namely T, which is analytic in $\mathbb{C} \setminus [1/8, \infty)$. We now assume that $a \ne 0$ and $|a| < 4/\sqrt{3}$. This guarantees that $A(a)$ is well-defined, where the series A is defined in Theorem 14.

Let us first study the singularities of the series $\bar\mu \equiv \bar\mu(z, u)$ defined as the unique formal power series in z satisfying

$$\bar\mu = (u - 1)\frac{(1 + z^2)(1 + \bar\mu z)(1 + \bar\mu z^2)(1 + \bar\mu z^3)}{(1 + z)(1 + z + z^2)(1 - z)^3(1 - \bar\mu z^2)}.$$

Note that $\bar\mu$ has polynomial coefficients in u, and vanishes when $u = 1$. Assume that u is a fixed real number close to, but different from, 1. Recall that, as all algebraic formal power series, $\bar\mu(t, u)$ has a positive radius of convergence. Let us perform a classical analysis to detect its possible singularities. These singularities are found in the union of two sets $\mathcal{S}_1$ and $\mathcal{S}_2$:

• $\mathcal{S}_1$ is the set of non-zero roots of the dominant coefficient of the equation defining $\bar{\mu}$. That is, $\mathcal{S}_1 = \{\pm i\}$,

• $\mathcal{S}_2$ is the set of the roots of the discriminant of the equation defining $\bar{\mu}$. For $u = 1 + x$ and x small, these roots are found to be

$$z = \pm 1, \quad z = -1 + O(x), \quad z = e^{\pm 2i\pi/3} + O(x), \quad z = 1 + \omega 12^{1/6} x^{1/3} + O(|x|^{2/3}),$$

where ω satisfies $\omega^6 = 1$. (The term ω allows us to write loosely $x^{1/3}$ without saying which determination of the cubic root we take.)
Observe that the moduli of all these "candidates for singularities" go to 1 as x goes to 0.

Now the series $\mu = \mu(t, u)$ involved in the expression (11) of $S_j(t, u)$ satisfies

$$\mu(t, u) = \bar{\mu}(Z(t), u)$$

where $Z(t)$ is defined by (5). In other words, we could have defined the series $\bar{\mu}$ by

$$\bar{\mu}(z, u) = \mu\left(\frac{z(1 + z^2)}{(1 + z)^4}, u\right).$$

Recall that Z is analytic in the domain $\mathcal{D} = \mathbb{C} \setminus [1/8, \infty)$. Take $u = u_n = e^{an^{-3/4}} = 1 + x$, with $x = an^{-3/4}(1 + o(1))$. By Lemma 11, for n large, the only values of $\mathcal{S}_1 \cup \mathcal{S}_2$ that may be reached by $Z(t)$, for $t \in \mathcal{D}$, are of the form

$$z = 1 + \omega 12^{1/6} a^{1/3} n^{-1/4} + O(n^{-1/2}).$$

In view of (5), these values of $Z(t)$ are reached for

$$t = \frac{1}{8} - \frac{\omega^4 (12)^{2/3}}{128} \frac{a^{4/3}}{n} + O(n^{-5/4}).$$

Since $|a| < 4/\sqrt{3}$, these values of t are at distance less than $1/((8 + \epsilon)n)$ of $1/8$, for some $\epsilon > 0$, and hence outside the domain $\mathcal{E}_n$. Consequently, $\mu(t, u_n)$ is analytic inside $\mathcal{E}_n$.

We now want to bound $\mu(t, u_n)$ inside $\mathcal{I}_n$. Let $t_n \in \mathcal{I}_n$ be such that

$$|\mu(t_n, u_n)| = \max_{t \in \mathcal{I}_n} |\mu(t, u_n)|.$$

In particular, $|\mu(t_n, u_n)| \geq |\mu((1 - 1/n)/8, u_n)|$. In order to evaluate the latter quantity, note that $Z((1 - 1/n)/8) = 1 - 2n^{-1/4} + O(n^{-1/2})$. Thanks to the closed form expression of μ given in Proposition 5, and to the expression (43) of the series A, we see that $\mu((1 - 1/n)/8, u_n) \to A(a)$. Since $a \neq 0$, $A(a) \neq 0$, and for n large enough,

$$|\mu(t_n, u_n)| \geq |\mu((1 - 1/n)/8, u_n)| = |A(a)| + o(1) > 0. \tag{45}$$

Recall that all the sets $\mathcal{I}_n$ are included in a ball of finite radius centered at the origin. Let α be an accumulation point of the sequence t_n. Then $|\alpha| \leq 1/8$.

Assume first that $\alpha \neq 1/8$. Then there exists N such that α is in $\mathcal{E}_n$ for all $n \geq N$, that is, in the analyticity domain of $\mu(\cdot, u_n)$. Let $t_{n_1}, t_{n_2}, \ldots$ converge to α. By continuity of μ in t and u, we have

$$\mu(t_{n_i}, u_{n_i}) \to \mu(\alpha, 1) = 0.$$

This contradicts (45). Hence the only accumulation point of t_n is $1/8$, and t_n converges to $1/8$. Let us thus write

$$t_n = \frac{1}{8}\left(1 - \frac{x_n}{n}\right).$$

We have $x_n = o(n)$, but also $|x_n| > 1$ since t_n belongs to $\mathcal{I}_n$. We wish to estimate $\mu(t_n, u_n)$. From the singular behaviour of Z (Lemma 11), we derive

$$Z(t_n) = 1 - 2\left(\frac{x_n}{n}\right)^{1/4} + O\left(\left(\frac{x_n}{n}\right)^{1/2}\right).$$

Moreover,

$$u_n - 1 = an^{-3/4}\left(1 + O(n^{-3/4})\right).$$

This gives

$$\frac{u_n-1}{(1-Z)^3} = \frac{a}{8x_n^{3/4}}\left(1 + O\left(\left(\frac{x_n}{n}\right)^{1/4}\right)\right).$$

If the sequence x_n was unbounded, then there would exist a subsequence x_{n_i} converging to infinity. Then $(u_{n_i}-1)/(1-Z)^3$ would tend to 0. The closed form expression of μ given in Proposition 5 implies that $\mu(t_{n_i}, u_{n_i})$ would tend to 0, contradicting (45). Hence the sequence x_n is bounded, and one derives from the explicit expressions of μ and A that

$$\mu(t_n, u_n) = A(ax_n^{-3/4}) + o(1).$$

Since A is bounded by $2-\sqrt{3}$ inside its disk of convergence, $|\mu(t_n, u_n)|$ is certainly smaller than some α for $\alpha < 1$ and n large enough. This concludes the proof of the second statement of Lemma 16.

By continuity of $\mu(t, u_n)$, this function of t is still bounded by 1 (in modulus) is a neighborhood of $\mathcal{I}_n$. Recall also that the modulus of $Z(t)$ never reaches 1 for $t \in \mathbb{C} \setminus [1/8, \infty)$. The form (11) then implies that $S_j(t, u_n)$ is an analytic function of t in a neigbourhood of $\mathcal{I}_n$.

■

Let us now go back to the expression (44) of the Laplace transform of $Y_n(j)$. Thanks to the lemma we have just proved, we can use the Cauchy formula to extract the coefficient of t^n in $S_j(t, u_n)$. We use the following expression of S_j:

$$S_j = T + T\frac{(1-Z)^2(1+Z+Z^2)\mu Z^j}{(1+\mu Z^{j+1})(1+\mu Z^{j+3})},$$

which is easily derived from (11). Thus

$$[t^n]S_j(t,u_n) = 2^n C_n + \frac{1}{2i\pi}\int_{\mathcal{C}_n} T\frac{(1-Z)^2(1+Z+Z^2)\mu Z^j}{(1+\mu Z^{j+1})(1+\mu Z^{j+3})}\frac{dt}{t^{n+1}}.$$

Again, we split the contour $\mathcal{C}_n$ into two parts $\mathcal{C}_n^{(1)}$ and $\mathcal{C}_n^{(2)}$, shown in Figure 6. As in the proof of Theorem 9, the contribution of $\mathcal{C}_n^{(1)}$ is easily seen to be $o(8^n/n^m)$ for all $m > 0$, thanks to the results of Lemmas 11 and 16. On $\mathcal{C}_n^{(2)}$, one has

$$t = \frac{1}{8}\left(1 + \frac{z}{n}\right)$$

where z lies in the truncated Hankel contour $\mathcal{H}_n$. Conversely, let $z \in \mathcal{H}$. Then $z \in \mathcal{H}_n$ for n large enough, and, in addition to the estimations (29) already used in the proof of Theorem 9, one finds

$$\mu(t, u_n) = A(a(-z)^{-3/4})(1 + o(1)), \tag{46}$$

where $A(x)$ is the series defined by (42). After a few reductions, one finally obtains

$$[t^n]S_j(t,u_n) = 2^n C_n + \frac{12.8^n n^{-3/2}}{i\pi}\int_{\mathcal{H}} \frac{A(a(-z)^{-3/4})\exp(-2\lambda(-z)^{1/4})\sqrt{-z}e^{-z}}{(1+A(a(-z)^{-3/4})\exp(-2\lambda(-z)^{1/4}))^2}dz + o(8^n n^{-3/2}).$$

It remains to normalize by $2^n C_n = 8^n n^{-3/2}/\sqrt{\pi}$, and then to set $v = (-z)^{1/4}$ to obtain the expected expression for the limit of the Laplace transform of $Y_n(j)$, with $j = \lfloor \lambda n^{1/4} \rfloor$.

The limit Laplace transform $L(\lambda, a)$ is clearly continuous at $a = 0$, and equals 1 at this point. A version of Lévy's continuity theorem [14, Thm. 9.8.2] adapted to Laplace transforms implies that the sequence $Y_n(j)$ converges in distribution to a limit random variable $Y(\lambda)$ having Laplace transform $L(\lambda, \cdot)$.

From the convergence of the Laplace transform in a neighbourhood of the origin, it is easy to derive that for every k, the sequence of random variables $Y_n(j)^k$ is *uniformly integrable*. But then the convergence in distribution implies the convergence of the moments [4, Thm. 5.4]. This concludes the proof of Theorem 14.

4.2. **Proof of Proposition 12**

When $\lambda = 0$, the limiting Laplace transform reduces to

$$L(0,a) = 1 + \frac{48}{i\sqrt{\pi}} \int_\Gamma \frac{A(a/v^3)}{(1+A(a/v^3))^2} v^5 e^{v^4} dv = 1 + \frac{12}{i\sqrt{\pi}} \int_\Gamma \frac{\chi(a/v^3)}{1+\chi(a/v^3)} v^5 e^{v^4} dv$$

where $\chi(x)$ is the unique series in x satisfying

$$\chi = \frac{x}{6}(1+\chi)^{3/2}.$$

The Lagrange inversion formula [25, p. 38] gives, for $k \geq 1$,

$$[x^k]\frac{\chi(x)}{1+\chi(x)} = \frac{1}{6^k}\frac{\Gamma(3k/2-1)}{k!\Gamma(k/2)}.$$

Consequently,

$$L(0,a) = 1 + \frac{12}{i\sqrt{\pi}} \int_\Gamma \sum_{k\geq 1} \frac{1}{6^k}\frac{\Gamma(3k/2-1)}{k!\Gamma(k/2)} a^k v^{5-3k} e^{v^4} dv.$$

The convergence is absolute, so that we can exchange the sum and the integral:

$$L(0,a) = 1 + \frac{12}{i\sqrt{\pi}} \sum_{k\geq 1} \frac{1}{6^k}\frac{\Gamma(3k/2-1)}{k!\Gamma(k/2)} a^k \int_\Gamma v^{5-3k} e^{v^4} dv.$$

Using (31), and picking the coefficient of a^k, we find that the kth moment of the random variable $Y(0)$ is

$$\mathbb{E}(Y(0)^k) = \frac{\sqrt{\pi}}{6^{k-1}} \frac{\Gamma(3k/2-1)}{\Gamma(k/2)\Gamma((3k-2)/4)}.$$

The duplication formula,

$$2^{2s-1}\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}\,\Gamma(2s),$$

applied to $s = (3k-2)/4$, finally gives

$$\mathbb{E}(Y(0)^k) = \left(\frac{\sqrt{2}}{3}\right)^k \frac{\Gamma(1+3k/4)}{\Gamma(1+k/2)} = \lim_{n\to\infty} \mathbb{E}\left(Y_n(0)^k\right).$$

Since $Y(0)$ has a Laplace transform, it is uniquely determined by its moments [3, Thm. 30.1]. But

$$m_k = \frac{\Gamma(1+3k/4)}{\Gamma(1+k/2)}$$

is known to be the kth moment of $T^{-1/2}$, where T follows a unilateral stable law of parameter $2/3$ (see [9, p. 111]). Proposition 12 follows.

4.3. **Proof of Proposition 13**

We have derived above the moments of $Y(0)$ from the expression of its Laplace transform. This extends to the moments of $Y(\lambda)$, for $\lambda > 0$: for $k \geq 1$,

$$\mathbb{E}(Y(\lambda)^k) = \frac{48.k!}{i\sqrt{\pi}} \int_\Gamma [a^k] \frac{A(a/v^3)e^{-2\lambda v}}{(1 + A(a/v^3)e^{-2\lambda v})^2} v^5 e^{v^4} dv.$$

Since $A(x) = x/24 + O(x^2)$, the case $k = 1$ of the above identity reads

$$\mathbb{E}(Y(\lambda)) = \frac{2}{i\sqrt{\pi}} \int_\Gamma e^{-2\lambda v} v^2 e^{v^4} dv.$$

In the above expression, expand the exponential as a series. The convergence of the sum and integral is absolute, so that one can exchange them. This gives:

$$\mathbb{E}(Y(\lambda)) = \frac{2}{i\sqrt{\pi}} \sum_{m \geq 0} \frac{(-2\lambda)^m}{m!} \int_\Gamma v^{m+2} e^{v^4} dv.$$

Using (31) (which is valid for any s with the convention $1/\Gamma(-n) = 0$ for $n \in \mathbb{N}$), this can be rewritten as

$$\mathbb{E}(Y(\lambda)) = \sqrt{\pi} \sum_{m \geq 0} \frac{(-2\lambda)^m}{m!\Gamma((1-m)/4)} = \frac{1}{\sqrt{\pi}} \sum_{m \geq 0} \frac{(-2\lambda)^m}{m!} \Gamma\left(\frac{m+3}{4}\right) \cos\left(\frac{(m+1)\pi}{4}\right).$$

The last equality follows from the complement formula,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{47}$$

5. **A global limit law, and the distribution function of the ISE**

In Section 4, we have derived from Proposition 4 some *local* limit results; for instance, a limit law for $X_n(0)/n^{3/4}$, the (normalized) number of nodes labelled 0. In this section, we proceed with a similar study, which aims at deriving from Proposition 7 a *global* limit result — in particular, the limit law of $X_n^+(0)/n$, the normalized number of nodes having a non-negative label. The technique is copied on Section 4, and we do not give all the details.

For $j \in \mathbb{Z}$, let $X_n^+(j)$ denote the random variable equal to the number of nodes having label at least j in a random tree of $\mathcal{T}_{0,n}$. Let us define a normalized version of $X_n^+(j)$ by

$$Y_n^+(j) = \frac{X_n^+(j)}{n}.$$

These quantities are related to the series $R_j(t, u)$ studied in Proposition 7. In particular,

$$\mathbb{E}\left(e^{aY_n^+(j)}\right) = \mathbb{E}\left(e^{an^{-1}X_n^+(j)}\right) = \frac{[t^n]R_j(t, e^{a/n})}{2^n C_n}.$$

We extend the definition of X_n^+ and Y_n^+ to real values in a natural way by setting $X_n^+(x) = X_n^+(\lceil x \rceil)$ and $Y_n^+(x) = Y_n^+(\lceil x \rceil)$. Let $\lambda \geq 0$. The aim of this section is to prove that $Y_n^+(\lambda n^{1/4})$ converges in distribution, as n goes to infinity, to a random variable $Y^+(\lambda)$ that we describe by its Laplace transform. This is achieved in Theorem 19 below, but we first want to present two consequences of this theorem, which have a simpler formulation. The first consequence is a striking limit law for $Y_n^+(0)$. Recall that, up to a normalization by n, this random variable gives the number of nodes having a non-negative label in a tree rooted at 0.

Proposition 17 (The number of non-negative nodes). *As n goes to infinity, the random variable $Y_n^+(0)$ converges in law to the uniform distribution on* $[0, 1]$.

This proposition will be proved in Section 5.2. The second consequence of Theorem 19 is an explicit expansion in λ of the limiting first moment of $Y_n^+(\lambda n^{1/4})$.

Proposition 18 (The first moment). *Let $\lambda \geq 0$. Then, as n goes to infinity,*

$$\mathbb{E}\left(Y_n^+(\lambda n^{1/4})\right) \to \frac{1}{2\sqrt{\pi}} \sum_{m\geq 0} \frac{(-2\lambda)^m}{m!} \cos\left(\frac{m\pi}{4}\right) \Gamma\left(\frac{m+2}{4}\right).$$

This proposition will be proved in Section 5.3.

Let us, finally, state our main theorem, from which the two above propositions derive.

Theorem 19 (A global limit law). *Let $\lambda \geq 0$. The sequence $Y_n^+(\lambda n^{1/4})$ converges in distribution to a random variable $Y^+(\lambda)$ whose Laplace transform is given, for $|a| < 1$, by*

$$\mathbb{E}\left(e^{aY^+(\lambda)}\right) = G(\lambda, a),$$

where

$$G(\lambda, a) = 1 + \frac{48}{i\sqrt{\pi}} \int_\Gamma \frac{B(a/v^4)e^{-2\lambda v}}{(1 + B(a/v^4)e^{-2\lambda v})^2} v^5 e^{v^4} dv,$$

$$B(x) = -\frac{(1-D)(1-2D)}{(1+D)(1+2D)}, \qquad D = \sqrt{\frac{1+\sqrt{1-x}}{2}}, \tag{48}$$

and the integral is taken over

$$\Gamma = \{1 - te^{-i\pi/4}, t \in (\infty, 0]\} \cup \{1 + te^{-i\pi/4}, t \in [0, \infty)\}.$$

Moreover, the Laplace transform of $Y_n^+(\lambda n^{1/4})$ converges pointwise to $G(\lambda, \cdot)$ on the interval $(-1, 1)$. The convergence of moments holds as well.

The sequence $Y_n^+(-\lambda n^{1/4})$ converges in distribution to the random variable $1 - Y^+(\lambda)$.

This theorem will be proved in the next subsection. In view of the following proposition, it tells us about the law of the distribution function of the ISE.

Proposition 20 (The tail distribution function of the ISE). *Let $g_{\mathrm{ISE}}(\lambda) = \mu_{\mathrm{ISE}}(\lambda, +\infty)$ denote the tail distribution function of the ISE. Then for $\lambda \geq 0$,*

$$g_{\mathrm{ISE}}(\lambda) \stackrel{\mathrm{d}}{=} Y^+(\lambda/\sqrt{2}),$$

where the law of the variable $Y^+(\lambda)$ is given in Theorem 19. In particular, $g_{\mathrm{ISE}}(0)$ is uniformly distributed on $[0, 1]$. The random variable $g_{\mathrm{ISE}}(-\lambda)$ has the same distribution as $1 - Y^+(\lambda/\sqrt{2})$.

Comments

1. The law of $g_{\mathrm{ISE}}(0)$ was already given by Aldous [1, Eq. (12)].

2. Let us add a few words on the series B and D to convince ourselves that the integral giving $G(\lambda, a)$ is well-defined as long as $|a| < 1$. Let $E(x) = 1 - D(x)$. Then E admits the following expansion:

$$E(x) = 1 - \sqrt{1 - \frac{1 - \sqrt{1-x}}{2}} = 2\sum_{n\geq 1} C_{n-1} \left(\frac{1-\sqrt{1-x}}{8}\right)^n$$

where C_n is the nth Catalan number. Similarly,

$$1 - \sqrt{1-x} = 2\sum_{n\geq 1} C_{n-1} 4^{-n} x^n,$$

and these two identities imply that $E(x)$ has non-negative coefficients. Moreover, its radius of convergence is easily seen to be 1, so that $|E(x)| \leq E(1) = 1 - 1/\sqrt{2}$ for $|x| \leq 1$. Moreover, expressing B in terms of E gives:

$$B = \frac{E(1-2E)}{(2-E)(3-2E)},$$

which shows that $B(x)$ is also analytic for $|x| < 1$ and satisfies in this domain

$$|B(x)| \leq \frac{E(1)(1+2E(1))}{(2-E(1))(3-2E(1))} = 22\sqrt{2} - 31 = 0.11...$$

For $v \in \Gamma$, $|v| \geq 1$ and $\Re(v) \geq 1$. This implies that

$$\frac{B(a/v^4)e^{-2\lambda v}}{(1+B(a/v^4)e^{-2\lambda v})^2}$$

is uniformly bounded on Γ, and $G(\lambda, a)$ is well-defined.

5.1. **Proof of Theorem 19**

Let $j = \lceil \lambda n^{1/4} \rceil$. Given that the product forms for the series S_j and R_j are very similar, it is not surprising that we use an approach copied on that of the previous section. We start from

$$\mathbb{E}(e^{aY_n^+(j)}) = \mathbb{E}(u_n^{X_n^+(j)}) = \frac{[t^n]R_j(t, u_n)}{2^n C_n},$$

with $u_n = e^{a/n}$. For technical reasons, we choose to modify slightly the integration contour of Figure 6. The Hankel part of this contour, which was lying at distance 1/8 of the real axis, is now moved a bit further, at distance 1/6 of the real axis. More precisely, the new contour $\overline{\mathcal{C}}_n$ consists of two parts $\overline{\mathcal{C}}_n^{(1)}$ and $\overline{\mathcal{C}}_n^{(2)}$ such that

- $\overline{\mathcal{C}}_n^{(1)}$ is an arc of radius $(1 + \log^2 n/n)/8$, centered at the origin;
- $\overline{\mathcal{C}}_n^{(2)}$ is a Hankel contour around 1/8, at distance $1/(6n)$ of the real axis, which meets $\overline{\mathcal{C}}_n^{(1)}$ at both ends.

We first need to prove that the series $R_j(t, u_n)$ is analytic in a neighborhood of $\overline{\mathcal{I}}_n$, the region lying inside the integration contour $\overline{\mathcal{C}}_n$. The following lemma is the counterpart of Lemma 16.

Lemma 21 (**Analytic properties of ν and R_j**). *Let a be a real number such that $|a| < 1$. Then $\nu(t, u_n)$ is analytic in a neighborhood of $\overline{\mathcal{I}}_n$. Its modulus in $\overline{\mathcal{I}}_n$ is smaller than α, for some $\alpha < 1$ independent of a and n. The series $R_j(t, u_n)$ is also analytic in a neighborhood of $\overline{\mathcal{I}}_n$.*

Proof. Again, the lemma is obvious if $a = 0$. We thus assume $a \neq 0$ and $|a| < 1$.

Let us first study the singularities of the series $\bar{\nu} \equiv \bar{\nu}(z, u)$ defined by

$$\bar{\nu}(z, u) = \nu\left(\frac{z(1+z^2)}{(1+z)^4}, u\right).$$

According to Proposition 7, $\bar{\nu}$ is a formal power series in z with polynomial coefficients in u, and by (5), one has:

$$\nu(t, u) = \bar{\nu}(Z(t), u).$$

In the course of the proof of Proposition 7, we have obtained a polynomial equation $P(\nu, Z, u) = 0$, of degree 4 in ν, relating $\nu(t, u), Z(t)$ and the variable u. This equation is not written in the paper (it is a bit too big), but it can be easily obtained using the expression of ν given in Proposition 8. By definition of $\bar{\nu}$, we have $P(\bar{\nu}, z, u) = 0$.

Assume that u is a fixed real number close to 1. That is, $u = 1 + x$, with x small. In order to study the singularities of $\bar{\nu}$, we look again at the zeroes of the leading coefficient of P, and at the zeroes of its discriminant. This gives several candidates for the singularities of $\bar{\nu}(z, u)$, which we classify in three series according to their behaviour when x is small. First, some candidates tend to a limit that is different from 1,

$$z = -1, \quad z = \pm i, \quad z = e^{\pm 2i\pi/3}, \quad z = e^{\pm 2i\pi/3} + O(x), \quad z = -1 + O(x).$$

Then, some candidates tend to 1 and lie at distance at most $|x|^{1/4}$ of 1 (up to a multiplicative constant):

$$z = 1 + \omega(cx)^{1/4} + O(\sqrt{|x|}),$$

where ω is a fourth root of unity and c is in the set $\{0, 16, 64/3, -16/3\}$. Finally, some candidates tend to 1 but lie further away from 1 (more precisely, at distance $|x|^{1/6}$):

$$z = 1 + 2e^{i\pi/6}\omega' x^{1/6} + O(|x|^{1/3}),$$

where ω' is a sixth root of unity.

Let us now consider $\nu(t,u) = \bar{\nu}(Z(t), u)$ with $u = u_n = e^{a/n} = 1 + x$, where $x = a/n(1 + o(1))$. Recall that Z is analytic in $\mathbb{C} \setminus [1/8, \infty)$. By Lemma 11, the series $Z(t)$ never approaches any root of unity different from 1. Hence for n large enough, $Z(t)$ never reaches any of the candidates z of the first series.

The candidates of the second series are of the form

$$z = 1 + \omega(ac/n)^{1/4} + O(n^{-1/2})$$

for some constant c, with $|c| \le 64/3$, depending on the candidate. By (5), $Z(t)$ may only reach these values for

$$t = \frac{1}{8} - \frac{ac}{128n} + O(n^{-5/4}).$$

Since $|a| < 1$, there exists $\epsilon > 0$ such that these values lie at distance less that $1/((6+\epsilon)n)$ of $1/8$, that is, outside a neighborhood of the domain $\overline{\mathcal{I}}_n$.

The candidates of the third series are more worrying: $Z(t)$ may reach them for

$$t = \frac{1}{8} - \frac{\omega''}{8}(a/n)^{2/3} + O(n^{-5/6}), \tag{49}$$

where ω'' is a cubic root of unity, and these values may lie inside $\overline{\mathcal{I}}_n$. If $a > 0$ and $\omega'' = e^{\pm 2i\pi/3}$, or if $a < 0$ and $\omega'' = e^{2i\pi/3}$, the modulus of the above value of t is found to be $1/8(1 + cn^{-2/3} + o(n^{-2/3}))$, for some *positive* constant c: this is larger than the radius of the contour $\overline{\mathcal{C}}_n$, which implies that t lies outside a neighborhood of $\overline{\mathcal{I}}_n$. However, if $a > 0$ and $\omega'' = 1$, or if $a < 0$ and $\omega'' = 1$ or $e^{-2i\pi/3}$, the above value of t lies definitely inside $\overline{\mathcal{I}}_n$. Its modulus is $1/8(1 - cn^{-2/3} + o(n^{-2/3}))$, for some positive constant c.

In order to rule out the possibility that $\nu(t, u_n)$ has such a singularity, we are going to prove, by having a close look at the expression of ν given in Proposition 8, that the radius of convergence of $\nu(t, u_n)$ is at least $1/8 - O(1/n)$. We use below the notation of Proposition 8.

Clearly, the series $V(t, u_n)$ has radius of convergence $\min(1/8, 1/(8u_n))$. In particular, this radius is at least $\rho_n := 1/(8(1 + |x|))$ (with $u_n = 1 + x$). Moreover, the series V admits the following expansion

$$V(t, 1+x) = \frac{1}{4}\left(1 - \sqrt{1 - \frac{8tx}{1-8t}}\right) = \frac{1}{2}\sum_{n\ge 1} C_{n-1}\left(\frac{2tx}{1-8t}\right)^n,$$

where C_n is the nth Catalan number. This shows that $V(t, 1 + |x|)$ is a series in t with *positive* coefficients and that for all t such that $|t| \le \rho_n$,

$$|V(t, 1+x)| \le V(|t|, 1 + |x|) \le V\left(\frac{1}{8(1+|x|)}, 1 + |x|\right) = \frac{1}{4}.$$

The next step is to prove that $\Delta(t, u_n)$ never vanishes for $|t| \le \rho_n$. Indeed,

$$\Delta = (1 - V)^2 - 4WV^2,$$

where $W \equiv W(t)$ is the formal power series in t defined by (24). This series has radius $1/8$, and non-negative coefficients. Hence for all t such that $|t| \le 1/8$, one has $|W(t)| \le$

$W(1/8) = 1/4$. Consequently, for $|t| \leq \rho_n$,

$$|\Delta(t, 1+x)| \geq (1 - |V(t, 1+x)|)^2 - 4|W(t)||V(t, 1+x)|^2 \geq \left(1 - \frac{1}{4}\right)^2 - \frac{1}{16} = \frac{1}{2}.$$

Hence $\Delta(t, u_n)$ does not vanish in the centered disk of radius ρ_n. It follows that the series $P(t, u_n)$ is analytic inside this disk.

According to the expression of ν given in Proposition 8, the series $\nu(t, u_n)$ is meromorphic for $|t| \leq \rho_n$. The final question we need to answer is whether ν has poles in this disk, and where. Returning to the polynomial P such that $P(\nu, Z, u) = 0$ shows that this can only happen if the coefficient of ν^4 in this polynomial vanishes. But this can only occur if $z = Z(t)$ has one of the following forms:

$$z = \pm i, \quad z = e^{\pm 2i\pi/3} + O(x), \quad z = -1 + O(x), \quad z = 1 + \omega(64x/3)^{1/4} + O(x^{1/2}).$$

As argued above, only the last value of z is likely to be reached by $Z(t)$, and this may only occur if

$$t = \frac{1}{8} - \frac{1}{6}(a/n) + O(n^{-5/4}).$$

Consequently, the radius of $\nu(t, u_n)$ is at least $1/8 - O(1/n)$, and this proves that the values (49) that have been shown to lie in the centered disk of radius $1/8$, are not, after all, singularities of $\nu(t, u_n)$. This completes our proof that $\nu(t, u_n)$ is analytic in a neighborhood of $\overline{\mathcal{I}}_n$.

We now want to bound $\nu(t, u_n)$ inside $\overline{\mathcal{I}}_n$. From now on, we can walk safely in the steps of the proof of Lemma 16. Let $t_n \in \overline{\mathcal{I}}_n$ be such that

$$|\nu(t_n, u_n)| = \max_{t \in \overline{\mathcal{I}}_n} |\nu(t, u_n)|.$$

We first give a lower bound for this quantity, by estimating $\nu(t, u_n)$ for $t = 1/8 - 1/(6n)$. This is easily done by combining the closed form expressions of ν (Proposition 8) and B (Theorem 19). One obtains:

$$|\nu(t_n, u_n)| \geq |\mu(1/8 - 1/(6n), u_n)| = |B(3a/4)| + o(1) > 0.$$

This lower bound is then used to rule out the possibility that the sequence t_n has an accumulation point different from $1/8$. Thus t_n converges to $1/8$, and one can write

$$t_n = \frac{1}{8}\left(1 - \frac{x_n}{n}\right).$$

We have $x_n = o(n)$, but also $|x_n| > 4/3$ since t_n belongs to $\overline{\mathcal{I}}_n$. We want to estimate $\nu(t_n, u_n)$. Since

$$Z(t_n) = 1 - 2\left(\frac{x_n}{n}\right)^{1/4} + O\left(\left(\frac{x_n}{n}\right)^{1/2}\right)$$

and

$$u_n - 1 = a/n\,(1 + O(1/n)),$$

one has

$$\frac{u-1}{(1-Z)^4} = \frac{a}{16x_n}\left(1 + O\left(\left(\frac{x_n}{n}\right)^{1/4}\right)\right).$$

The closed form expressions of ν and B imply that the sequence x_n is bounded and

$$\nu(t_n, u_n) = B(a/x_n) + o(1).$$

Since B is bounded by 0.12 inside its disk of convergence, $|\nu(t_n, u_n)|$ is certainly smaller than some α for $\alpha < 1$ and n large enough. This concludes the proof of the second statement of Lemma 21.

By continuity of $\nu(t, u_n)$, this function of t is still bounded by 1 (in modulus) is a neighborhood of $\overline{\mathcal{I}}_n$. Recall also that the modulus of $Z(t)$ never reaches 1 for $t \in \mathbb{C} \setminus [1/8, \infty)$.

The form (16) then implies that $R_j(t, u_n)$ is an analytic function of t in a neigborhood of $\overline{\mathcal{I}}_n$.

■

Once this rather painful lemma is at last established, the rest of the proof of Theorem 19 copies the end of the proof of Theorem 14, with S_j, μ and A respectively replaced by R_j, ν and B. The counterpart of (46) is

$$\nu(t, u_n) = B(-a/z)(1 + o(1)).$$

Recall that the Hankel part of the contour $\overline{\mathcal{C}}_n$ is now at distance $1/(6n)$ of the real axis. Hence, when n goes to infinity, one finds

$$[t^n]R_j(t, u_n) = 2^n C_n + \frac{12.8^n n^{-3/2}}{i\pi} \int_{4/3\mathcal{H}} \frac{B(-a/z)\exp(-2\lambda(-z)^{1/4})\sqrt{-z}e^{-z}}{(1 + B(-a/z)\exp(-2\lambda(-z)^{1/4}))^2} dz + o(8^n n^{-3/2}).$$

After normalizing by $2^n C_n$ and setting $v = (-z)^{1/4}$, this gives

$$\mathbb{E}(e^{aY_n^+(j)}) \to 1 + \frac{48}{i\sqrt{\pi}} \int_{(4/3)^{1/4}\Gamma} \frac{B(a/v^4)e^{-2\lambda v}}{(1 + B(a/v^4)e^{-2\lambda v})^2} v^5 e^{v^4} dv,$$

but the analyticity properties of the integrand allow us to replace the integration contour by Γ.

5.2. **Proof of Proposition 17**

When $\lambda = 0$, the limiting Laplace transform reduces to

$$G(0, a) = 1 + \frac{48}{i\sqrt{\pi}} \int_\Gamma \frac{B(a/v^4)}{(1 + B(a/v^4))^2} v^5 e^{v^4} dv = 1 + \frac{4}{3i\sqrt{\pi}} \int_\Gamma \frac{\chi(a/v^4)(3 - \chi(a/v^4))}{1 + \chi(a/v^4)} v^5 e^{v^4} dv$$

where $\chi(x)$ is the unique formal power series in x satisfying

$$\chi = \frac{x}{4}(1 + \chi)^2.$$

The Lagrange inversion formula gives, for $k \geq 1$,

$$[x^k]\frac{\chi(x)(3 - \chi(x))}{1 + \chi(x)} = \frac{6}{4^k} \frac{(2k - 2)!}{(k - 1)!(k + 1)!}.$$

Consequently,

$$G(0, a) = 1 + \frac{8}{i\sqrt{\pi}} \int_\Gamma \sum_{k\geq 1} \frac{1}{4^k} \frac{(2k - 2)!}{(k - 1)!(k + 1)!} a^k v^{5-4k} e^{v^4} dv.$$

The convergence is absolute, so that we can exchange the sum and the integral:

$$G(0, a) = 1 + \frac{8}{i\sqrt{\pi}} \sum_{k\geq 1} \frac{1}{4^k} \frac{(2k - 2)!}{(k - 1)!(k + 1)!} a^k \int_\Gamma v^{5-4k} e^{v^4} dv.$$

Using (31), and picking the coefficient of a^k, we find that the kth moment of the random variable $Y^+(0)$ is

$$\mathbb{E}(Y(0)^k) = \frac{8}{i\sqrt{\pi}} \frac{1}{4^k} \frac{(2k - 2)!}{(k - 1)!(k + 1)} \frac{i\pi}{2\Gamma(k - 1/2)} = \frac{1}{k + 1}.$$

The unique distribution having its kth moment equal to $1/(k+1)$ is the uniform distribution on $[0, 1]$. Proposition 17 follows.

5.3. **Proof of Proposition 18**

We have derived above the moments of $Y^+(0)$ from the expression of its Laplace transform. This extends to the moments of $Y^+(\lambda)$, for $\lambda > 0$: for $k \geq 1$,

$$\mathbb{E}(Y^+(\lambda)^k) = \frac{48.k!}{i\sqrt{\pi}} \int_\Gamma [a^k] \frac{B(a/v^4)e^{-2\lambda v}}{(1+B(a/v^4)e^{-2\lambda v})^2} v^5 e^{v^4} dv.$$

Since $B(x) = x/48 + O(x^2)$, the case $k = 1$ of the above identity gives

$$\mathbb{E}(Y^+(\lambda)) = \frac{1}{i\sqrt{\pi}} \int_\Gamma e^{-2\lambda v} v e^{v^4} dv.$$

In the above expression, expand the exponential as a series. The convergence of the sum and integral is absolute, so that one can exchange them. This gives:

$$\mathbb{E}(Y^+(\lambda)) = \frac{1}{i\sqrt{\pi}} \sum_{m\geq 0} \frac{(-2\lambda)^m}{m!} \int_\Gamma v^{m+1} e^{v^4} dv.$$

Using (31), this can be rewritten as

$$\mathbb{E}(Y^+(\lambda)) = \frac{\sqrt{\pi}}{2} \sum_{m\geq 0} \frac{(-2\lambda)^m}{m!\Gamma((2-m)/4)} = \frac{1}{2\sqrt{\pi}} \sum_{m\geq 0} \frac{(-2\lambda)^m}{m!} \Gamma\left(\frac{m+2}{4}\right) \cos\left(\frac{m\pi}{4}\right).$$

The last equality follows from the complement formula (47).

5.4. **The distribution function of the** ISE

Let us finally prove Proposition 20.

Let μ_n be a sequence of random probability measures on $\mathbb{R}$ converging weakly to a probability measure μ. Let F_n denote the (random) distribution function of μ_n: for $\lambda \in \mathbb{R}$,

$$F_n(\lambda) = \mu_n(-\infty, \lambda].$$

Similarly, let F denote the distribution function of μ. It is not very hard to prove that, for all $\lambda \in \mathbb{R}$ such that $\mu\{\lambda\} = 0$, $F_n(\lambda)$ converges in distribution to $F(\lambda)$. (We prove this in the appendix of the paper, but it is certainly written somewhere in the literature.)

Let us now apply this general result to our context. The probability measure μ_n is given by (1), with $c = \sqrt{2}$. It is known to converge to the random measure μ_{ISE}. Assume for the moment that this measure does not assign a positive weight to any point. Then, with the above notation, $F_n(\lambda)$ converges in distribution to $F(\lambda)$, for all $\lambda \in \mathbb{R}$. But, given the definition (1) of μ_n,

$$F_n(\lambda) = 1 - \mu_n(\lambda, \infty) = 1 - \frac{1}{n+1} X_n^+(\lambda n^{1/4}/\sqrt{2}) + \frac{1}{n+1} X_n(\lambda n^{1/4}/\sqrt{2}),$$

where the definition of X_n is extended to all reals by $X_n(x) = 0$ if $x \in \mathbb{R}\setminus\mathbb{Z}$. By Theorems 14 and 19, the right-hand side converges in distribution to $1 - Y^+(\lambda/\sqrt{2})$. Consequently, the tail distribution function of the ISE (that is, $\mu_{\text{ISE}}(\lambda, \infty)$) has the same law as $Y^+(\lambda/\sqrt{2})$.

It remains to prove that μ_{ISE} does not weight points positively (almost surely). Let $\lambda \in \mathbb{R}$. Then

$$\mathbb{P}(\mu_{\text{ISE}}\{\lambda\} > 0) = 0 \Longleftrightarrow \mathbb{E}(\mu_{\text{ISE}}\{\lambda\}) = 0. \tag{50}$$

Let $\epsilon > 0$, and let $h_{\lambda,\epsilon}$ be the function plotted on the right-hand side of Figure 9. Then

$$\begin{aligned}
\mu_{\text{ISE}}\{\lambda\} &= \lim_{\epsilon\to 0^+} \int_\mathbb{R} h_{\lambda,\epsilon}(y) d\mu_{\text{ISE}}(y) \\
&= \lim_{\epsilon\to 0^+} \int_0^1 h_{\lambda,\epsilon}(r(t)) dt \qquad \text{(by (41))} \\
&= \int_0^1 \mathbb{1}_{\lambda=r(t)} dt.
\end{aligned}$$

Taking expectations, we obtain

$$\mathbb{E}(\mu_{\text{ISE}}\{\lambda\}) = \mathbb{E}\left(\int_0^1 \mathbb{1}_{\lambda=r(t)}dt\right) = \int_0^1 \mathbb{P}(\lambda = r(t))dt.$$

But $\mathbb{P}(\lambda = r(t)) = 0$ for all $t \in (0,1)$ and λ, since $r(t)$ has a density with respect to the Lebesgue measure for all t. By (50), we conclude that μ_{ISE} does not weight points positively.

The last statement of Proposition 20 is then easily proven, using a the symmetry of μ_{ISE} and the fact that it does not assign a positive probability to any point.

6. Other tree models and universality

6.1. Trees with increments $0, \pm 1$

We consider in this section a slight variation on the previous family of trees: the increments of the labels along edges may now be $0, \pm 1$. This family of trees has attracted a lot of interest in relation to planar maps [6, 8, 11, 24].

6.1.1. ***Enumerative results.*** As above, let $T_j \equiv T_j(t)$ be the generating function of labelled trees in which all labels are at most j, counted by their number of edges. Let $S_j \equiv S_j(t,u)$ be the generating function of labelled trees, counted by the number of edges (variable t) and the number of nodes labelled j (variable u). Finally, let $R_j \equiv R_j(t,u)$ be the generating function of labelled trees, counted by the number of edges and the number of nodes having label j at least. As above, it is easy to write an infinite system of equations defining any of the families T_j, S_j or R_j. The only difference with our first family of trees is that a third case arises in the decomposition of trees illustrated by Figure 4: the leftmost child of the root may have label j. In particular, the generating function $T \equiv T(t)$ counting plane labelled trees now satisfies

$$T = 1 + 3tT^2,$$

while for $j \geq 0$,

$$T_j = 1 + t(T_{j-1} + T_j + T_{j+1})T_j. \tag{51}$$

The equations of Lemmas 3 and 6 are modified in a similar way. The three infinite systems of equations thus obtained can be solved using the same techniques as in Section 2. The solutions are expressed in terms of the above series $T \equiv T(t)$ and of the unique formal power series $Z \equiv Z(t)$, with constant term 0, satisfying

$$Z = t\,\frac{(1 + 4Z + Z^2)^2}{1 + Z + Z^2}. \tag{52}$$

Observe that T and Z are related by:

$$T = \frac{1 + 4Z + Z^2}{1 + Z + Z^2}.$$

We state without proof the counterparts of Propositions 2, 4 and 7.

Proposition 22 (**Trees with small labels** [5, 6]**).** *Let $T_j \equiv T_j(t)$ be the generating function of trees having no label greater than j. Then T_j is algebraic of degree (at most) 2. In particular,*

$$T_0 = 1 - 16\,t + 18\,tT_0 - 27\,t^2 {T_0}^2.$$

Moreover, for all $j \geq -1$,

$$T_j = T\,\frac{(1 - Z^{j+1})(1 - Z^{j+4})}{(1 - Z^{j+2})(1 - Z^{j+3})},$$

where $Z \equiv Z(t)$ is given by (52).

Remarks

1. As observed in [6, p. 645], there is an "invariant" function attached to equations of the form (51): for $j \geq 0$,

$$I(T_{j-1}, T_j) = I(T_j, T_{j+1})$$

where I is now given by

$$I(x, y) = xy(1 - t(x + y)) - x - y.$$

As explained in the remark that follows Propositions 4 and 5, this can be used to derive rapidly from (51) the value of T_0.

2. As was the case for trees with increments ± 1, the trees counted by T_0 (equivalently, the trees having only non-negative labels) are closely related to planar maps. More precisely, there is a one-to-one correspondence between non-negative trees having n edges and planar maps having n edges [8, 11]. The coefficients of $T_0(t)$ are also remarkably simple:

$$T_0(t) = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2} = \sum_{n \geq 0} \frac{2.3^n}{(n+1)(n+2)} \binom{2n}{n} t^n.$$

A combinatorial explanation for the algebraicity of T_0 is given in [11].

Proposition 23 (**The number of nodes labelled** j)**.** *For any $j \in \mathbb{Z}$, the generating function $S_j \equiv S_j(t, u)$ that counts labelled trees by the number of edges and the number of nodes labelled j is algebraic of degree at most 4 over $\mathbb{Q}(T, u)$ (and hence has degree at most 8 over $\mathbb{Q}(t, u)$). More precisely,*

$$\frac{9T^4(u-1)^2}{(T - S_0)^2} = 9T^2 - 2T(T-1)(2T+1)S_0 + (T-1)^2 S_0^2$$

and all the S_j belong to $\mathbb{Q}(t, u, S_0)$. Moreover, for all $j \geq 0$,

$$S_j = T\, \frac{(1 + \mu Z^j)(1 + \mu Z^{j+3})}{(1 + \mu Z^{j+1})(1 + \mu Z^{j+2})},$$

where $Z \equiv Z(t)$ is given by (52) and $\mu \equiv \mu(t, u)$ is the unique formal power series in t satisfying

$$\mu = (u-1) \frac{(1 + Z + Z^2)(1 + \mu Z)^2 (1 + \mu Z^2)^2}{(1 + Z)^2 (1 - Z)^3 (1 - \mu^2 Z^3)}.$$

The series $\mu(t, u)$ has polynomial coefficients in u, and satisfies $\mu(t, 1) = 0$. It has degree 4 over $\mathbb{Q}(Z, u)$ and 16 over $\mathbb{Q}(t, u)$.

Proposition 24 (**The number of nodes labelled** j **or more**)**.** *Let $j \in \mathbb{Z}$. The generating function $R_j(t, u) \equiv R_j$ that counts labelled trees by the number of edges and the number of nodes labelled j or more is algebraic over $\mathbb{Q}(t, u)$, of degree at most 8. It has degree at most 2 over $\mathbb{Q}(T, \tilde{T})$, where $T \equiv T(t)$ and $\tilde{T} \equiv T(tu)$. More precisely, it belongs to the extension of $\mathbb{Q}(T, \tilde{T})$ generated by*

$$\sqrt{4(T + \tilde{T})^2 - T\tilde{T}(4 + 3T\tilde{T})}.$$

Moreover, for all $j \geq 0$,

$$R_j = T\, \frac{(1 + \nu Z^j)(1 + \nu Z^{j+3})}{(1 + \nu Z^{j+1})(1 + \nu Z^{j+2})},$$

where $Z \equiv Z(t)$ is given by (52) and $\nu \equiv \nu(t, u)$ is a formal power series in t, with polynomial coefficients in u, which is algebraic of degree 4 over $\mathbb{Q}(u, Z)$, and of degree 16 over $\mathbb{Q}(t, u)$. This series satisfies $\nu(t, 1) = 0$. The first terms in its expansion are:

$$\nu(t, u) = (u-1)\Big(1 + 3\,ut + \big(15\,u + 14\,u^2\big)\,t^2 + \big(104\,u + 117\,u^2 + 83\,u^3\big)\,t^3 + O(t^4)\Big).$$

6.1.2. ***Limit laws.*** We now endow the set of labelled trees having n edges with the uniform distribution, and consider the same random variables as for our first family of trees: M_n, the largest label, $X_n(j)$, the number of nodes having label j, and finally $X_n^+(j)$, the number of nodes having label j at least.

Again, we can prove that $M_n n^{-1/4}$ converges in law to $N_{\text{ISE}}/\sqrt{3}$, where N_{ISE} is the supremum of the support of the ISE, and that for all $\lambda \in \mathbb{R}$, the sequence $X_n^+(\lambda n^{1/4})/n$ converges in law to $g_{\text{ISE}}(\sqrt{3}\lambda)$ where g_{ISE} is the tail distribution function of the ISE. The arguments are the same as for our first class of trees (Sections 3.5 and 5.4).

Hence we could just as well have started from the enumerative results of Section 6.1.1, rather than from those of Section 2, to characterize the laws of N_{ISE} and $g_{\text{ISE}}(\lambda)$ (Propositions 10 and 20). More remarkably, we have performed on $X_n(j)$ an analysis similar to that of Section 4, and obtained *the same* local limit law. In other words, for all $\lambda \geq 0$, the sequence $X_n(\lfloor \lambda n^{1/4} \rfloor) n^{-3/4}$ converges in law to $\sqrt{3} f_{\text{ISE}}(\sqrt{3}\lambda)$ where f_{ISE} is the conjectured density of the ISE, given in Conjecture 15.

In all three cases, the convergence of the moments holds as well.

6.2. **Naturally embedded binary trees**

We study in the section the incomplete binary trees[5] carrying their natural labelling, as shown on the right of Figure 1. Such trees are either empty, or have a root, to which a left and right subtree (both possibly empty) are attached. A (minor) difference with the two previous families of trees is that the main enumeration parameter is now the number of nodes rather than the number of edges.

6.2.1. ***Enumerative results.*** Let $T_j \equiv T_j(t)$ be the generating function of (naturally labelled) binary trees in which all labels are at most j, counted by their number of nodes. Let $S_j \equiv S_j(t, u)$ be the generating function of binary trees, counted by the number of nodes (variable t) and the number of nodes labelled j (variable u). Finally, let $R_j \equiv R_j(t, u)$ be the generating function of binary trees, counted by the number of nodes and the number of nodes having label j at least. It is easy to write an infinite system of equations defining any of the families T_j, S_j or R_j. The decomposition of trees that was crucial in Section 2 is now replaced by the decomposition of Figure 11. The generating function $T \equiv T(t)$ counting naturally labelled binary trees satisfies

$$T = 1 + tT^2,$$

(as it should!) while for $j \geq 0$,

$$T_j = 1 + tT_{j-1}T_{j+1}. \tag{53}$$

Note that the initial condition is now $T_{-1} = 1$ (accounting for the empty tree).

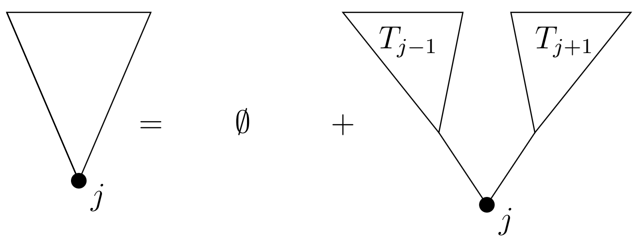

FIGURE 11. The decomposition of naturally labelled binary trees rooted at j.

[5]The author has obtained similar, but slightly heavier results for embedded *complete* binary trees.

The equations of Lemmas 3 and 6 respectively become:

$$S_j = \begin{cases} 1 + tS_{j-1}S_{j+1} & \text{if } j \neq 0, \\ 1 + tuS_1^2 & \text{if } j = 0, \end{cases} \tag{54}$$

while

$$R_j = 1 + tR_{j-1}R_{j+1} \quad \text{for } j \geq 1, \tag{55}$$

and

$$R_{-j}(t,u) = R_{j+1}(tu, 1/u) \quad \text{for all } j \in \mathbb{Z}. \tag{56}$$

The three infinite systems of equations thus obtained can be solved using the same techniques as in Section 2. The solutions are expressed in terms of the above series $T \equiv T(t)$ and of the unique formal power series $Z \equiv Z(t)$, with constant term 0, satisfying

$$Z = t\,\frac{\left(1+Z^2\right)^2}{1-Z+Z^2}. \tag{57}$$

Observe that T and Z are related by:

$$T = \frac{1+Z^2}{1-Z+Z^2}.$$

We state without proof the counterparts of Propositions 2, 4 and 7. Once again, the results below are dying for combinatorial explanations!

Proposition 25. *Let $T_j \equiv T_j(t)$ be the generating function of binary trees having no label greater than j. Then T_j is algebraic of degree (at most) 2. In particular,*

$$T_0 = \frac{(1-4t)^{3/2} - 1 + 8t - 2t^2}{2t(1+t)}.$$

Moreover, for all $j \geq -1$,

$$T_j = T\,\frac{(1-Z^{j+2})(1-Z^{j+7})}{(1-Z^{j+4})(1-Z^{j+5})},$$

where $Z \equiv Z(t)$ is given by (57).

It is easy to check that the above series T_j satisfy the equations (53) and the initial condition $T_{-1} = 1$. The method we used to *discover* this product form is again borrowed from [5].

Remark. For this family of trees as well, we have found an "invariant" function attached to equations of the form (53): for $j \geq 0$,

$$I(T_{j-1}, T_j) = I(T_j, T_{j+1})$$

where

$$I(x,y) = (x+y)\,t^2 + \frac{\left(x^2 - x - y + y^2\right)t}{xy} + \frac{-1+x+y}{xy}.$$

This can be used to derive rapidly from (53) the value of T_0.

Proposition 26 (The number of nodes labelled j). *For any $j \in \mathbb{Z}$, the generating function $S_j \equiv S_j(t,u)$ that counts binary trees by the number of nodes and the number of nodes labelled j is algebraic of degree at most 4 over $\mathbb{Q}(T,u)$ (and thus has degree at most 8 over $\mathbb{Q}(t,u)$). More precisely,*

$$\frac{T^2(u-1)^2}{u(T-S_0)^2} = \frac{(T-1)^4 S_0^2 - 2TS_0(T-1)^2(3-9T+7T^2) + T^2(T^2+T-1)^2}{(T-1)(S_0-1)(T^2+TS_0-S_0)^2}$$

and all the series S_j belong to $\mathbb{Q}(t,u,S_0)$. Moreover, for all $j \geq 0$,

$$S_j = T\,\frac{(1+\mu Z^j)(1+\mu Z^{j+5})}{(1+\mu Z^{j+2})(1+\mu Z^{j+3})},$$

where $Z \equiv Z(t)$ is given by (57) and $\mu \equiv \mu(t,u)$ is the unique formal power series in t satisfying

$$\mu = (u-1)\frac{Z(1+\mu Z)^2(1+\mu Z^2)(1+\mu Z^6)}{(1+Z)^2(1+Z+Z^2)(1-Z)^3(1-\mu^2 Z^5)}.$$

The series $\mu(t,u)$ has polynomial coefficients in u, and satisfies $\mu(t,1)=0$. It has degree 4 over $\mathbb{Q}(Z,u)$ and 16 over $\mathbb{Q}(t,u)$.

Comment on the proof. The proof is similar to the proof of Proposition 7 until one computes the equation satisfied by S_0. But then, the relation $S_0 = 1 + tS_1^2$ does not allow us to conclude that S_1 belongs to $\mathbb{Q}(t,u,S_0)$. Instead, we compute the algebraic equation satisfied by S_1. It is found to have degree 4 over $\mathbb{Q}(u,T)$. The above relation between S_0 and S_1 shows that S_0 belongs to the extension of $\mathbb{Q}(t,u)$ generated by S_1. Comparing the degrees implies finally that $\mathbb{Q}(t,u,S_0) = \mathbb{Q}(t,u,S_1)$. Then (54) shows, by induction on j, that all the series S_j belong to this field.

Proposition 27 (**The number of nodes labelled j or more).** *Let $j \in \mathbb{Z}$. The generating function $R_j(t,u) \equiv R_j$ that counts binary trees by the number of nodes and the number of nodes labelled j or more is algebraic over $\mathbb{Q}(t,u)$. More precisely, R_0 has degree 16 over $\mathbb{Q}(t,u)$ and degree 4 over $\mathbb{Q}(T,\tilde{T})$, with $\tilde{T} = T(tu)$, and all the series R_j belong to $\mathbb{Q}(T,\tilde{T},R_0) = \mathbb{Q}(t,u,R_0)$. Moreover, for all $j \geq 0$,*

$$R_j = T\,\frac{(1+\nu Z^j)(1+\nu Z^{j+5})}{(1+\nu Z^{j+2})(1+\nu Z^{j+3})},$$

where $Z \equiv Z(t)$ is given by (52) and $\nu \equiv \nu(t,u)$ is a formal power series in t, with polynomial coefficients in u, which is algebraic of degree 8 over $\mathbb{Q}(u,Z)$ and 32 over $\mathbb{Q}(t,u)$. This series satisfies $\nu(t,1)=0$. The first terms in its expansion are:

$$\nu(t,u) = (u-1)\Big(t + (u+1)t^2 + (2u^2+3u+3)t^3 + O(t^4)\Big).$$

Comment on the proof. The proof is similar to the proof of Proposition 7 until one computes the equation satisfied by R_0. One finds that R_0 has degree 4 over $\mathbb{Q}(T,\tilde{T})$, and degree 16 over $\mathbb{Q}(t,u)$. Using (56), one then derives an equation satisfied by R_1. Strangely enough, it turns out that the minimal polynomials of R_0 and R_1 over $\mathbb{Q}(T,\tilde{T})$ (or over $\mathbb{Q}(t,u)$) *are the same*. The two series are of course different:

$$\begin{aligned} R_0(t,u) &= 1 + tu + u(1+u)t^2 + u(2u^2+2u+1)t^3 + u(1+u)(4u^2+u+2)t^4 + O(t^5), \\ R_1(t,u) &= 1 + t + (1+u)t^2 + (u^2+2u+2)t^3 + (1+u)(2u^2+u+4)t^4 + O(t^5). \end{aligned}$$

Let $P(x)$ be the minimal polynomial of R_0 and R_1 over $\mathbb{K} \equiv \mathbb{Q}(T,\tilde{T})$. We want to prove that R_1 belongs to the extension of $\mathbb{K}$ generated by R_0. Note that this property does not simply follow from the fact that R_0 and R_1 are conjugate roots of P. For instance, for a *generic* polynomial P of degree 4 over $\mathbb{K}$, with Galois group $\mathcal{S}_4$, the four extensions of $\mathbb{K}$ generated by the roots of P are different. We are going to determine the Galois group of our polynomial P, using the general strategy described in [2, p. 141–142]. The resolvent cubic of P, which we denote R below, is found to factor into a linear term and a quadratic one. Hence the Galois group of R over $\mathbb{K}$ has order 2. This implies that the Galois group G of P over $\mathbb{K}$ is either the cyclic group of order 4 or the dihedral group of order 8. In the former case, the four extensions of $\mathbb{K}$ generated by the roots of P coincide (and are equal to the splitting field of P) and we are done. In the latter case, there exists a labelling of the four roots of P, say X_0, X_1, X_2, X_3, such that the group G, seen as a subgroup of the permutations of $\{0,1,2,3\}$, is

$$G = \{\mathrm{id}, (0,1), (2,3), (0,1)(2,3), (0,2)(1,3), (0,3)(1,2), (0,2,1,3), (0,3,1,2)\}.$$

Then

• the simple extensions of $\mathbb{K}$ generated by the X_i satisfy $\mathbb{K}(X_0) = \mathbb{K}(X_1)$ and $\mathbb{K}(X_2) = \mathbb{K}(X_3)$,

• the root of the resolvent that belongs to $\mathbb{K}$ is $Y = X_0X_1 + X_2X_3$.

The root of R that belongs to $\mathbb{K} = \mathbb{Q}(T, \tilde{T})$ is found to be $Y = t^{-3}/(u(1+u)) + O(t^{-2})$. We already know two roots of P, namely R_0 and R_1, which are equal to $1+O(t)$. The other two roots are respectively of the form $Q_2 = -t^{-2}/u+O(t^{-1})$ and $Q_3 = -t^{-1}/(1+u)+O(1)$. From the value of Y, we conclude that the above properties hold with $X_0 = R_0$, $X_1 = R_1$, $X_2 = Q_2$ and $X_3 = Q_3$. In particular, R_0 and R_1 belong to the same extension of degree 4 of $\mathbb{Q}(T, \tilde{T})$.

Then, an induction on j, based on (55), implies that for all $j \geq 2$, the series R_j belongs to the extension of $\mathbb{K} = \mathbb{Q}(T, \tilde{T})$ generated by R_0. Since $R_1(t, u) = R_0(tu, 1/u)$ and $\mathbb{K}(R_0) = \mathbb{K}(R_1)$, the field $\mathbb{K}(R_0)$ is invariant under the transformation $A(t, u) \mapsto A(tu, 1/u)$. This property, combined with (56), implies that for $j \geq 1$, the series R_{-j} belongs to $\mathbb{K}(R_0)$.

6.2.2. ***Limit laws.*** We now endow the set of binary trees having n nodes with the uniform distribution, and consider the same random variables as for above: M_n, the largest label, $X_n(j)$, the number of nodes having label j, and $X_n^+(j)$, the number of nodes having label j at least.

Again, we can prove that $M_n n^{-1/4}$ converges in law to N_{ISE}, where N_{ISE} is the supremum of the support of the ISE, and that for all $\lambda \in \mathbb{R}$, the sequence $X_n^+(\lambda n^{1/4})/n$ converges in law to $g_{\text{ISE}}(\lambda)$ where g_{ISE} is the tail distribution function of the ISE. The arguments are the same as for our first class of trees (Sections 3.5 and 5.4). Hence we could just as well have started from the enumerative results of Section 6.2.1, rather than from those of Section 2, to characterize the laws of N_{ISE} and $g_{\text{ISE}}(\lambda)$ (Propositions 10 and 20).

More remarkably, we have performed on $X_n(j)$ an analysis similar to that of Section 4, and obtained *the same* local limit law. In other words, for all $\lambda \geq 0$, the sequence $X_n(\lfloor \lambda n^{1/4} \rfloor) n^{-3/4}$ converges in law to $f_{\text{ISE}}(\lambda)$ where f_{ISE} is the conjectured density of the ISE, given in Conjecture 15.

In all three cases, the convergence of the moments holds as well.

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Appendix: convergence of random distribution functions

We want to prove the result stated without proof at the beginning of the proof of Section 5.4.

Recall that a sequence of real random variables Z_n converges in law to another random variable Z if and only if for all $x \in \mathbb{R}$ such that $\mathbb{P}(Z = x) = 0$,

$$\lim_n \mathbb{P}(Z_n \le x) = \mathbb{P}(Z \le x).$$

This implies the so-called *portmanteau* inequality: for *all* $x \in \mathbb{R}$,

$$\mathbb{P}(Z < x) \le \liminf \mathbb{P}(Z_n \le x) \le \limsup \mathbb{P}(Z_n \le x) \le \mathbb{P}(Z \le x). \tag{58}$$

Let us now use the notation of Section 5.4. The convergence of μ_n to μ implies that for any bounded Lipschitz function f on $\mathbb{R}$ [27, p. 71–74]:

$$\int_{\mathbb{R}} f(x) d\mu_n(x) \xrightarrow{\mathrm{d}} \int_{\mathbb{R}} f(x) d\mu(x).$$

Let $\lambda \in \mathbb{R}$ and let $f_{\lambda,\epsilon}$ and $g_{\lambda,\epsilon}$ be the functions plotted in Figure 9. Then

$$\int_{\mathbb{R}} g_{\lambda,\epsilon}(y) d\mu_n(y) \le F_n(\lambda) = \mu_n(-\infty, \lambda] \le \int_{\mathbb{R}} f_{\lambda,\epsilon}(y) d\mu_n(y).$$

Hence, for all $x \in \mathbb{R}$,

$$\mathbb{P}\left(\int_{\mathbb{R}} f_{\lambda,\epsilon}(y) d\mu_n(y) \le x\right) \le \mathbb{P}(F_n(\lambda) \le x) \le \mathbb{P}\left(\int_{\mathbb{R}} g_{\lambda,\epsilon}(y) d\mu_n(y) \le x\right).$$

Since μ_n converges to μ, and $g_{\lambda,\epsilon}$ is a bounded Lipschitz function,

$$\int_{\mathbb{R}} g_{\lambda,\epsilon}(y)d\mu_n(y) \xrightarrow{\mathrm{d}} \int_{\mathbb{R}} g_{\lambda,\epsilon}(y)d\mu(y).$$

A similar result holds for the integral involving $f_{\lambda,\epsilon}$. Thus (58) implies

$$\mathbb{P}\left(\int_{\mathbb{R}} f_{\lambda,\epsilon}(y)d\mu(y) < x\right) \le \liminf \mathbb{P}(F_n(\lambda) \le x) \le \limsup \mathbb{P}(F_n(\lambda) \le x) \le \mathbb{P}\left(\int_{\mathbb{R}} g_{\lambda,\epsilon}(y)d\mu(y) \le x\right).$$

The integral occurring in the rightmost expression of this inequality is bounded from below by $\mu(-\infty, \lambda - \epsilon]$, while the integral involving $f_{\lambda,\epsilon}$ is bounded from above by $\mu(-\infty, \lambda + \epsilon]$. Hence

$$\mathbb{P}(\mu(-\infty, \lambda+\epsilon] < x) \le \liminf \mathbb{P}(F_n(\lambda) \le x) \le \limsup \mathbb{P}(F_n(\lambda) \le x) \le \mathbb{P}(\mu(-\infty, \lambda-\epsilon] \le x).$$

Taking the limit $\epsilon \to 0^+$ gives:

$$\mathbb{P}(\mu(-\infty, \lambda] < x) \le \liminf \mathbb{P}(F_n(\lambda) \le x) \le \limsup \mathbb{P}(F_n(\lambda) \le x) \le \mathbb{P}(\mu(-\infty, \lambda) \le x).$$

If, in addition, the measure μ does not assign a positive probability to λ, the rightmost expression in the above inequality equals $\mathbb{P}(\mu(-\infty, \lambda] \le x)$. The inequality becomes

$$\mathbb{P}(F(\lambda) < x) \le \liminf \mathbb{P}(F_n(\lambda) \le x) \le \limsup \mathbb{P}(F_n(\lambda) \le x) \le \mathbb{P}(F(\lambda) \le x),$$

so that for all x such that $\mathbb{P}(F(\lambda) = x) = 0$,

$$\lim \mathbb{P}(F_n(\lambda) \le x) = \mathbb{P}(F(\lambda) \le x).$$

That is, $F_n(\lambda)$ converges in law to $F(\lambda)$.

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