Bounds on primitives of differential forms and group cocycles

Jean-Claude Sikorav

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0. Introduction

We investigate the relations between the existence of a "primitive" with given bounds and the satisfaction of weighted isoperimetric inequalities. In one direction, the relation follows from various versions of Stokes' formula. In the other, it uses a Hahn-Banach type argument. We shall consider three frameworks:

1) Riemannian manifolds and primitives of exact differential forms. If V is a Riemannian manifold and $\omega \in \mathcal{E}^q(V)$ is a differential form of degree q, we define its norm at the point $x \in V$ of by $||\omega||(x) = \max\{\omega_x(v_1, \cdots, v_q) \mid v_i \in T_xV, ||v_i|| \leq 1\}$.

Question 1. *Let V be a Riemannian manifold. Let $\omega \in \mathcal{E}^q(V)$ be exact, of degree $q \geq 2$ and let $\varphi : V \to \mathbb{R}_+$ be a continuous function. When does there exist $\tau \in \mathcal{E}^{q-1}(V)$ such that $d\tau = \omega$ and $||\tau|| \leq \varphi$?*

A case of special interest will be $V = \widetilde{M}$, the universal covering of a compact Riemannian manifold, and ω comes from a closed form on M.

2) Cellular [in particular simplicial] complexes and primitives of exact cochains.

Question 2. *Let X be a cellular complex. Let $u \in C^q(X;\mathbb{R})$ be an exact q-cochain for some $q \geq 2$, and let $f \in C^{q-1}(X;\mathbb{R}_+)$ be a nonnegative cellular $(q-1)$-cochain (function on the $(q-1)$-cells). When does there exist $t \in C^{q-1}(X;\mathbb{R})$ such that $dt = u$ and $|t| \leq f$?*

The answer to Question 2 is an immediate application of Hahn-Banach.

3) Groups and primitives of exact cochains.

Question 3. *Let (G, S) be a group equipped with a finite generating system. Let b be a q-cocycle on G for some $q \geq 2$, and let F be a function from G to $\mathbb{R}_+$. When does there exist a $(q-1)$-cochain $a \in C^{q-1}(G;\mathbb{R})$ such that $da = b$ and*

$$|a(g, g\overline{s}_1, g\overline{s}_1\overline{s}_2, \cdots, g\overline{s}_1 \cdots \overline{s}_{q-1})| \leq F(g)?$$

Special case $q = 2$. *Let $b : G^3 \to \mathbb{R}$ be a 2-cocycle, ie $b(g_1, g_2, g_3) - b(g_0, g_2, g_3) + b(g_0, g_1, g_3) - b(g_0, g_1, g_2) = 0$, and let F be a nonnegative function on G. When does there exist $a : G^2 \to \mathbb{R}$ such that $a(g_1, g_2) - a(g_0, g_2) + a(g_0, g_1) = b(g_0, g_1, g_2)$ and*

$$|a(g, g\overline{s})| \leq F(g)?$$

We first answer Question 1 in terms of weighted isoperimetric inequalities given by Stokes' formula.

Theorem 1. *Let $\omega \in \mathcal{E}^q(V)$ with $q \geq 2$, and let $\varphi : V \to \mathbb{R}_+$ be continuous. Assume that for every real smooth singular q-chain c one has*

$$I_c(\omega) \leq \mathrm{M}_\varphi(I_{\partial c}).$$

Then for every $\varepsilon > 0$, there exists $\tau \in \mathcal{E}^{q-1}(V)$ such that $d\tau = \omega$ and $||\tau|| \leq \varphi + \varepsilon$.

Here I_c is the integration current associated with c, and $\mathrm{M}_\varphi(T)$ its *weighted mass* of a current (see the definitions in section 1). For instance, if $\varphi = 1$ and ∂c has no geometric cancellations, $\mathrm{M}_\varphi(I_{\partial c})$ is its $(q-1)$-dimensional volume.

Corollary. *The "smallest" norm of a primitive of ω is*

$$\inf\{||\tau||_\infty \mid d\tau = \omega\} = \sup_T \frac{T(\omega)}{\mathrm{M}(\partial T)} = \sup_c \frac{I_c(\omega)}{\mathrm{M}(I(\partial c))}.$$

In the case of volume forms, the result is much nicer.

Theorem 2. *Let V be an oriented Riemannian manifold of dimension n and let ω be a nonnegative smooth n-form (in particular a volume form). Let $\varphi = V \to \mathbb{R}_+$ be continuous. Assume that, for every compact domain $\Omega \subset V$ with smooth boundary,*

$$\int_\Omega \omega \leq \mathrm{vol}_\varphi(\partial\Omega) = \int_{\partial\Omega} \varphi d\sigma$$

where $d\sigma$ is the $(n-1)$-dimensional measure on $\partial\Omega$.

Then for every continuous $\varepsilon > 0$, there exists $\tau \in \mathcal{E}^{n-1}(V)$ such that $d\tau = \omega$ and $||\tau|| \leq \varphi + \varepsilon$.

From Theorem 1 we deduce a comparison predicted by Gromov [G2, p.98] between the cofilling function and (a suitable version of) the filling area. For the definitions, see section 4.

Theorem 3. *Let V be a Riemannian manifold such that $H_1(V;\mathbb{R}) = 0$ and V is* quasihomogeneous*: there exists $C > 0$ and for every $x, y \in V$ a C-bilipschitz homeomorphism $h : V \to V$ with $d(h(x), y) \leq C$.*

Then

$$\mathrm{Cof}(R) \sim \frac{\mathbb{R}FA(R)}{R},$$

where $\varphi(R) \sim g(R)$ means that there exists $C_1, C_2 > 0$ such that $C_1\varphi(C_1 R) \leq g(R) \leq C_2\varphi(C_2 R)$.

Remark. [G2] states that $\mathrm{Cof}(R) \sim \mathrm{FA}(R)/R$. The equivalence between FA and $\mathbb{R}FA$ for V the universal covering of a compact manifold is an old question [?].

Question 3 can be answered using Hahn-Banach. We give first the case $q = 2$:

Theorem 4. *Let b be a 2-cocycle on G, and let F be a function from G to $\mathbb{R}_+$. Then the following are equivalent:*

(i) There exists $a \in C^1(G;\mathbb{R})$ such that $da = b$ and $|a(g, g\overline{s}^{\pm 1})| \leq F(g)$.

(ii) For every $g \in G$ and every relation $w = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \in R$, one has, setting $g_i = g\overline{s}_1^{\varepsilon_1} \cdots \overline{s}_{i-1}^{\varepsilon_{i-1}}$ ($g_0 = 1$):

$$\Big|\sum_{i=1}^n b(1, g_i, g_{i+1}) - \sum_{\varepsilon_i = -1} b(1, g_{i+1}, g_i)\Big| \leq \sum_{i=1}^n F(g_i).$$

In particular, if b is bounded, it has a primitive satisfying $|a(g, g\overline{s}^{\pm 1})| \leq \delta_{\mathbb{R}}^{ab}(|g|)$, where $\delta_{\mathbb{R}}^{ab}$ is the Abelianized and regularized Dehn function.

Contents

1. Spaces of currents with compact support. Approximation and regularization results

First, $\mathcal{E}(V) = \bigoplus_{q=0}^{n} \ \mathcal{E}^q(V)$ is the topological vector spaces of smooth differential forms. Its dual space is $\mathcal{E}'(V) = \oplus \ \mathcal{E}'_q(V)$. Recall [dR] that $\mathcal{E}_q(V)$ is reflexive. The elements of $\mathcal{E}'(V)$ will be called *currents with compact support*. This is a slight (but usual) abuse since the topology is distinct from that induced by the space of currents (the dual of forms with with compact support). It will cause no preoblem since we shall never use currents without compact support.

We now consider special subspaces of $\mathcal{E}(V)$.

1) Currents of *finite mass*, where the mass $\mathrm{M}(T)$ is defined by $\mathrm{M}(T) = \sup\{T(\varphi) \mid ||\varphi|| \leq 1\}$. By the representation theorem of Federer [F1], these are the same as compactly supported *measure-type* currents:

$$T_\xi(\varphi) = \int_V \varphi_x(\xi_x) \ d\nu(x),$$

where $\xi : M \to \Lambda^q(TM)$ is a measurable field of q-vectors, compactly supported, and such that $||\xi|| \in L^1(V)$. Note that $\mathrm{M}(T_\xi) = ||\xi||_{L^1}$.

Weighted mass. If φ is a nonnegative function on V, we can define the *weighted mass* of $T \in \mathcal{E}'_q(V)$:

$$\mathrm{M}_\varphi(T) = \sup\{T(\omega) \mid \omega \in \mathcal{E}^q(V) \ , \ ||\omega|| \leq \varphi\}.$$

In particular, if $f = 1$ this is the usual mass.

We denote $\mathcal{M}_q(V) \subset \mathcal{E}'_q(V)$ the subspace of measure-type currents. Following Federer, one defines $\mathrm{N}(T) = \mathrm{M}(T) + \mathrm{M}(\partial T)$, and calls T *normal* if $\mathrm{N}(T)$ is finite. We denote by $\mathcal{N}_q(V)$ the space of compactly supported normal q-currents, and $\mathcal{N}_{q,K}(V)$ the space of those with support in the compact subset K.

Flat chains and locally flat cochains [W] [F1] [F2]. For $K \subset V$ compact, one defines the flat semi-norm

$$F_K(T) = \sup\{T(\varphi) \mid \varphi \in \mathcal{E}^q(V) \ , \ \max(||\varphi||_K, ||d\varphi||_K) \leq 1\}.$$

Then, by [F1] (p. 367), $F_K(T) = \inf\{\mathrm{M}(T - \partial S) + \mathrm{M}(S) \mid \mathrm{supp}(S) \subset K\}$. One defines $\mathcal{F}_{q,K}(V)$ as the F_K-closure of $\mathcal{N}_{q,K}(V)$. The space of *flat* q*-chains* is $\mathcal{F}_q(V) = \cup_K \mathcal{F}_{q,K}(V)$, union over all compact subsets K.

A *locally flat q-cochain* is a linear form ℓ on $\mathcal{F}_q(V)$ (or on $\mathcal{N}_q(V)$) which is F_K-bounded on every $\mathcal{F}_{q,K}$ (or on every $\mathcal{N}_{q,K}$). Such a cochain is equivalent to a *locally flat form* of degree q, ie $\lambda \in L^\infty_{loc}\mathcal{E}^q(V)$ (coefficients measurable and locally bounded) such that there exists $\mu \in L^\infty_{loc}\mathcal{E}^{q+1}(V)$ (necessarily unique) which satisfies $T(\mu) = \partial T(\lambda)$ for every $T \in \mathcal{N}_k(V)$ ($d\lambda = \mu$ in the sense of distributions).

The correspondence $\lambda \leftrightarrow \ell$ is given by $\ell(T_\xi) = \int_V \lambda(\xi)d\nu$ if ξ is a compactly supported field of q-vectors. We define $T_\xi(\lambda) = \ell(T_\xi)$, thus $T(\lambda)$ is defined if $\mathrm{M}(T) < \infty$.

We denote by $\mathcal{F}^*_{loc}(V)$ the space of locally flat forms.

2) *Smooth currents* are currents of the form T_ξ where ξ is a smooth (compactly supported) field of q-vectors. These are also called *diffuse* currents [Su]. We denote $\mathcal{S}'_q(V) \subset \mathcal{E}'_q(V)$ the subspace of smooth currents.

3) *Currents associated to singular chains:* if $c = \sum_{i=1}^k a_i\sigma_i$ is a real Lipschitz singular chain, one associates the integration current

$$I_c(\varphi) = \sum_i a_i \int_{\Delta^q} \sigma_i^*\varphi.$$

Note that this time, $c \mapsto I_c$ is not injective. Note also that $I_{\partial c} = \partial I_c$ and that $\mathrm{M}(I_c) \le \sum_i |a_i|\mathrm{vol}(\sigma_i)$, with equality if there is no geometric cancellation between the σ_i, eg if there images are disjoint.

We denote $\mathcal{C}^{\mathrm{Lip}}_q(V) \subset \mathcal{E}'_q(V)$ the subspace of currents associated to Lipschitz singular chains, and similarly $\mathcal{C}^{C^k}_q(V)$ for $k \in \mathbb{N} \cup \{\infty\}$.

We shall need the following density and regularization results.

1) **Density of smooth singular chains.** *Let T be a normal current on V with support contained in the interior of a compact set K. Then for every $\varepsilon > 0$ there exists a smooth singular $\mathbb{R}$-chain c with values in K, such that*

$$\begin{aligned} &F_K(I_c - T) < \varepsilon \\ &\mathrm{M}(I_c) < \mathrm{M}(T) + \varepsilon \\ &\mathrm{M}(\partial I_c) < \mathrm{M}(\partial T) + \varepsilon. \end{aligned}$$

Proof. Federer ([F1], Theorem 4.2.24) proves the case $V = \mathbb{R}^n$ and T normal, with c polyhedral. The last two inequalities are replaced by $\mathrm{N}(T') \le \mathrm{N}(T) + \varepsilon$, but actually he proves the more precise inequalities stated here.

In general, we embed isometrically $i : V \to \mathbb{R}^N$, and work in an arbitrarily small compact tubular neighbourhood $\widehat{K}$ of $i(K)$, equipped with a smooth projection $\pi : \widehat{K} \to K$ with $||\pi - \mathrm{Id}|| < \varepsilon, ||D\pi|| \le 1+\varepsilon$. Let p be a polyhedral chain in $\widehat{K}$ satisfying $F_K(I_p - T) < \varepsilon/2$, $\mathrm{M}(I_p) < \mathrm{M}(T) + \varepsilon/2$, $\mathrm{M}(\partial I_p) < \mathrm{M}(\partial T) + \varepsilon/2$.

Then $I_{\pi\circ p} = \pi_*(I_p)$, $\mathrm{M}(\pi_* I_p) \le (1+\varepsilon)\mathrm{M}(I_p) < \mathrm{M}(T) + \varepsilon$, and similarly $\mathrm{M}(\partial\pi_* I_p) = \mathrm{M}(\pi_*\partial I_p) < \mathrm{M}(\partial T) + \varepsilon$, $F_K(\pi_* I_p - T) = F_K(\pi_*(I_p - T)) \le (1+\varepsilon)F_K(I_p - T)$. Thus $c = \pi \circ p$ is the desired singular chain.

2) In his book [dR], de Rham proves a regularization theorem for currents. It is easy to adapt his proof (p. 72-83) in the dual setting of locally flat forms, to obtain the following result. See also [F2] in the case where V is an open set in $\mathbb{R}^n$.

Regularization of locally flat forms. *Let $\rho : V \to \mathbb{R}^*_+$ be continuous. There exists a linear chain map of degree* 0, $\mathcal{R}^*_\rho : \mathcal{F}^*_{loc}(V) \to \mathcal{E}^*(V)$, *and a homotopy* $\mathcal{R}^*_\rho - \mathrm{Id} = d\mathcal{H}^*_\rho + \mathcal{H}^*_\rho d$, *with the following properties:*

$$||\mathcal{R}^*_\rho\omega(x)|| \leq (1+\rho(x))\ ||\omega|B(x,\rho(x))||$$
$$||\mathcal{H}^*_\rho\omega(x)|| \leq \rho(x)\ ||\omega|B(x,\rho(x))||.$$

Also, if ω is already smooth, $\mathcal{R}^*_\rho\omega \to \omega$ in the C^∞_{loc} topology if $\rho \to 0$ in the compact-open topology.

2. Answer to Question 1

Let $\omega \in \mathcal{E}^q(V)$ for some $q \geq 2$. Assume that it has a primitive $\tau \in \mathcal{E}^{q-1}(V)$ such that $||\tau|| \leq \varphi$. If $T \in \mathcal{E}'_q(V)$, the "Stokes identity" $T(\omega) = T(d\tau) = (\partial T)(\tau)$ implies the weighted isoperimetric inequality

$$T(\omega) \leq \mathrm{M}_\varphi(\partial T).$$

In particular, if $T = I_c$ is associated to a singular chain, this is the inequality of Theorem 1. This theorem states that the converse is almost true.

Lemma 1. *Under the hypothesis of Theorem 1, we have*

$$(\forall T \in \mathcal{N}_q(V)) \qquad T(\omega) \leq \mathrm{M}_\varphi(\partial T).$$

Proof of the lemma. Let $T \in \mathcal{N}_q(V)$. Let $K \subset V$ be a compact set such that $\mathrm{supp}(T) \subset \mathrm{Int}(K)$. By the density of smooth singular chains, for every $\varepsilon > 0$ there exists a smooth singular $\mathbb{R}$-chain c with values in K, such that

$$\begin{aligned} &F_K(I_c - T) < \varepsilon \\ &\mathrm{M}(I_c) < \mathrm{M}(T) + \varepsilon \\ &\mathrm{M}_\varphi(\partial I_c) < \mathrm{M}_\varphi(\partial T) + \varepsilon. \end{aligned}$$

The first inequality says that there exists S with $\mathrm{M}(T - I_c - \partial S) + \mathrm{M}(S) < \varepsilon$. Since ω is closed,

$$\begin{aligned} T(\omega) &= I_c(\omega) + (T - I_c - \partial S)(\omega) \leq I_c(\omega) + \varepsilon||\omega||_K \\ &\leq \mathrm{M}_\varphi(I_{\partial c}) + \varepsilon||\omega||_K \qquad \text{(by the hypothesis)} \\ &\leq \mathrm{M}_\varphi(\partial T) + \varepsilon + \varepsilon||\omega||_K. \end{aligned}$$

Since this holds for every $\varepsilon > 0$, Lemma 1 follows.

Proof of Theorem 1. If $S \in \partial\mathcal{N}_q(V)$, define $\bar{t}(S) = T(\omega)$ for any $T \in \mathcal{N}_q(V)$ such that $\partial T = S$. This is well defined since ω is exact. Moreover, for every $S = \partial T \in \partial\mathcal{N}_q(V)$, Lemma 1 says that $\bar{t}(S) \leq \mathrm{M}_\varphi(\partial T) = \mathrm{M}_\varphi(S)$. By Hahn-Banach, $\bar{t}$ can be extended to a linear form t on $\mathcal{N}_{q-1}(V)$ such that

$$(\forall S \in \mathcal{N}_{q-1}(V)) \qquad t(S) \leq \mathrm{M}_\varphi(S).$$

Thus t is defined by a L^∞_{loc} form τ_0, satisfying $||\tau_0|| \leq \varphi$ ae. The identity $t(\partial T) = T(\omega)$ if $T \in \mathcal{N}_q(V)$ means that τ_0 is locally flat and $d\tau_0 = \omega$ in the sense of distributions.

Using the regularization theorem of section 1, define

$$\tau = \mathcal{R}^*_\rho\tau_0 - \mathcal{H}^*_\rho\omega = \tau_0 + d\mathcal{H}^*_\rho\tau_0$$

for some $\rho \in C^0(V, \mathbb{R}^*_+)$. Then τ is smooth and $d\tau = d\tau_0 = \omega$. Moreover, for every $x \in V$ one has

$$||\tau(x)|| \leq (1 + \rho(x))\big)\ ||\varphi|B(x, \rho(x))|| + \rho(x)\ ||\omega|B(x, \rho(x))||.$$

If ρ decreases sufficiently fast, the right-hand side is $\leq \varphi(x) + \varepsilon$ for very $x \in V$, qed.

We now state and prove a "localized" generalization.

Theorem 1'. *Let $U \subset V$ be an open subset, and let $\omega \in \mathcal{E}^q(V)$ with $q \geq 2$, and let $\varphi : U \to \mathbb{R}_+$ be continuous, where $U \subset V$ is open. Assume that ω is exact and $I_c(\omega) \leq \mathrm{M}_\varphi(I_{\partial c})$ for every real smooth singular q-chain c on V with boundary in U.*

Then for every $\varepsilon > 0$ and every compact $A \subset U$, there exists a smooth form $\tau \in \mathcal{E}^{q-1}(V)$ such that $d\tau = \omega$ on V and $||\tau|| \leq \varphi + \varepsilon$ on A.

Lemma 2. *Let $K \subset V$ be compact. There exists a positive continuous function F on V with the following property.*

For every $q-1$-current S_1 on V of finite mass which is homologous to a current with with support in K, there exists $T_1 \in \mathcal{N}_q(V)$ such that $\partial T_1 = S_1 + S_2$ with $\mathrm{supp}(S_2) \subset K$ and $\mathrm{M}_{||\omega||}(T_1) + \mathrm{M}_\varphi(S_2) \leq \mathrm{M}_F(S_1)$.

Proof of Theorem 1'. Let T be an element of $\mathcal{N}_q(V)$. We apply the lemma to a compact $K \subset U$ such that $A \subset \mathrm{Int}(K)$, and $S_1 = \partial T \setminus K$. Then

$$\begin{aligned}\partial(T - T_1) &= (\partial T \cap K) + S_1 - \partial T_1 \\ &= (\partial T \cap K) - S_2.\end{aligned}$$

This is supported in K and a fortiori in U, thus by the hypothesis and Lemma 1, one has

$$\begin{aligned}(T - T_1)(\omega) &\leq \mathrm{M}_\varphi((\partial T \cap A) - S_2) \\ &\leq \mathrm{M}_\varphi(\partial T \cap K) + \mathrm{M}_\varphi(S_2)\end{aligned}$$

Thus

$$\begin{aligned}T(\omega) &\leq \mathrm{M}_\varphi(\partial T \cap A) + T_1(\omega) + \mathrm{M}_\varphi(S_2) \\ &\leq \mathrm{M}_\varphi(\partial T \cap K) + \mathrm{M}_F(\partial T \setminus K).\end{aligned}$$

There exists a continuous ψ such that $\psi = \varphi$ on A, $\psi \geq \varphi$ on $K \setminus A$, and $\psi = F$ on $V \setminus K$. Then $T(\omega) \leq \mathrm{M}_\psi(\partial T)$, thus Theorem 1 implies that there exists $\tau \in \mathcal{E}^q(V)$ with $d\tau = \omega$ and $||\tau|| \leq \psi + \varepsilon$. This implies Theorem 1'.

3. The case of volume forms

Proof of theorem 2. Using the density of smooth currents and Theorem 1, it suffices to prove that $T_h(\omega) \leq \mathrm{M}_\varphi(\partial T_h)$ for every current of the form $T_h(\varphi) = \int_V h\varphi$, where h is a smooth function with compact support. Then

$$V_\varphi(\partial T_h) = \sup\{\int_V hd\tau \mid ||\tau|| \leq f\} = \sup\{\int_V dh \wedge \tau \mid ||\tau|| \leq f\} = \int_V ||dh||\varphi\ \nu$$

where ν is the Riemannian volume form.

By the coarea formula [F] applied to $|h|$, $\int_V ||dh||f\nu = \int_0^{+\infty}(\int_{|h|=t} \varphi d\sigma) \wedge dt$. For almost all t, $\Omega_t = \{|h| \geq t\}$ is a smooth compact domain with boundary $\{|h| = t\}$. The hypothesis implies

$$V_\varphi(\partial T_h) \geq \int_0^{+\infty} (\int_{|h|\geq t} \omega) \wedge dt = \int_V |h|\omega.$$

Since this is $\geq T_h(\omega)$, Theorem 3 is proved.

4. Filling and cofilling invariants

We recall here several definitions given by Gromov in [G2], chap.5 (and some variants).

Let $\gamma : S^1 \to V$ be a rectifiable loop, homologous to zero. The *filling area* Fill Area(γ) is the infimum of the area of an integer singular 2-chain c with boundary γ. If we take the infimum over all real chains, we obtain the *real filling area* $\mathbb{R}$Fill Area(γ), which is defined as soon as γ is real-homologous to zero. If γ is integer-homologous to zero, $\mathbb{R}\text{Fill Area}(\gamma) = \lim_n \text{Fill Area}(\gamma^n)/n$.

We can define analogously $\mathbb{R}$Fill Area(b) for any real singular boundary. It clearly depends only on I_b. In fact, one can define (in any dimension) the *filling mass* of a boundary current:

$$\text{Fill Mass}(S) = \inf\{\text{M}(T) \mid T \in \partial\mathcal{E}'_q(V) \text{ and } \partial T = S\}.$$

By the density theorem, $\text{Fill Mass}(I_b) = \mathbb{R}\text{Fill Area}(b)$.

The following result is proved in [F2], 4.13. Actually, it is only stated for locally flat forms, but regularization immediately gives the result with smooth forms (cf also [GLP], 4.35).

Whitney's duality. *If $S_0 \in \partial\mathcal{E}'_q(V)$,*

$$\text{Fill Mass}(S_0) = \sup\{S_0(\tau) \mid \tau \in \mathcal{E}^{q-1}(V)\ ,\ ||d\tau|| \leq 1\}.$$

We recall the proof for the convenience of the reader. The argument is quite close to the proof of Theorem 1.

The inequality $\geq$ is an immediate consequence of Stokes. To prove $\leq$, we have to find for every $\varepsilon > 0$ a smooth form τ such that $||d\tau|| \leq 1$ and $S_0(\tau) \geq \text{FillMass}(S_0) - \varepsilon$. It suffices to find a locally flat form with these properties, then regularization will give the desired smooth one.

Actually we can then take $\varepsilon = 0$. Indeed, by Hahn-Banach there exists a linear form t on $\mathcal{F}_{q-1}(V)$ such that $t(S_0) = \text{Fill Mass}(S_0)$ and $|t(S)| \leq \text{Fill Mass}(S)$ for every S which is a boundary. This is equivalent to a flat form τ such that $S_0(\tau) = \text{Fill Mass}(S_0)$ and $|S(\tau)| \leq \text{Fill Mass}(S)$ for every S which is a boundary, which in turn is equivalent to: $|T(d\tau)| \leq \text{M}(T)$ for every T, ie $||d\tau|| \leq 1$.

Cofilling function. Fix x_0 in V. Gromov defines *the* cofilling function as " the infimum of all functions" $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that every exact 2-form ω on V with $||\omega|| \leq 1$ has a primitive τ on V satisfying $||\tau(x)|| \leq f(d(x_0, x))$.

To make this more precise, we say that such a function f is *a* cofilling function, and we define $\text{Cof}_q(R)$ as the infimum of all $C \geq 0$ such that every exact 2-form ω on V with $||\omega|| \leq 1$ has a primitive τ on V satisfying $||\tau|| \leq C$ on $B'(x_0, R)$.

For $q = 2$, we set $\text{Cof} = \text{Cof}_2$. Under reasonable assumptions, we shall see that $C\text{Cof}(CR)$ is a cofilling function for some constant C, which will justifiy Gromov's definition.

We begin by a general geometric characterization of Cof_q.

Proposition 1. *For every $R \geq 0$,*

$$\text{Cof}_q(R) = \sup\{\frac{\text{Fill Mass}(S)}{\text{M}(S)} \mid S \in \partial\mathcal{E}'_q(V)\ ,\ \text{supp}(S) \subset B(x_0, R)\}.$$

Proof. Using Whitney's duality, it suffices to prove that, for every $\omega \in d\mathcal{E}^q(V)$ with $||\omega|| \leq 1$, one has

$$\inf_{\tau, d\tau=\omega} \max_{B'(x_0,R)} ||\tau|| = \sup\{\frac{T(\omega)}{\mathrm{M}(\partial T)} \mid T \in \mathcal{E}'_q(V)\ ,\ \mathrm{supp}(\partial T) \subset B(x_0,R)\}.$$

The inequality $\geq$ is obvious by Stokes. To prove $\leq$, denote by $\mathcal{R}$ the right-hand-side. We need to find, for every $\varepsilon > 0$, a primitive τ with $||\tau|| \leq \mathcal{R} + \varepsilon$ on $B(x_0, R)$. This results from Theorem 1' with $U = B(x_0, R)$ and $\varphi \equiv \mathcal{R}$.

Now we suppose $q = 2$, and $H_1(V; \mathbb{R}) = 0$.

Proposition 2. *If $H_1(V; \mathbb{R}) = 0$,*

$$\mathrm{Cof}(R) = \sup\{\frac{\mathbb{R}\mathrm{Fill\ Area}(\gamma)}{\ell(\gamma)} \mid \gamma \in \mathrm{Lip}(S^1, B(x_0,R))\}.$$

Remark. One may replace Lip by C^∞.

Proof. The hypothesis implies that $I_\gamma \in \partial\mathcal{E}'_2(V)$ for every Lipschitz loop γ. In Proposition 1, we may restrict by density and homogeneity to $S = \sum_{i=1}^k I_{\gamma_i}$ where $\gamma_i \in \mathrm{Lip}(S^1, B(x_0,R))$.

If $\mathrm{M}(S) < \sum \ell(\gamma_i)$, the loops have common parts which cancel. By approximation, we may assume that this common part is defined on unions of segments. By surgery, one has $S = \sum I_{\gamma'_j}$ with no cancellations. Thus we may assume that $\mathrm{M}(S) = \sum \ell(\gamma_i)$

Then $\mathrm{Fill\ Mass}(S) = \mathbb{R}\mathrm{Fill\ Area}(\sum \gamma_i) \leq \sum_{i=1}^k \mathbb{R}\mathrm{FillArea}(\gamma_i)$ and thus

$$\frac{\mathrm{Fill\ Mass}(S)}{\mathrm{M}(S)} \leq \frac{\sum_{i=1}^k \mathbb{R}\mathrm{FillArea}}{\sum \ell(\gamma_i)} \leq \frac{\max_i \mathbb{R}\mathrm{FillArea}(\gamma_i)}{\ell(\gamma_i)}.$$

This proves Proposition 2.

Real filling area function. This is the function $\mathbb{R}FA : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\mathbb{R}FA(R) = \sup \mathbb{R}\mathrm{Fill\ Area}(T_c) \mid c \in \mathrm{Lip}(S^1, M)\ ,\ [c] = 0 \in H_1(V,\mathbb{R})\ ,\ \ell(c) \leq R\}.$$

Using c^n, one sees that $\mathbb{R}FA(nR) \geq n\mathrm{FA}(R)$ if $n \in \mathbb{N}$, thus $\frac{\mathbb{R}FA(R)}{R}$ is "almost non-decreasing": $\frac{\mathbb{R}FA(r)}{r} \leq 2\frac{\mathbb{R}FA(R)}{R}$ if $r < R$.

Theorem 3. *(i) If $H_1(V; \mathbb{R}) = 0$, $\mathrm{Cof}(R) \leq 2\frac{\mathbb{R}FA(3R)}{R}$.*

(ii) If moreover V is C-quasihomogeneous and $R \geq 2C$, $\mathrm{Cof}(R) \geq C^{-3}\frac{\mathbb{R}FA(R)}{R}$. Thus $\mathrm{Cof}(R) \sim \frac{\mathbb{R}FA(R)}{R}$.

Proof

(i) Let γ be a loop in $B(x_0, R)$. If $\ell(\gamma) \leq R$,

$$\frac{\mathbb{R}\mathrm{FillArea}(\gamma)}{\ell(\gamma)} \leq \sup_{r \leq R} \frac{\mathbb{R}FA(r)}{r} \leq 2\frac{\mathrm{FA}(R)}{R}.$$

If $\ell(\gamma) > R$, we take $x_1, \cdots, x_k \in c$ with k the smallest integer $\geq \ell(\gamma)/R$, such that the length of the arc $\gamma_i = x_i x_{i+1}$ on γ is at most R, where we identify $x_{k+1} = x_1$. We define an oriented loop $\gamma_i' = f_i * \gamma_i * f_{i+1}^{-1}$ where f_i is a path from x_0 to x_i of length $\leq R$. Then $\ell(\gamma_i') \leq 3R$ and $I_\gamma = \sum_{i=1}^k I_{\gamma_i}$, thus

$$\mathbb{R}\text{FillArea}(\gamma) \leq \sum_{i=1}^{k} \mathbb{R}FA(\gamma_i) \leq k\mathbb{R}FA(3R) \leq (\frac{\ell(\gamma)}{R} + 1)\mathbb{R}FA(3R).$$

Thus

$$\frac{\mathbb{R}\text{FillArea}(\gamma)}{\ell(\gamma)} \leq \frac{\mathbb{R}FA(3R)}{R}(1 + \frac{R}{\ell(\gamma)}) \leq 2\frac{\mathbb{R}FA(3R)}{R}.$$

Taking the supremum over all γ and using Proposition 2, we obtain (i).

(ii) Let $\gamma \in \text{Lip}(S^1, V)$ be a loop of length $\leq R$. Its diameter is at most $R/2$, thus the quasihomogeneity gives $\gamma' = \varphi \circ \gamma$ with values in $B(x_0, R/2 + C)$, of length $\leq CR$. It also implies $\mathbb{R}\text{FillArea}(\gamma) \leq C^2 \mathbb{R}\text{FillArea}(\gamma')$. For $R \geq 2C$, $\gamma'(S^1) \subset B(x_0, R)$, thus $\mathbb{R}\text{FillArea}(\gamma') \leq \text{Cof}(R)\ell(\gamma') \leq CR\text{Cof}(R)$.

Finally, $\mathbb{R}\text{FillArea}(\gamma) \leq C^3 R\text{Cof}(R)$, which gives (ii).

Proposition 3. *We make the assumptions (i) and (ii) of Theorem 3. Define*

$$\varphi(x) = 3C^2 \frac{\mathbb{R}FA(6d(x_0, x))}{d(x_0, x)}.$$

Then Fill Mass$(S) \leq \text{M}_\varphi(S)$ *for every* $S \in \partial\mathcal{E}_2'(V)$.

Proof. As in Proposition 2, we first reduce to the case where $S = I_\gamma$ with γ a loop. Then using the quasihomogeneity, we may assume that

$$\gamma(S^1) \subset B(x_0, \ell(\gamma)/2 + C) \setminus B'(x_0, 1) \subset B(x_0, \ell(\gamma)) \setminus B'(x_0, 1),$$

and also that $\gamma(0) \in B(x_0, C)$. This will increase the constant N by at most a factor C^2.

We may assume that $\gamma : [0, \ell(\gamma)] \to V$ is parametrized by arclength. We define $t_0 = 0$ and $t_i = t_{i-1} + \frac{1}{2}d(x_0, \gamma(t_{i-1}))$ as long as $t_i \leq \ell(\gamma)$. Since $d(x_0, \gamma(t_i)) \geq 1$, this is possible up to a maximal $i = N$. We obtain thus N consecutive arcs $I_i = \gamma|[t_{i-1}, t_i]$, $1 \leq i \leq N$.

Let c_i be a minimal geodesic from x_0 to $\gamma(t_i)$, and let γ_i the loop $c_{i-1} * (\gamma|I_i) * c_i^{-1}$. Define also $\gamma_0 = c_N * \gamma|[t_N, \ell(\gamma)] * c_0^{-1}$. Set

$$d_i = d(x_0, \gamma(t_i)) \ , \ \ell_i = \ell(\gamma_i) = t_i - t_{i-1} \ , \ \ell_0 = \ell(\gamma) - t_N$$
$$\delta_i = d(x_0, \gamma(I_i)) \ , \ \Delta_i = \max_{t \in I_i} d(x_0, \gamma(t)).$$

Then $\ell_i = \frac{1}{2}d_{i-1}$, thus

$$\delta_i \geq d_{i-1} - \ell_i = \ell_i$$
$$d_i \leq \Delta_i \leq d_{i-1} + \ell_i = 3\ell_i \leq 3\delta_i.$$

Thus

$$\mathbb{R}\text{FillArea}(\gamma) \leq \sum_{i=0}^{N} \mathbb{R}\text{FillArea}(\gamma_i) \leq \mathbb{R}FA(d_N + \ell_0 + d_0) + \sum_{i=1}^{N} \mathbb{R}FAd_{i-1} + \ell_i + d_i)$$
$$\leq \mathbb{R}FA(d_N + \ell_0 + d_0) + \sum_{i=1}^{N} \frac{3\mathbb{R}FA(6\delta_i)\ell_i}{\Delta_i}$$

Moreover, $d_0 \leq C$, $\ell_0 < \frac{1}{2} d_N$ by maximality. Thus

$$d_N \leq d(\gamma(t_0), \gamma(t_N)) + C \leq \ell_0 + C < \frac{1}{2} d_N + C,$$

so $d_N < 2C$, $d_N + \ell_0 + d_0 \leq 2C + C + C = 4C$.

Defining $\psi(x) = \dfrac{3\mathbb{R}FA(6d(x_0, x))}{d(x_0, x)}$, we have

$$\begin{aligned}
\text{Fill Mass}(I_\gamma) = \mathbb{R}\text{FillArea}(\gamma) &\leq \mathbb{R}FA(4C) + \sum_{i=1}^{N} \min_{t \in [t_{i-1}, t_i]} \psi(\gamma(t)) \ (t_i - t_{i-1}) \\
&\leq \mathbb{R}FA(4C) + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \psi(\gamma(t)) \ dt \\
&= \mathbb{R}FA(4C) + \int_0^\ell \psi(\gamma(t)) \ dt \\
&= \mathbb{R}FA(4C) + \mathrm{M}_\psi(I_\gamma).
\end{aligned}$$

Replacing γ by γ^n and making $n \to +\infty$, we deduce $\text{FillMass}(I_\gamma) \leq \mathrm{M}_\psi(I_\gamma)$. Thus for every $S \in \partial\mathcal{E}_2'(V)$, we have $\text{Fill Mass}(S) \leq C^2 \mathrm{M}_\psi(I_\gamma)$. This proves Proposition 3.

Corollary. *Assume that $H_1(V; \mathbb{R}) = 0$ and that V is C-quasihomogeneous. Then every exact 2-form with norm ≤ 1 has a primitive such that $||\tau(x)|| \leq 4C^2 \dfrac{\mathbb{R}FA(6d(x_0, x))}{d(x_0, x)}$. In other words, $f(x) = 4C^2 \dfrac{\mathbb{R}FA(6d(x_0, x))}{d(x_0, x)}$ is a cofilling function.*

By Theorem 3,(ii), it is the "smallest" cofilling function up to equivalence.

5. Primitives of cocycles of degree 2 on a group

Recall that a q-cochain $u \in C^q(G;\mathbb{R})$ on the group G is a function $u : G^{q+1} \to \mathbb{R}$. The differential is defined by

$$du(g_0, \cdots, g_{q+1}) = \sum_{i=0}^{q+1} (-1)^i u(g_0, \cdots, \widehat{g}_i, \cdots, g_q).$$

Recall that the subcomplex of G-invariants cochains $C^*_{inv}(G;\mathbb{R})$ gives rise to the group cohomology $H^*(G,\mathbb{R})$.

Recall the statement of Theorem 4.

Theorem 4. *Let b be a 2-cocycle on G, and let F be a function from G to $\mathbb{R}_+$. Then the following are equivalent:*

(i) There exists $t \in C^1(G;\mathbb{R})$ such that $da = b$ and $|a(g, g\overline{s})| \leq F(g)$.

(ii) For every $g \in G$ and every relation $w = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \in R$, one has, setting $g_i = g\overline{s}_1^{\varepsilon_1} \cdots \overline{s}_{i-1}^{\varepsilon_{i-1}}$ ($g_0 = 1$):

$$\Big|\sum_{i=1}^{n} b(1, g_i, g_{i+1}) - \sum_{\varepsilon_i=-1} b(1, g_{i+1}, g_i)\Big| \leq \sum_{i=1}^{n} F(g_i).$$

Using the canonical primitive $a_0(g,h) = b(g_0, g_1)$, we can write $a = a_0 + dm$ with $m : G \to \mathbb{R}$ (ie $a(g_0, g_1) = a_0(g_0, g_1) + m(g_1) - m(g_0)$). Setting then $\alpha_0(g, s) = a_0(g, g\overline{s})$, we see that the significant data is $\alpha_0 : G \times S \to \mathbb{R}$, which we can view as function on the edges of the Cayley graph. We can restate Theorem 4 as follows.

Theorem 4'. *Let α_0 be a function on $G \times S$, and let F be a function from G to $\mathbb{R}_+$. Then the following are equivalent:*

(i) There exists $m : G \to \mathbb{R}$ such that $|\alpha_0(g, s) + m(g\overline{s}) - m(g)| \leq F(g)$.

(ii) For every $g \in G$ and every relation $w = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \in R$, one has, setting $g_i = g\overline{s}_1^{\varepsilon_1} \cdots \overline{s}_{i-1}^{\varepsilon_{i-1}}$ ($g_0 = 1$):

$$\Big|\sum_{\varepsilon_i=1} \alpha_0(g_i, s_i) - \sum_{\varepsilon_i=-1} \alpha_0(g_{i+1}, s_i)\Big| \leq \sum_{i=1}^{n} F(g_i)$$

Proof of Theorem 4'. We consider m as a linear form on $\mathbb{R}[G]$. By Hahn-Banach, (i) is equivalent to

$$(i)' \qquad \sum_{i=1}^{n} \tau_i(g_i\overline{s}_i - g_i) = 0 \Rightarrow \Big|\sum_{i=1}^{n} \tau_i \alpha_0(g_i, s_i)\Big| \leq \sum_{i=1}^{n} |\tau_i| F(g_i),$$

where the τ_i are nonzero real numbers.

1) Suppose that (i) is true. The hypothesis of (ii) implies $\sum_{i=1}^n \varepsilon_i(g_i\overline{s}_i^{\varepsilon_i} - g_i) = 0$. Then (i)' with $\tau_i = \varepsilon_i$ gives (ii).

2) Suppose that (ii) is true, and that

$$(1) \qquad \sum_{i=1}^{n} \tau_i(g_i\overline{s}_i - g_i) = 0.$$

We want to prove that

$$(2) \qquad \Big|\sum_{i=1}^{n}\tau_i\alpha_0(g_i,s_i)\Big| \le \sum_{i=1}^{n}|\tau_i|F(g_i).$$

We argue by induction over n, the result being trivial for $n=0$. We may assume that $\tau_1>0$ and that $|\tau_1|$ is minimal.

The term $\tau_1 g_1\overline{s}_1$ must cancel with some other, ie there exists $i=i_2$ such that either ($g_1\overline{s}_1=g_i$ with $\tau_i\tau_1>0$), or ($g_1\overline{s}_1=g_i\overline{s}_i$ with $\tau_i\tau_1<0$). Continuing with the term $\tau_i g_i\overline{s}_i$ or $\tau_i g_i$ respectively, we define inductively $i_1=1,i_2,i_3,\cdots$ and $\varepsilon_1=1,\varepsilon_2,\cdots$, such that, for all k, one has

$$g_1\overline{s}_{i_1}^{\varepsilon_1}\overline{s}_{i_2}^{\varepsilon_2}\overline{s}_{i_3}^{\varepsilon_3}\cdots\overline{s}_{i_k}^{\varepsilon_k}=\begin{cases} g_{i_{k+1}} & \text{if } \varepsilon_{k+1}=1\\ g_{i_{k+1}}\overline{s}_{i_{k+1}} & \text{if } \varepsilon_{k+1}=-1\end{cases}$$
$$\varepsilon_k=\operatorname{sgn}(\tau_{i_k}).$$

Let k be the smallest integer such that $i_{k+1}=i_1=1$. If we have $i_\ell=i_m$ for some $1\le\ell<m\le k$, we can suppress the indexes i_r with r between $\ell+1$ and m. Thus we can assume that all the i_r are distinct. Since $g_{k+1}=g_1$, we have $s_{i_1}^{\varepsilon_1}\cdots s_{i_k}^{\varepsilon_k}\in R$, with $\epsilon_1=1$. This implies

$$\sum_{\varepsilon_r=1}(g_{i_r}\overline{s}_{i_r}-g_{i_r})-\sum_{\varepsilon_r=-1}(g_{i_{r+1}}\overline{s}_{i_r}-g_{i_{r+1}})=0.$$

Changing the numbering of the g_i, we can rewrite this equality and (1) as

$$\sum_{i=1}^{k}\varepsilon_i(g_i\overline{s}_i-g_i)=0$$
$$\sum_{i=1}^{n}\tau_i(g_i\overline{s}_i-g_i)=0.$$

We also have $\varepsilon_i=\operatorname{sgn}(\tau_i)$. Combining the two, we get

$$\sum_{i=2}^{k}(\tau_i-\varepsilon_i\tau_1)(g_i\overline{s}_i-g_i)+\sum_{i=k+1}^{n}=0.$$

The inductive hypothesis implies

$$\Big|\sum_{i=2}^{k}(\tau_i-\varepsilon_i\tau_1)\alpha_0(g_i,s_i)+\sum_{i=k+1}^{n}\tau_i\alpha_0(g_i,s_i)\Big|\le\sum_{i=2}^{k}|\tau_i-\varepsilon_i\tau_1|F(g_i)+\sum_{i=k+1}^{n}|\tau_i|F(g_i).$$

By (ii), the property $s_{i_1}^{\varepsilon_1}\cdots s_{i_k}^{\varepsilon_k}\in R$, with $\epsilon_1=1$ implies (with the new numbering)

$$\Big|\sum_{\varepsilon_i=1}\varepsilon_i\alpha_0(g_i,s_i)\Big|\le\sum_{r=1}^{k}F(g_{i_r}).$$

Finally, the hypotheses imply $|\tau_i-\varepsilon\tau_1|+|\tau_1|=|\tau_i|$, thus combining the last two inequalities gives (i)'. This finishes the proof of Theorem 4.

Remark. The proof of $(ii) \Rightarrow (i)'$ is related to the property $\ker(\partial_1) = \mathrm{im}(\theta)$ in the "Hopf" exact sequence

$$0 \to R^{ab} \xrightarrow{\theta} \mathbb{Z}[G]^p \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

or its tensorization over the reals. Here the relation module R_{ab} is the abelianization of $R \subset F(S) = F_p$, the relation subgroup. The G-action comes from conjugation in F_p, thus $\theta([gwg^{-1}]) = g\theta([w])$.

Define

$$x = \sum_{i=1}^{n} \tau_i g_i e_{s_i} \in \mathbb{R}[G]^S = \mathbb{R}[G]^p.$$

The relation $\sum_{i=1}^{n} \tau_i(g_i\overline{s}_i - g_i) = 0$ translates to $\partial_1(x) = 0$. Thus there exists a finite family $(r_q, \mu_q) \in R \times \mathbb{R}$ such that

$$\theta(\sum_q \mu_q \overline{r}_q) = \sum \tau_i g_i. \tag{4}$$

By [Brow] p.45, an explicit formula for $\theta([w])$, where r is the relation $s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}$, is

$$\theta([w]) = \sum_{s\in S} \overline{\frac{\partial r}{\partial s}}\, e_s = \sum_{\varepsilon_i = 1} g_i e_{s_i} - \sum_{\varepsilon_i = -1} g_{i+1} e_{s_i},$$

where $g_i = \overline{s}_1^{\varepsilon_1} \cdots \overline{s}_{i-1}^{\varepsilon_{i-1}}$.

In other words, θ is induced by the derivation $d : F \to \mathbb{Z}[G]^p$ such that $d(s_i) = e_i$.

Then the formula $(*)$ translates into a decomposition of the identity $\sum \tau_i(g_i\overline{s}_i - g_i) = 0$ into a combination of identities $(\sum \varepsilon_i(\widetilde{g}_i - \widetilde{g}_i\overline{s}_i) = 0)$ associated to the relations.

6. Relation with the ℓ_1-norm of Gersten and the homological Dehn function

Assume now that $G = \langle s_1, \cdots, s_p \mid r_1, \cdots, r_q \rangle$ be a finitely presented group. Consider the exact sequence associated to the cellular homology of $\widetilde{M}$, where M is the 2-complex defined by the presentation:

$$\mathbb{Z}[G]^q \xrightarrow{\partial_2} \mathbb{Z}[G]^p \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

or its tensorization over the reals. The Hopf exact sequence gives an isomorphism $\theta : R_{ab} \simeq \ker \partial_1 \subset \mathbb{Z}[G]^p$ (see the previous section), and we have $\theta([r_i]) = \partial_2(f_i)$ for $1 \leq j \leq q$.

The group $\mathbb{Z}[G]$ (or the vector space $\mathbb{R}[G]$) is equipped with the ℓ_1-norm $|\sum_g \tau_g g|_1 = \sum_g |\tau_g|$. This extends to $\mathbb{Z}[G]^q$, $\mathbb{Z}[G]^p$. Then one can define another norm on $\ker \partial_1 = \mathrm{im}(\partial_2)$:

$$||z|| = \inf\{|c|_1 \mid c \in \mathbb{Z}[G]^q \ , \ \partial_2 c = z\}$$

(If we work with integer coefficients, we have a minimum).

If $w \in R$ is a relation, $[w] \in R^{ab} = \mathrm{im}(\partial_2)$. S. Gersten in [Gersten 1990] gives the following definition:

$$||[w]|| = \inf\{|c|_1 \mid c \in \mathbb{Z}[G]^q \ , \ \partial_2 c = [w]\}.$$

One checks that, if the coefficients are integers, this is equal to the abelianized isoperimetric function of [BMS]:

$$||[w]|| = \Delta^{ab}(w) = \min\{m \mid w \in \prod_{i=1}^{m} u_i r_{j_i}^{\varepsilon_i} u_i^{-1} [R, R]\}.$$

We shall need the stable version

$$\Delta^{ab}_{\mathbb{R}}(w) = \lim_{n \to \infty} \frac{\Delta^{ab}(w^n)}{n}.$$

1-cycle associated to a relation. Here it suffices that $G = F(S)/R$ be finitely generated. The space of k-chains is $C_k(G) = \mathbb{Z}[G^{k+1}]$. As a $\mathbb{Z}[G]$-module, it free with the standard basis

$$[g_1|\cdots|g_k] = (1, g_1, g_1 g_2, \cdots, g_1 g_2 \cdots, g_n).$$

If $w = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \in R$, we define $g_i = \overline{s}_1^{\varepsilon_1} \cdots \overline{s}_{i-1}^{\varepsilon_{i-1}}$ and

$$I_w = \sum_{i=1}^{n} (g_i, g_{i+1}) = \sum_{i=1}^{n} g_i [\overline{s}_i] \in C_1(G).$$

In other words, $I_w = \eta(\theta([w]))$, where $\eta : \mathbb{Z}[G]^p \to C_1(G)$ is $\mathbb{R}[G]$-linear and $\eta(e_i) = [\overline{s}_i]$.

Clearly, I_w is a cycle, ie $I_w \in Z_1(G; \mathbb{R})$, and I_w only depends on $[w] \in R^{ab}$. Then one has simply $I_w = \sum_{i=1}^{n} [g_i, g_{i+1}]$.

The complex $C_*(G; \mathbb{R})$ is exact, thus there exists $T \in C_2(G; \mathbb{R})$ with $\partial T = I_w$.

Proposition. *The map $[w] \mapsto I_w$ is injective from R^{ab} to $Z_1(G)$.*

Proof. View w as a loop starting from 1 in the Cayley graph of (G, S). The property $I_w = 0$ means that w has an algebraic coefficient 1 on each edge. This means that it is homologous to zero, ie $w \in [R, R]$, or $[w] = 0$, qed.

Question. Où y a-t-il une référence à ça dans la littérature ?

Corollary. *If $w \in R$,*

$$\Delta^{ab}_{\mathbb{R}}(w) = \max\{a(I_w) \mid t \in C^1(G;\mathbb{R}) \text{ and } (\forall (g,j)\ |a(gI_{r_j})| \leq 1\}.$$

Remark. Note the similarity with (i) in the lemma of section 4.

Proof of the corollary. Let $w \in R$. By the lemma,

$$w \in \prod_{i=1}^{m} u_i r_{j_i}^{\varepsilon_i} u_i^{-1}[R,R] \Leftrightarrow I_w = \sum_{i=1}^{m} \varepsilon_i g_i I_{r_{j_i}}.$$

Thus $\Delta^{ab}(w) = \min\{m \in \mathbb{N} \mid I_w = \sum_{i=1}^{m} \varepsilon_i g_i I_{r_{j_i}}\}$, which implies

$$\Delta^{ab}_{\mathbb{R}}(w) = \inf\{\sum |\tau_i| \mid I_w = \sum \tau_i g_i I_{r_{j_i}}\},$$

the sums being finite and with real coefficients. The corollary is an immediate consequence of Hahn-Banach.

7. Filling and cofilling in groups

Here $G = \langle s_1, \cdots, s_p \mid r_1, \cdots, r_q \rangle$ is a group equipped with a finite presentation. This gives a norm function for each 2-cocycle $b \in Z_2(G)$: if $b = da$, one sets

$$||b||(g) = \max_j |a(gI_{r_j})|.$$

Since I_w is closed, it is a boundary $I_w = \partial_2(c_w)$, thus $a(gI_{r_j}) = b(gc_{r_j})$ depends only on b.

Cofilling function. For $n \in \mathbb{N}$, we define $\mathrm{Cof}(n)$ as the infimum of all $C \geq 0$ such that every cocycle b on G with $||b|| \leq 1$ has a primitive a satisfying $||u_a|| \leq C$ on $B_S(n)$, ie $|a(g, g\overline{s}^{\pm}1)| \leq C$ if $|g| \leq n$.

Lemma. *For every $n \in \mathbb{N}$, one has*

$$\mathrm{Cof}(n) = \sup\{\frac{\Delta^{ab}_{\mathbb{R}}(w)}{|w|} \mid w \in R \ , \ |w| \leq n\}.$$

Proof. Recall the corollary in section 6:

$$\Delta^{ab}_{\mathbb{R}}(w) = \max\{a(I_w) \mid t \in C^1(G;\mathbb{R}) \ , \ ||da|| \leq 1\}.$$

Thus it suffices to prove that, for every $b \in dC^1(G;\mathbb{R})$ with $||b|| \leq 1$, one has

$$\inf_{a, da=b} \max_{B_S(n)} ||a|| = \sup\{\frac{b(T)}{|\partial T|_1} \mid T \in C_2(B_S(n))\}.$$

Call $\mathcal{L}$ the left-hand side and $\mathcal{R}$ the right-hand side. The inequality ($\mathcal{R} \leq \mathcal{L}$) is obvious by Stokes. To prove that ($\mathcal{L} \leq \mathcal{R}$), we need to find a primitive a with $||a|| \leq \mathcal{R}$ on $B_S(n)$. For this, we apply Theorem 4 with $F = \mathcal{R}$ on $B_S(n)$ and $F = \infty$ elsewhere.

It suffices to have $|b(T)| \leq \mathcal{R}|\partial T|_1$ for every $T \in C_2(B_S(n))$. We have $\partial T = \sum \tau_i I_{w_i}$ with $|\partial T|_1 = \sum |\tau_i||w_i|$, thus we may assume $\partial T = I_w$. Then

$$|b(T)| \leq \Delta^{ab}_{\mathbb{R}}(w) \leq \mathcal{R}|I_w|_1 = \mathcal{R}|\partial T|_1,$$

which proves the lemma.

Thus we obtain the *homological Dehn function*, or *abelian isoperimetric function* [BMS]:

$$\delta^{ab}(n) = \sup\{\Delta^{ab}(w) \mid |w| \leq n\} = \sup\{||z|| \mid z \in \ker \partial_1 \ , \ |z|_1 \leq n\}.$$

Again, there are two versions, with integer or real coefficients.

Let $w = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ be a relation, and let $w = \prod_{k=1}^N x_k r_{j_k}^{\eta_k} x_k^{-1}$ be a decomposition into elementary relations modulo $[R, R]$. Assume that N is minimum, ie $N = \Delta^{ab}(w)$. Then $\theta([w]) = \sum_{k=1}^N \eta_k \theta(x_k[r_{j_k}])$, and the condition on b to have a primitive bounded by F becomes

$$|\sum_{k=1}^N \eta_k b(x_k c_{r_{j_k}})| \leq \sum_{\varepsilon_i = 1} F(g_i) + \sum_{\varepsilon_i = -1} F(g_{i+1}).$$

Let $M = \max(|[c_{r_j}]|)$, then the left-hand-side is bounded by $M\Delta^{ab}(w)$. Replacing w by w^n and making $n \to +\infty$, we see that it is in fact bounded by $\Delta^{ab}_R(w)$.

Primitive of a bounded cocycle. In order for Question 2 to have a positive answer for every $a \in Z^2G$ with $||a||| \leq 1$, it suffices that, for every relation $w \in R$, one have

$$\Delta_R^{ab}(w) \leq M^{-1}\big(\sum_{i=1}^{n} F(g_i)\big).$$

Special case: constant bounds. Let $f = A$ be constant in Question 2. Then Theorem 4 says that the answer is positive if, for every relation $w \in R$, one has $\Delta_R^{ab}(w) \leq AM^{-1}|w|$, ie $\delta_R^{ab}(n) \leq AM^{-1}n$.

Relation with hyperbolicity. By Mineyev, this is equivalent to the hyperbolicity of G.

Primitives of cocycles of degree > 2 on a group

Let q be an integer > 2. Let G be a group of type F_q, ie there exists a finite cell complex M such that $\pi_1 M = G$ and $\pi_i M = 0$ for $2 \leq i \leq q-1$. Alternatively, there exists a cell complex Y which is a $K(G,1)$ and has a finite q-skeleton.

8. Relation between Questions 1 and 2

Let V be a Riemannian manifold equipped with a geometrically bounded triangulation T. Let $I^* : C^*(T) \to \mathcal{E}^*(V)$ be the integration morphism. The following result is contained in substance in [Si]. The proof consists essentially in adding bounds to the proof of the theorem of de Rham given in [ST], p.165 sqq.

Proposition

(i) There exists a right inverse R^* for I^* which is a chain map ($R \circ d = d \circ R$) and satisfies

$$||R(u)_x|| + ||d(R(u))_x|| \leq C \max\{|u(\sigma)| \mid \sigma \subset B'(x,C)\}.$$

(ii)There exists a linear map $\Pi^q : \mathcal{B}^q(V) \cap \ker(I^q) \to \mathcal{E}^{q-1}(V)$ such that $\Pi^q(\omega)$ is a primitive of ω and

$$||\Pi^q(\omega)_x|| \leq C \max\{||\omega_y|| \mid y \in B'(x,C)\}.$$

Actually the right inverse has been defined by Whitney ([W] p.226), the new observation is (ii). In fact, a stronger and more natural property holds.

Proposition. *There exists a chain homotopy $H^* : R^*I^* - \mathrm{Id} \simeq 0$, ie a linear map $H^* = (H^q : \mathcal{E}^q(V) \to \mathcal{E}^{q+1}(V))$ of degree 1, with the property*

$$||H(\omega)_x|| \leq C \max\{||\omega_y||, ||d\omega_y|| \mid y \in B'(x,C)\}.$$

Corollary. *Let $\omega \in \mathcal{E}^q$ be an exact q-form on V, and let $t \in C^{q-1}(T;\mathbb{R})$ be a primitive of $I^q(\omega)$. Then ω has a primitive $\tau \in \mathcal{E}^{q-1}(V)$ such that*

$$||\tau_x|| \leq C(\max_{B'(x,C)} ||\omega|| + \max\{|t(\sigma)| \mid \sigma \subset B'(x,C)\}).$$

Proof of the corollary. Let $\omega_1 = \omega - dR(t)$, so that $I^q\omega_1) = 0$, and $\tau = d(R(t)) + \Pi(\omega_1)$. Then $d\tau = \omega$, and the estimates are immediate.

9. Relation between Questions 1, 2 and 3 for $q = 2$

Let M be a compact Riemannian manifold with infinite fundamental group, and $\pi : \widetilde{M} \to M$ be its universal covering.

Let T be a smooth triangulation of M, which we lift to $\widetilde{M}$. We associate to ω the 2-cochain $I_T(\omega)$.

Let X be a smooth cellulation of M, with only one 0-cell x_0. Thus $X^{(2)}$ defines a presentation of $\pi_1(M, x_0) = G$. Similarly, we lift X to $\widetilde{M}$ and define the 2-cochain $I_X(\omega$.

We have an action an action of $G = \pi_1(M, x_0)$ on $\widetilde{M}$. For each $g \in G$ choose a cellular path $\sigma(g)$ from $\widetilde{x}_0$ to $g\widetilde{x}_0$ representing g. This is the same as a normal form $\nu : G \to F$.

Let ω be an exact 2-form on $\widetilde{M}$ for some $q \geq 2$. We define a 2-cocycle $u \in C^2(G, \mathbb{R})$ by

$$u(g_0, g_1, g_2) = \int_{D(g_0,g_1,g_2)} \omega,$$

where $D(g_0, g_1, g_2) \subset \widetilde{M}$ is any cellular disk [C^1 map defined on D^2] bounded by the loop

$$\gamma(g_0, g_1, g_2) = g_0(\sigma(g_0^{-1}g_1) * \sigma(g_1^{-1}g_2) * \sigma(g_0^{-1}g_2)^{-1}.$$

This is well defined since $\int_\Sigma \omega = 0$ for every 2-sphere $\Sigma \subset \widetilde{M}$. [in fact for any surface]

A primitive of u is $t_0(g_0, g_1) = u(1, g_0, g_1) = \int_{D(g_0,g_1)} \omega$ where $D(g_0, g_1)$ is any disk bounded by the loop $\gamma(g_0, g_1) = \sigma(g_0) * \sigma(g_0^{-1}g_1) * \sigma(g_1)^{-1}$.

We want to relate the following properties:

(1) There exists $\tau \in \mathcal{E}^1(\widetilde{M})$ such that $d\tau = \omega$ and $||\tau|| \leq \varphi$.

(2) There exists $t \in C^1(\widetilde{T})$ such that $dt = I_X(\omega)$ and $|t| \leq f$.

(2') There exists $t \in C^1(\widetilde{X})$ such that $dt = I_T(\omega)$ and $|t| \leq f$.

(3) There exists $a \in C^1(G)$ such that $da = b$ and $|t(g, g\overline{s}^{\pm 1})| \leq F(g)$.

Proposition

(i) If (1) holds for some φ, (2) holds for $f(\sigma) = C \max\{\varphi(x) \mid \sigma \subset B'(x, C)\}$.

(ii) If (2) holds for f, (1) holds for $\varphi(x) = C \max\{f(\sigma) \mid \sigma \subset B'(x, C)\}$.

(iii) If (2) holds for f, (3) holds for $F(g) = C \max\{f(\sigma) \mid \sigma \subset \mathrm{st}^2(g\widetilde{x}_0)\}$.

(iii) If (3) holds for F, (2) holds for $f(\sigma) = C \max\{F(g) \mid \sigma \subset \mathrm{st}^2(g\widetilde{x}_0)\}$.

Proof. (i) is obvious: it suffices to take $t = I^1(\tau)$.

(ii) is an immediate consequence of the corollary in section 8.

(iii) and (iv). One defines G-equivariant chain maps $\psi_* : C_*(\widetilde{X}) \to C_*(G)$ and $\chi_* : C_*(G) \to C_*(\widetilde{X})$ in degrees ≤ 2 (cf. [Brown], p.46):

1) If instead of a triangulation we have a cellulation with $c_0 = 1$, $c_1 = p$, $c_2 = q$, we define

- $\psi_0 = \chi_0 = \mathrm{Id}$;
- $\psi_1 = \eta$, ie $\psi(e_i) = [\overline{s}_i]$, $1 \leq i \leq p$ (cf section 6);
- for each $g \in G$, choose a normal form $\nu(g) = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$ representing g, and set

$$\chi_1([1,g]) = d(\nu(g)) = \sum_{i=1}^{n} \frac{\partial(\nu(g))}{\partial s_i}$$
$$= \sum_{\varepsilon_i=1} g_i e_{s_i} - \sum_{\varepsilon_i=-1} g_i e_{s_i},$$

where $g_i = \overline{s}_1^{\varepsilon_1} \cdots \overline{s}_{i-1}^{\varepsilon_{i-1}}$ as usual.

- $\psi_2(f_j) = \text{Cone}(I_{r_j})$, $1 \le j \le q$, where $\text{Cone}(g,h) = (1,g,h) = [g|g^{-1}h]$; if $g_1, \cdots, g_n$ are associated to r_j as usual, $\psi_2(f_j) = \sum_{i=1}^{n} [g_i|\overline{s}_i^{\varepsilon_i}]$.

- for each $(g,h) \in G \times H$, we choose a decomposition

$$\nu(g)\nu(h)\nu(gh)^{-1} = \prod_k x_k r_{j_k}^{\varepsilon_k} x_k^{-1},$$

or

$$\nu(g)\nu(h)\nu(gh)^{-1} \equiv \prod_k x_k r_{j_k}^{\varepsilon_k} x_k^{-1} \text{ mod } [R,R].$$

Then we set

$$\chi_2([g|h]) = \sum \varepsilon_k \overline{x}_k \sigma_{j_k}.$$

By duality we have cochain maps ψ^* and χ^*. Then

Relation between the three questions for $q > 2$

We assume that $\pi^*\omega \in H^q(\widetilde{M};\mathbb{R})$ vanishes, ie there exists $u \in H^q(\pi_1 M;\mathbb{R})$ (unique) such that $i^*[u] = [\omega]$ where $i : M \to X(\pi_1 M, 1)$ is the natural map (defined up to homotopy).

Assume that $\pi_1 V$ is of type F_q [or $\pi_i V = 0$ for $2 \le i \le q-1$].

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